Bredon finiteness properties of groups acting on CAT(0)-spaces

Nansen Petrosyan

KU Leuven

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Goal: Discuss finiteness properties for $E_{\mathcal{FIN}}G$ and $E_{\mathcal{VC}}G$ when G acts isometrically and discretely on a CAT(0)-space.

1. Short introduction to classifying spaces;
2. Associated G-simplicial complex;
3. About proofs of Theorems A and B;
4. Linear groups over positive characteristic;
5. Mapping class group of closed oriented surfaces.

- joint work with Dieter Degrijse

All group considered will be discrete.

Let G be a group. A *family* of subgroups $\mathcal{F}$ of G is a collection of subgroups of G that is closed under conjugation and taking subgroups.

Definition. A *classifying space of G for the family $\mathcal{F}$*, also called a *model for $E_{\mathcal{F}}G$*, is a G-CW-complex X characterized by the properties:

(i) all isotropy subgroups of X are in $\mathcal{F}$;

(ii) for each $H \in \mathcal{F}$, the fixed point set X^H is contractible.

- A model for $E_{\mathcal{F}}G$ can be defined as a terminal object in the G-homotopy category of G-CW-complexes whose isotropy groups are in $\mathcal{F}$.

- A model for $E_{\mathcal{F}}G$ exists for any G and any $\mathcal{F}$.

Examples:

1. If $G \in \mathcal{F}$, then a point is a model for $E_{\mathcal{F}}G$.

2. If $\mathcal{F} = \{1\}$ - the trivial family of subgroups of G, then $E_{\mathcal{F}}G = EG$.

3. Let G be a connected Lie group and K be a maximal compact subgroup. If $\Gamma < G$ is discrete, then G/K is a model for $E_{\mathcal{FIN}}\Gamma$.

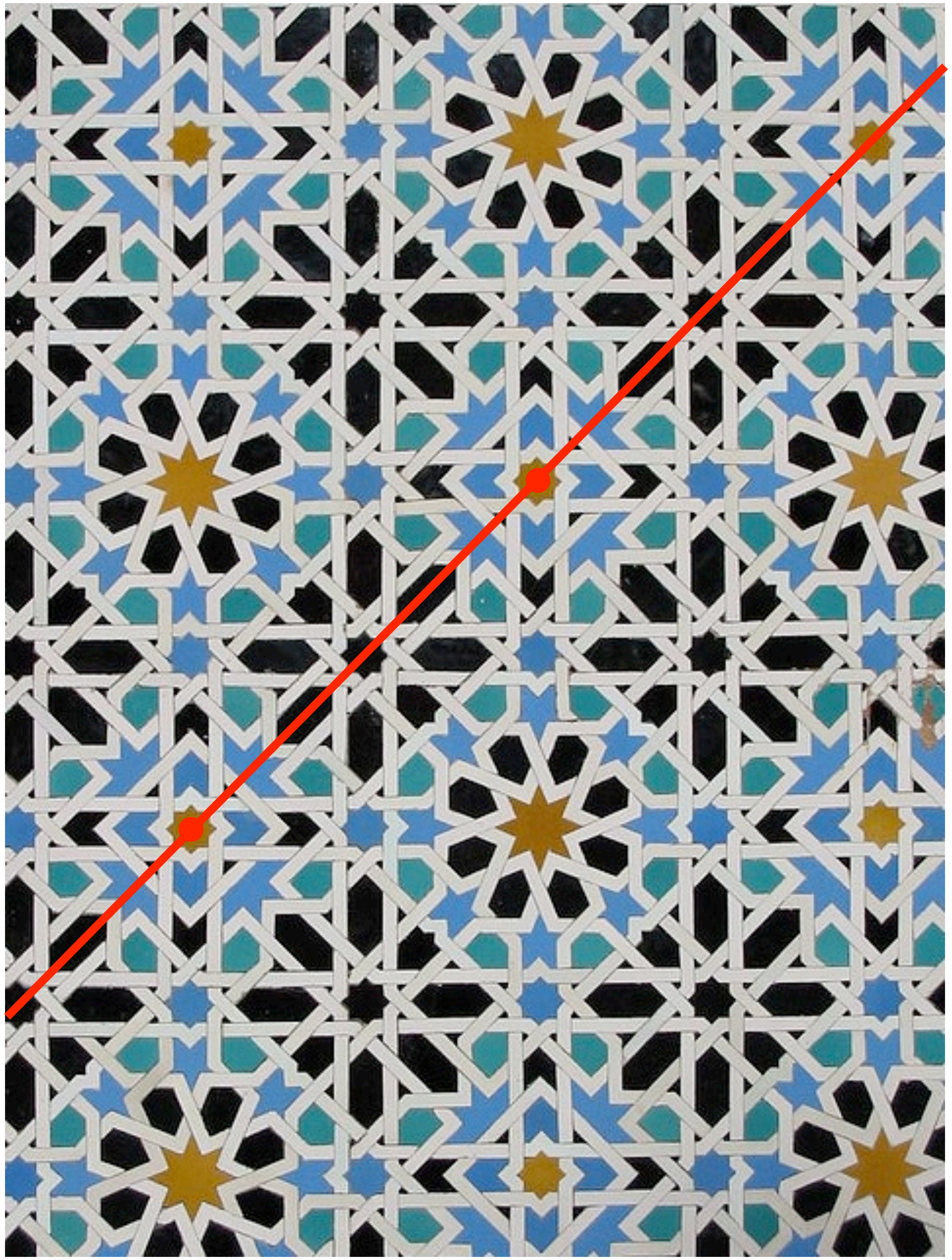
Ex. Let G be an n-dimensional crystallographic group.

- Then G acts isometrically, properly discontinuously and cocompactly on $\mathbb{E}^n$.

- All stabilizer subgroups are finite.

- The fixed point set of every finite subgroup of G is contractible.

- Hence, $\mathbb{E}^n$ is a *finite* model for $E_{\mathcal{FIN}}G$.



Main Motivation. $E_{\mathcal{FIN}}G$ and $E_{\mathcal{VC}}G$ appear in the Isomorphism Conjectures.

Question. What can we say about $E_{\mathcal{FIN}}$ and $E_{\mathcal{VC}}$ from isometric actions of groups on CAT(0)-spaces?

Starting point

Theorem (Lück, 2009). Let G be a group that acts properly and isometrically on a complete proper CAT(0)-space X. Let $d = 1$ or $d \geq 3$ such that top-dim$(X) \leq d$.

(i) Then there exists a model for $E_{\mathcal{FIN}}G$ of dimension at most d.

(ii) If in addition, G acts by semi-simple isometries, then there is a model for $E_{\mathcal{VC}}G$ of dimension at most $d+1$.

- Applies to crystallographic groups!

Question. What if the action on the CAT(0)-space is not proper?

- The key condition we will need is that the actions should be discrete.

Definition. We say that G acts *discretely* on a topological space X if the orbits Gx are discrete subsets of X for all $x \in X$.

- Cellular actions are discrete.

- An isometric group action on a metric space is proper if and only if it is discrete and all point stabilizers are finite.

Setting. G acts isometrically and discretely on a CAT(0)-space.

First step. Associate to the isometric action of a group on a metric space a certain simplicial action.

Proposition. Let X be a separable metric space of topological dimension at most n. Suppose G acts isometrically and discretely on X.

(i) Then there exists a simplicial G-complex Y of dimension at most n for which the stabilizers are the point stabilizers of X, together with a G-map $f : X \to Y$.

(ii) Moreover, if G act cocompactly, then Y/G is finite.

Sketch of proof.

For every $x \in X$, there exists an $\varepsilon > 0$ such that for all $g \in G$

$$g \cdot \mathrm{B}(x, \varepsilon) \cap \mathrm{B}(x, \varepsilon) \neq \emptyset \Leftrightarrow g \in G_x.$$

A *good open cover* $\mathcal{V}$ is a G-invariant open cover of X such that every $V \in \mathcal{V}$ satisfies:

there exists $x_V \in X$ such that for each $g \in G$

$$g \cdot V \cap V \neq \emptyset \Leftrightarrow g \cdot V = V \Leftrightarrow g \in G_{x_V}.$$

The *nerve* $\mathcal{N}(\mathcal{V})$ of $\mathcal{V}$ is the simplicial complex whose vertices are the elements of $\mathcal{V}$ and the pairwise distinct vertices $V_0, \ldots, V_d$ span a d-simplex if and only if $\cap_{i=0}^{d} V_i \neq \emptyset$.

- Since $\mathcal{V}$ is G-invariant, the action of G on X induces a simplicial action of G on $\mathcal{N}(\mathcal{V})$.

- Given $g \in G$, then $g \cdot (V_0, \ldots, V_d) = (V_0, \ldots, V_d)$ if and only if $(V_0, \ldots, V_d)$ is fixed pointwise by g.

Therefore, $\mathcal{N}(\mathcal{V})$ is a G-simplicial complex for which the stabilizers are point stabilizers of X.

(i) If $\dim(\mathcal{V}) \leq n$ then $\mathcal{N}(\mathcal{V})$ is of dimension at most n.

(ii) If the cover $\mathcal{V}$ has only finitely many G-orbits, then $\mathcal{N}(\mathcal{V})/G$ is finite.

In the rest of the proof we find a G-invariant good open cover of X that allows one to construct a G-map $f : X \to \mathcal{N}(\mathcal{V})$ and satisfying (i) and (ii).

$\square$

Theorem A. Let G be a group acting isometrically and discretely on a separable CAT(0)-space X of topological dimension n. Let $\mathcal{F}$ be a family such that $X^H \neq \emptyset$ for all $H \in \mathcal{F}$. Denote $d = \sup\{\mathrm{gd}_{\mathcal{F}\cap G_x}(G_x) \mid x \in X\}$. Then

$$\mathrm{gd}_{\mathcal{F}}(G) \leq \max\{3, n+d\}.$$

Sketch of proof.

Let $J_{\mathcal{F}}G$ be the terminal object in the G-homotopy category of $\mathcal{F}$-numerable G-spaces.

There exists a G-map $\varphi : E_{\mathcal{F}}G \to X \times J_{\mathcal{F}}G$ because $X \times J_{\mathcal{F}}G$ is a model for $J_{\mathcal{F}}G$ and $E_{\mathcal{F}}G$ is $\mathcal{F}$-numerable.

Theorem A. Let G be a group acting isometrically and discretely on a separable CAT(0)-space X of topological dimension n. Let $\mathcal{F}$ be a family such that $X^H \neq \emptyset$ for all $H \in \mathcal{F}$. Denote $d = \sup\{\mathrm{gd}_{\mathcal{F}\cap G_x}(G_x) \mid x \in X\}$. Then

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Sketch of proof.

$$\begin{tikzcd}
E_{\mathcal{F}}G \arrow[r, "\varphi"] \arrow[rrrd, bend right] & X \times J_{\mathcal{F}}G \arrow[r, "f\times \mathrm{Id}"] & Y \times J_{\mathcal{F}}G \arrow[r, "\mathrm{Id}\times\alpha"] & Y \times E_{\mathcal{F}}G \arrow[r, "\pi_2"] \arrow[d, "\simeq"'] & E_{\mathcal{F}}G \\
 & & & Z \arrow[ur, bend right] &
\end{tikzcd}$$

Z is a G-CW-complex of dimension $n + d$ and G-homotopy equivalent to $Y \times E_{\mathcal{F}}G$.

Theorem A. Let G be a group acting isometrically and discretely on a separable CAT(0)-space X of topological dimension n. Let $\mathcal{F}$ be a family such that $X^H \neq \emptyset$ for all $H \in \mathcal{F}$. Denote $d = \sup\{\mathrm{gd}_{\mathcal{F}\cap G_x}(G_x) \mid x \in X\}$. Then

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Sketch of proof.

$$\begin{tikzcd}[column sep=huge] E_{\mathcal{F}}G \arrow[r] \arrow[rr, bend right, "\simeq"'] & Z \arrow[r] & E_{\mathcal{F}}G \end{tikzcd}$$

Since $E_{\mathcal{F}}G$ is G-dominated by an $(n+d)$-dimensional G-CW-complex Z, it is G-homotopy equivalent to one of dimension $\max\{3, n+d\}$. $\square$

Question. What can we say about $E_{\mathcal{VC}}G$ at this point?

Answer. Not much, because when G acts isometrically on a CAT(0)-space an infinite cyclic subgroup C of G, we may have $X^C = \emptyset$.

General Strategy: Adapt a finite dimensional model for $E_{\mathcal{FIN}}G$ into a finite dimensional model for $E_{\mathcal{VC}}G$.

Construction of Lück and Weiermann

Let H be an infinite v-cyclic subgroup of G.

$$N_G[H] := \{x \in G \mid |H \cap H^x| = \infty\} = \mathrm{Comm}_G(H).$$

Let X be the cellular G-pushout:

$$\begin{tikzcd}
\coprod_{H\in\mathcal{I}} G \times_{\mathrm{N}_G[H]} E_{\mathcal{FIN}}\mathrm{N}_G[H] \arrow[r, "i"] \arrow[d, "\coprod_{H\in\mathcal{I}} \mathrm{id}_G \times_{\mathrm{N}_G[H]} f_{[H]}"] & E_{\mathcal{FIN}}G \arrow[d] \\
\coprod_{H\in\mathcal{I}} G \times_{\mathrm{N}_G[H]} E_{\mathcal{F}[H]}\mathrm{N}_G[H] \arrow[r] & {\color{red}X}
\end{tikzcd}$$

If each $f_{[H]}$ is a cellular $\mathrm{N}_G[H]$-map and i is an inclusion of G-CW-complexes, then X is a model for $E_{\mathcal{VC}}G$.

Theorem B. Let G be a countable group acting discretely by semi-simple isometries on a complete separable CAT(0)-space X of topological dimension n. Then

$$\mathrm{cd}_{\mathcal{VC}}(G) \leq n + \max\{\mathrm{st}_{vc}, \mathrm{vst}_{fin} + 1\},$$

where

- $\mathrm{st}_{vc} = \sup\{\mathrm{cd}_{\mathcal{VC}}(G_x) \mid x \in X\}$
- $\mathrm{vst}_{fin} = \sup\{\mathrm{cd}_{\mathcal{FIN}}(E) \mid E \in \mathcal{E}(G, X)\}$

and $\mathcal{E}(G, X)$ is the collection of all groups E that fit

$$1 \to N \to E \to F \to 1,$$

with $N \leq G_x$ for some $x \in X$ and F a subgroup of a finite dihedral group.

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Ex. Let G be a generalized Baumslag-Solitar group and X be the Bass-Serre tree.

The group G acts on X with infinite cyclic stabilizers.

Then $\mathrm{st}_{vc} = 0$ and $\mathrm{vst}_{fin} = 1$ and we get $\mathrm{gd}_{\mathcal{VC}}(G) \leq 3$.

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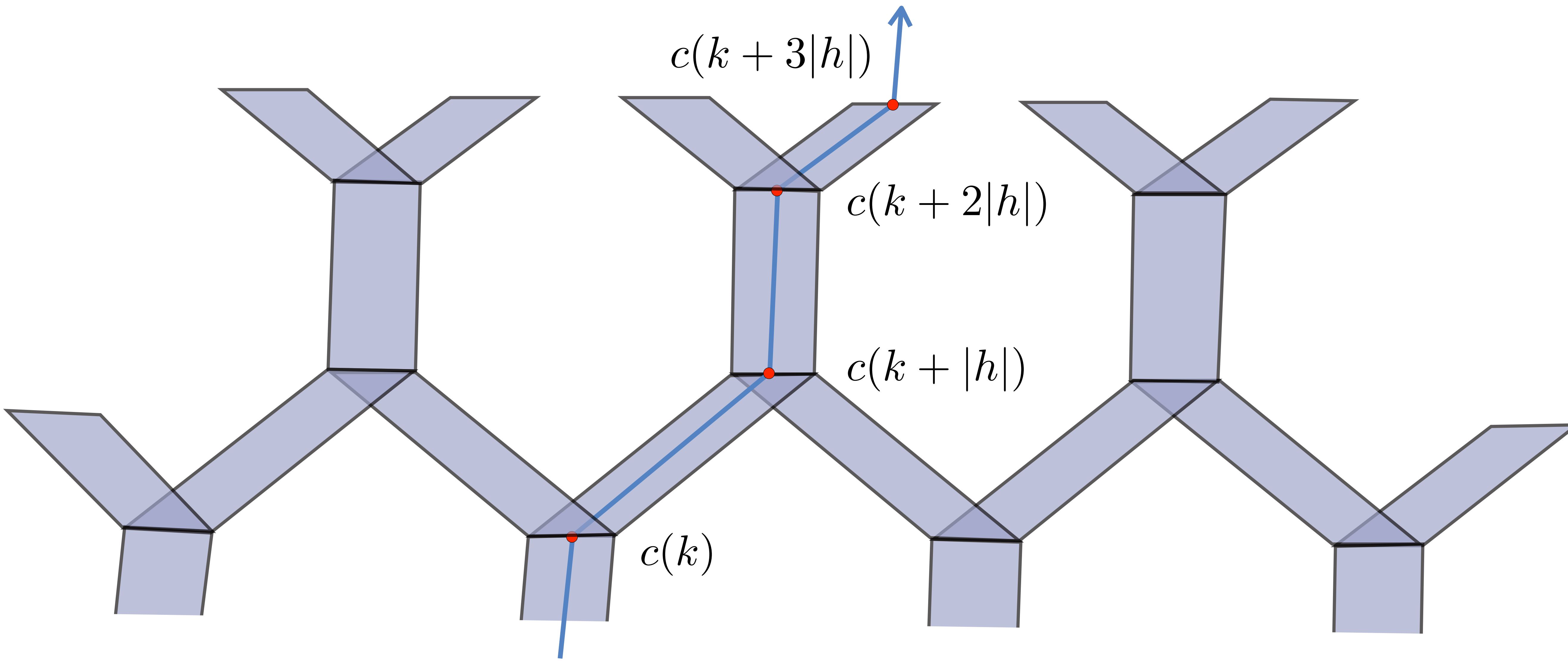
Idea of Proof.

- Apply Theorem A to get a model for $E_{\mathcal{FIN}}G$.

- Use Lück-Weiermann's construction to reduce the problem to bounding $\mathrm{cd}_{\mathcal{F}[H]}(\mathrm{N}_G[H])$ for each class $[H]$ where H is an infinite cyclic subgroup G.

- Consider 2 cases: a generator h of H is either an elliptic or a hyperbolic element.

Case 2: $H = \langle h \rangle$ and h is hyperbolic, i.e. has no fixed point.

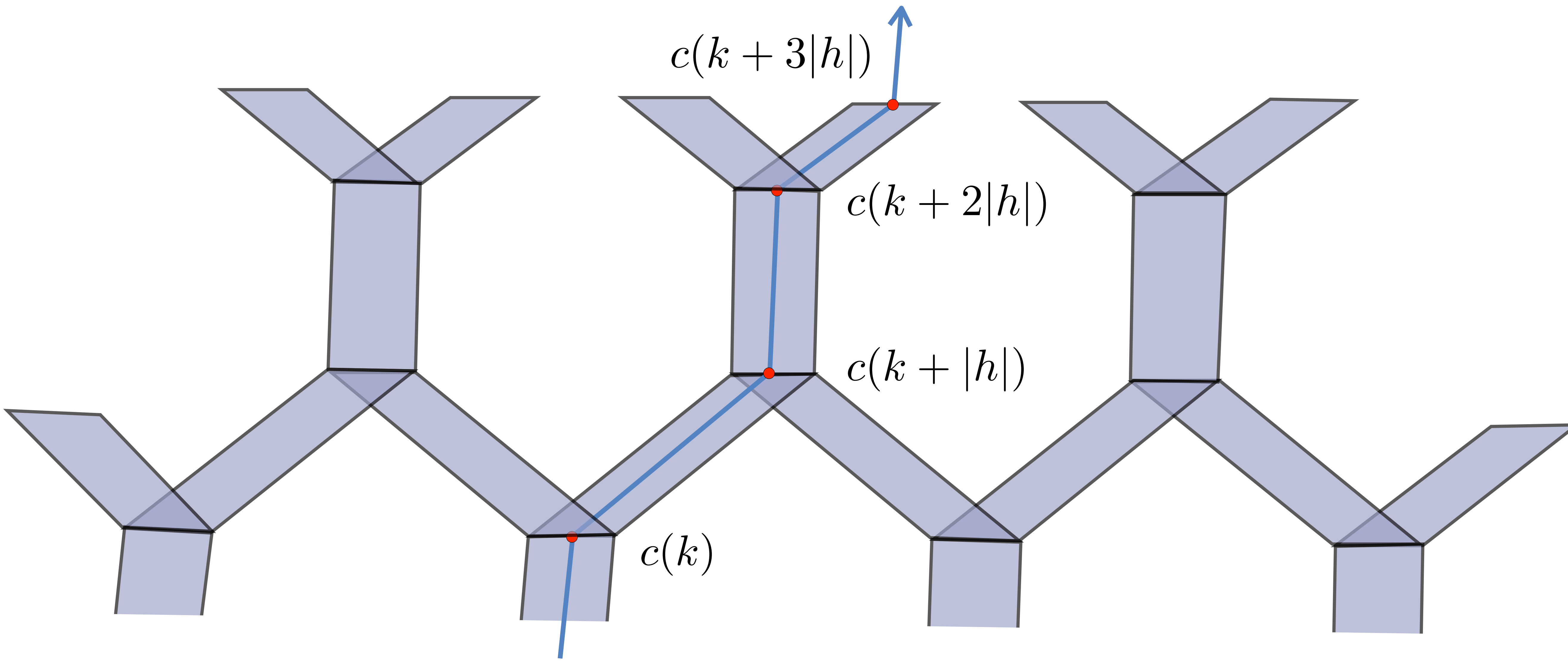


H acts on an axis $c(\mathbb{R})$ of h by $h \cdot c(t) = c(t + |h|)$ where $|h|$ is the translation length.

Let $g \in \mathrm{N}_G[H]$. Then $\exists l, m \neq 0$ such that $g^{-1}h^l g = h^m$.

This implies $|h^l| = |h^m|$. $\quad \Rightarrow \quad l = \pm m$.

Case 2: $H = \langle h \rangle$ and h is hyperbolic, i.e. has no fixed point.

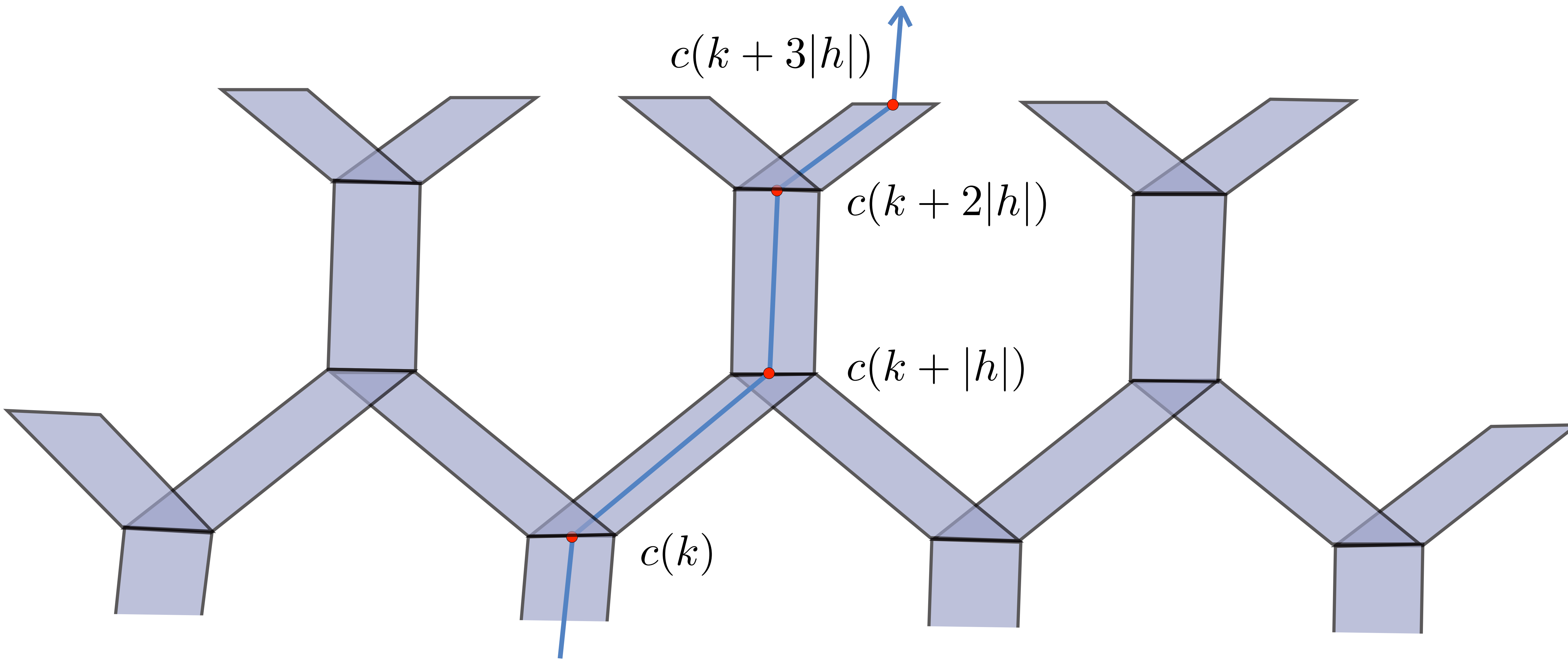


Now, let K be a f.g. subgroup of $\mathrm{N}_G[H]$ that contains H.

$\exists m \neq 0$ so that $g^{-1}h^m g = h^{\pm m}$ for all $g \in K$.

Because $\mathrm{N}_G[H] = \mathrm{N}_G[\langle h^m \rangle]$, we may assume $m = 1$.

Case 2: $H = \langle h \rangle$ and h is hyperbolic, i.e. has no fixed point.

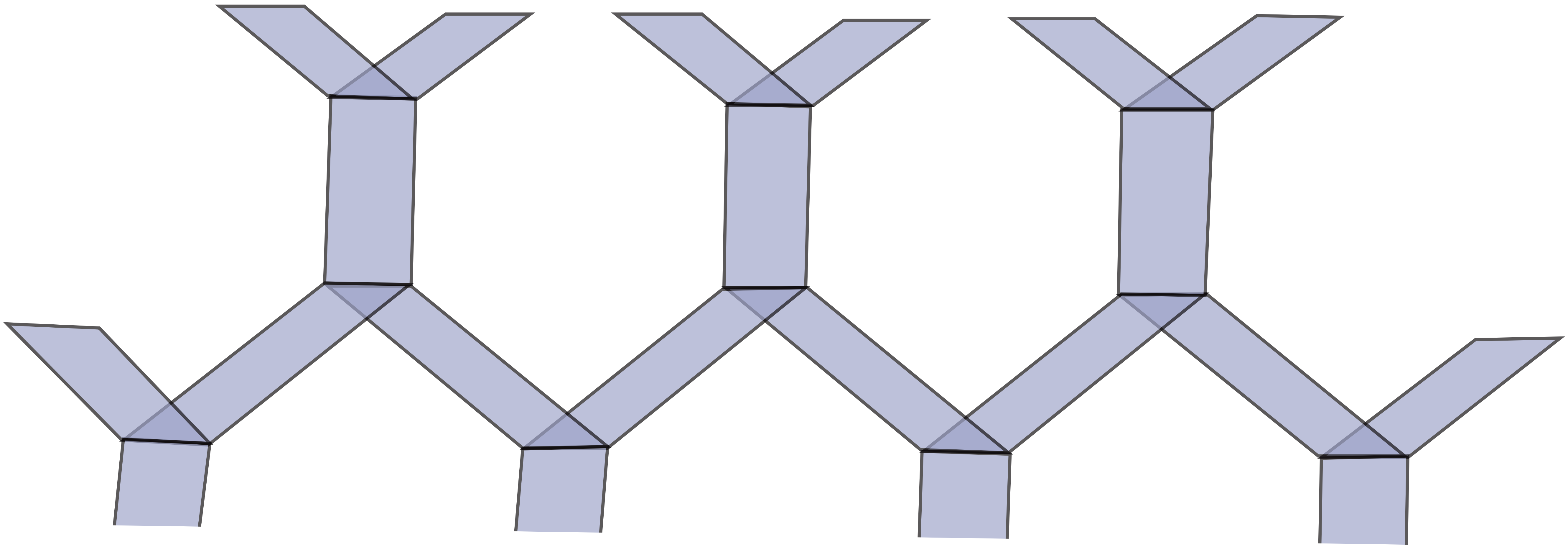


Hence, $H \lhd K$ where K is a f.g. subgroup of $\mathrm{N}_G[H]$.

$\Rightarrow \mathrm{cd}_{\mathcal{F}[H]\cap K}(K) \leq \mathrm{cd}_{\mathcal{FIN}}(K/H)$.

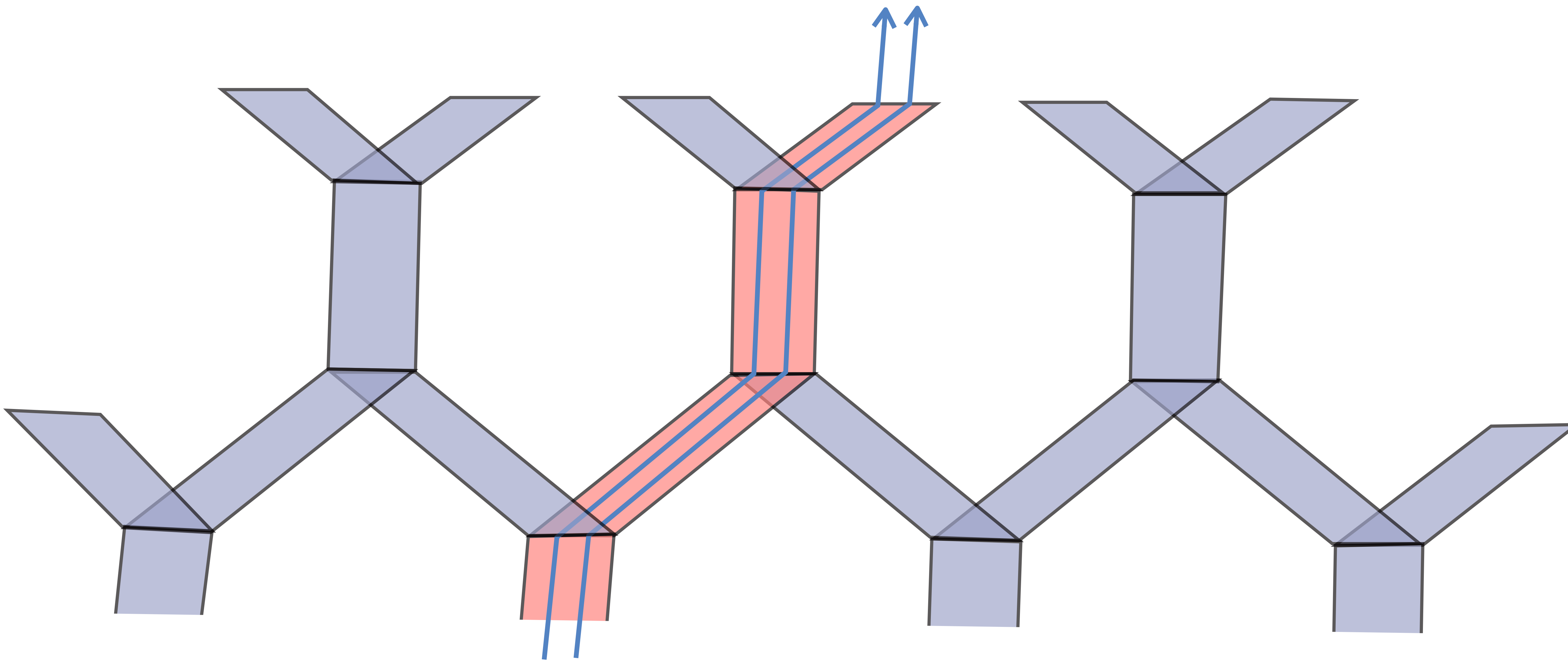
- It is left to bound $\mathrm{cd}_{\mathcal{FIN}}(K/H)$.

Case 2: $H = \langle h \rangle$ and h is hyperbolic, i.e. has no fixed point.



Recall that $\mathrm{Min}(h) = \{x \in X \mid d(h \cdot x, x) = |h|\}$.

Case 2: $H = \langle h \rangle$ and h is hyperbolic, i.e. has no fixed point.



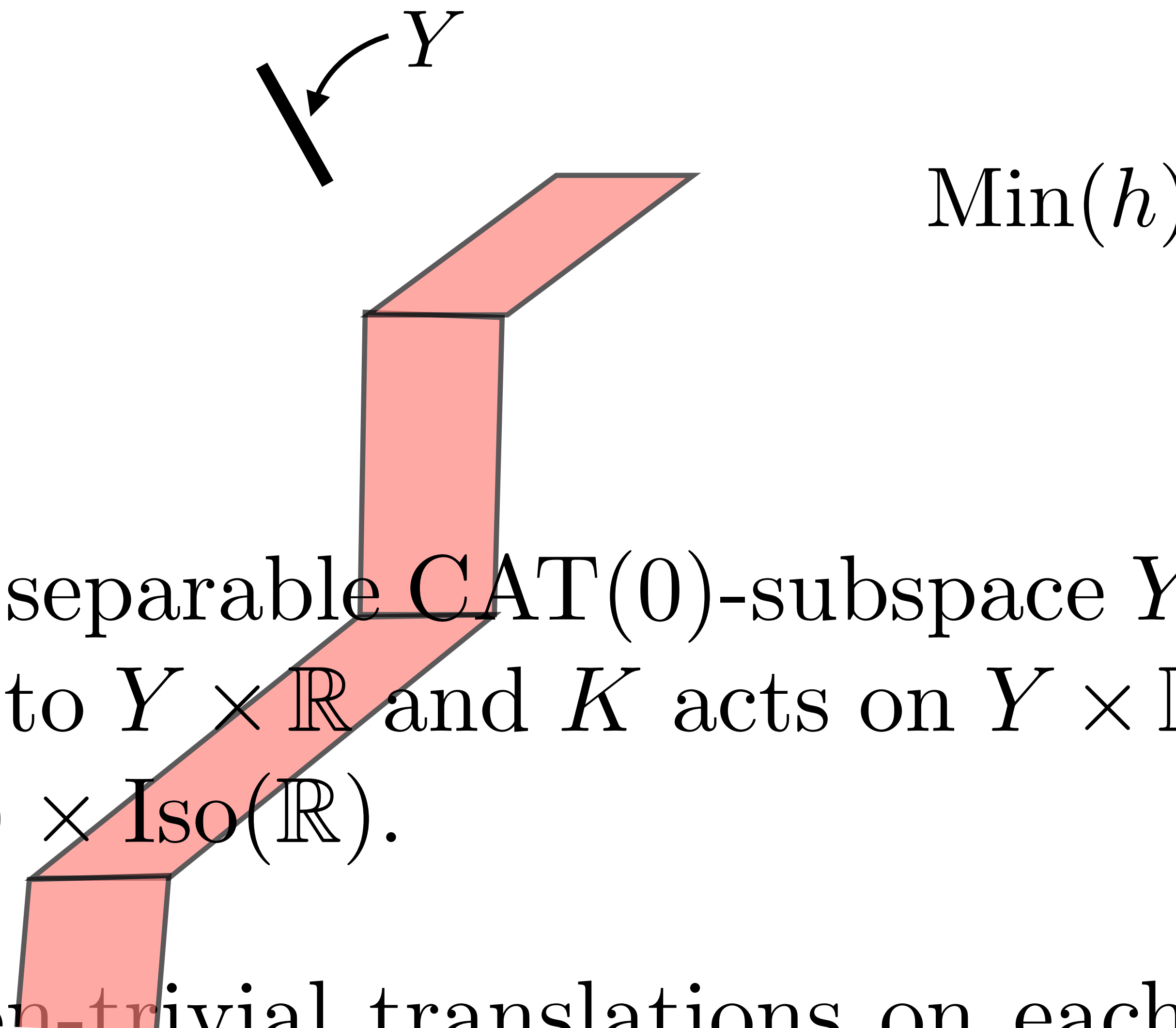
Recall that $\mathrm{Min}(h) = \{x \in X \mid d(h \cdot x, x) = |h|\}$.

It is a complete CAT(0)-space.

$\forall g \in K,\ ghg^{-1} = h^{\pm 1} \ \Rightarrow\ g \cdot \mathrm{Min}(h) = \mathrm{Min}(ghg^{-1}) = \mathrm{Min}(h)$

Moreover, K maps an axis of h to an axis of h.

Case 2: $H = \langle h \rangle$ and h is hyperbolic, i.e. has no fixed point.



There is a complete separable CAT(0)-subspace Y of X so that $\mathrm{Min}(h)$ is isometric to $Y \times \mathbb{R}$ and K acts on $Y \times \mathbb{R}$ via discrete isometries in $\mathrm{Iso}(Y) \times \mathrm{Iso}(\mathbb{R})$.

Since H acts by non-trivial translations on each axis, here it acts trivially on Y-factor and it acts cocompactly on $\mathbb{R}$-factor.

It follows that K/H acts isometrically and discretely on Y.

$$\text{Th.A} \;\Rightarrow\; \mathrm{cd}_{\mathcal{FIN}}(K/H) \leq n - 1 + \mathrm{vst}_{fin}.$$

$\square$

Corollary 1. Let G be a finitely generated subgroup of $\mathrm{GL}_n(F)$ where F is a field of positive characteristic. Then

$$\mathrm{gd}_{\mathcal{FIN}}(G) < \infty \quad \text{and} \quad \mathrm{gd}_{\mathcal{VC}}(G) < \infty.$$

Proof. The strategy is to obtain an action of G on a finite product of buildings.

Cornick-Kropholler Construction

- Can reduce to $G = \mathrm{SL}_n(S)$ where is S is a f.g. domain of characteristic $p > 0$.
- The ring S is a finitely generated domain and hence it is integral over some $\mathbb{F}_p[x_1, \ldots, x_s]$.

- There are finitely many discrete valuations of the fraction field E of S such that $S \cap \bigcap_{i=1}^{r} \mathcal{O}_{v_i} \subseteq L$, the algebraic closure of $\mathbb{F}_p$ in E and L is finite.

$\mathrm{SL}_n(\hat{E}_i)$ acts chamber transitively on the associated Euclidean building X_i of dimension $n-1$.

Let C_i be a chamber of X_i. Since X_i is a continuous image of the separable space $\mathrm{SL}_n(\hat{E}_i) \times C_i$ it is itself separable.

The restriction of this action to G has vertex stabilizers conjugate to a subgroup of $\mathrm{SL}_n(\mathcal{O}_{v_i})$.

Then G acts diagonally on

$$X = X_1 \times \ldots \times X_r$$

such that each stabilizer G_x of a vertex x of X lies inside

$$\mathrm{SL}_n(S) \cap \bigcap_{i=1}^{r} a_i^{-1} \mathrm{SL}_n(\mathcal{O}_{v_i}) a_i, \quad \text{for} \quad a_i \in \mathrm{SL}_n(E), \;\; i = 1, \ldots, r.$$

and therefore is locally finite.

$$\text{Th.A} \;\Rightarrow \quad \mathrm{cd}_{\mathcal{FIN}}(G) \le r(n-1) + 1.$$

$$\begin{aligned}\text{Th.B} \;\Rightarrow \quad \mathrm{cd}_{\mathcal{VC}}(G) &\le r(n-1) + \max\{\mathrm{st}_{vc}, \mathrm{vst}_{fin} + 1\}, \\ &\le r(n-1) + 2.\end{aligned}$$

$\square$

Corollary 2. Let $\mathrm{Mod}(S_g)$ be the mapping class group of a closed, connected and orientable surface of genus $g \geq 2$. Then

$$\mathrm{gd}_{\mathcal{VC}}(\mathrm{Mod}(S_g)) \leq 9g - 8.$$

Proof. Let $\mathcal{T}(S_g)$ be the Teichmüller space of S_g.

- With the natural topology $\mathcal{T}(S_g) \cong \mathbb{R}^{6g-6}$.

This is not enough as we need a CAT(0)-metric!

- Equipped with the Weil-Petersson metric, $\mathcal{T}(S_g)$ is a non-complete separable CAT(0)-space on which $\mathrm{Mod}(S_g)$ acts by isometries.

- The completion of $\mathcal{T}(S_g)$ with respect to the Weil-Petersson metric is the *augmented* Teichmüller space $\overline{\mathcal{T}}(S_g)$.

$\Rightarrow$ $(\overline{\mathcal{T}}(S_g), d_{WP})$ is a complete separable CAT(0)-space of dimension $6g-6$ on which $\mathrm{Mod}(S_g)$ acts (cocompactly) by isometries.

- By a theorem of Bridson (2010), this action is semi-simple.

- Claim 1: The stabilizers are virtually abelian of rank at most $3g-3$.

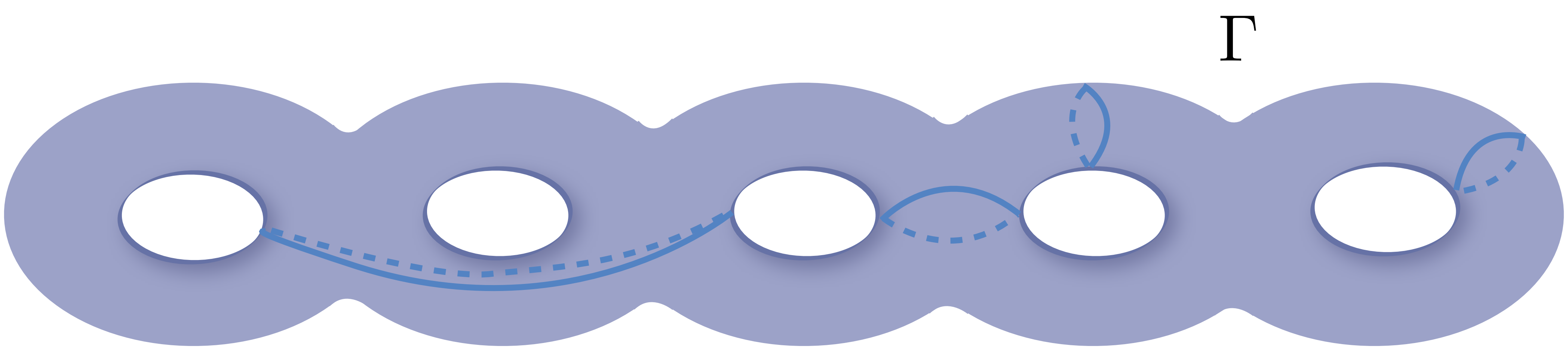
- Claim 2: The action is discrete.

Claim 1: The stabilizers are v-abelian of rank at most $3g-3$.

We only need to check stabilizers of points in $\overline{\mathcal{T}}(S_g) \smallsetminus \mathcal{T}(S_g)$.

It is a union of strata $\mathcal{S}_\Gamma$ corresponding to sets Γ of free homotopy classes of disjoint essential simple closed curves on S_g.

Let $x \in \mathcal{S}_\Gamma$ and let Δ_Γ be the group generated by the Dehn twists defined by the curves in Γ.



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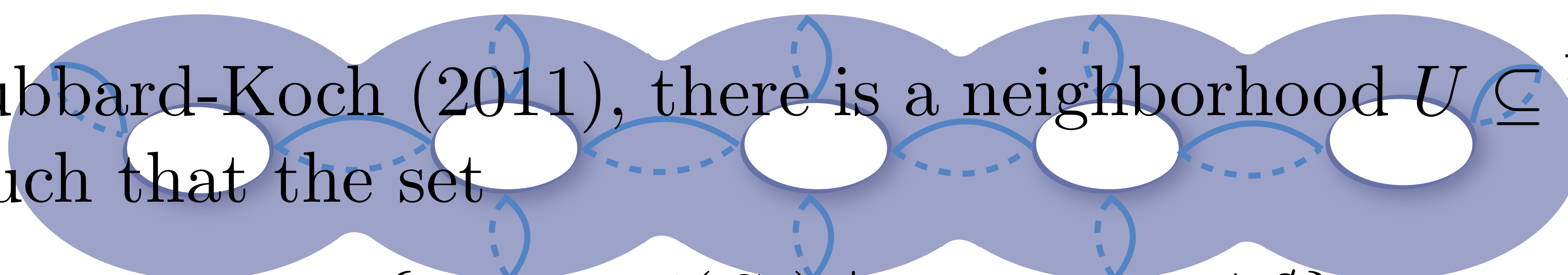

By Hubbard-Koch (2011), there is a neighborhood $U \subseteq \overline{\mathcal{T}}(S_g)$ of x such that the set

$$\{g \in \mathrm{Mod}(S_g) \mid g \cdot U \cap U \neq \emptyset\}$$

is a finite union of cosets of Δ_Γ.

Δ_Γ is free abelian of rank at most $3g-3$ and fixes $\mathcal{S}_\Gamma$ pointwise.

Now, applying Theorem B, we have

$$\begin{aligned}\mathrm{cd}_{\mathcal{VC}}(\mathrm{Mod}(S_g)) &\leq 6g-6+\max\{3g-3+1, 3g-3+1\},\\ &\leq 9g-8. \qquad \square\end{aligned}$$

- Recall that $\mathrm{Mod}(S_g)$ acts cocomactly on $(\overline{\mathcal{T}}(S_g), d_{WP})$.

Then,

$$E_{\mathcal{ST}}\mathrm{Mod}(S_g) \to \overline{\mathcal{T}}(S_g) \xrightarrow{f} Y \to E_{\mathcal{ST}}\mathrm{Mod}(S_g)$$

Corollary 3. $E_{\mathcal{ST}}\mathrm{Mod}(S_g)$ has a model of finite type.

$\square$