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Multiscale methods for parabolic and hyperbolic problems

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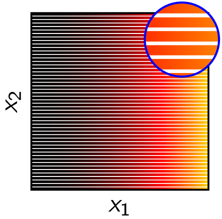
Problem: $\partial_t u^\varepsilon - div(\mathcal{A}^\varepsilon(x, \nabla u^\varepsilon)) = f \qquad \text{in } \Omega \times (0, T)$

Example from electromagnetic problems:

Laminated core: $\varepsilon = 1/50$, $T = 2$.
Magnetic law $\mathcal{A}^\varepsilon(x, \xi) = a^\varepsilon(x, \xi)\xi$ with

- iron: $a^\varepsilon(x, \cdot) \sim (1 + |\xi|^2)^{-0.485}$;
- insulation: $a^\varepsilon(x, \cdot) \sim cst$.



laminated core

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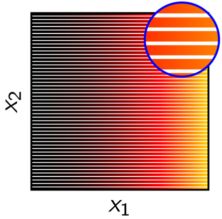
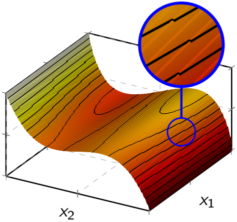


laminated core



fine scale solution u^ε
(standard FEM, 10^6 DOF).

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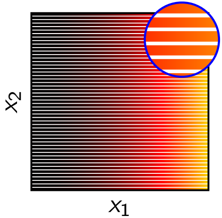

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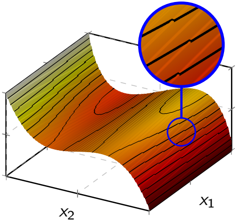

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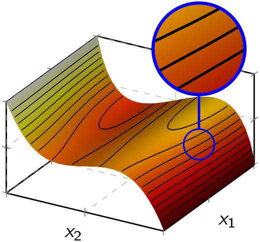
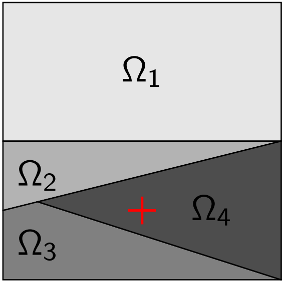


approximation of u^0
(FE-HMM, 10^3 DOF).

Example: Wave propagation in heterogeneous medium *(A.A, M. Grote, C. Stohrer, SIAM MMS 2014)*

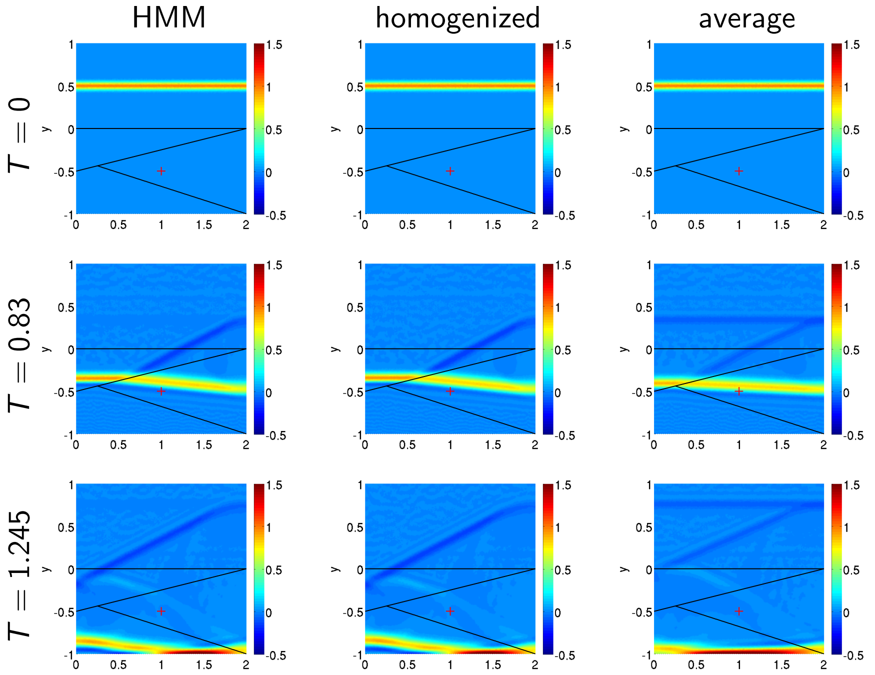
Problem: $\partial_{tt} u_\varepsilon - \nabla \cdot (a^\varepsilon(x) \nabla u_\varepsilon) = f$ in $\Omega \times (0, T)$

Example: Rock layer Ω divided into four distinct subdomains.



$$a^\varepsilon(x) = \begin{cases} 1 & x \in \Omega_1 \\ \sqrt{2} + \sin(\frac{2\pi x_2}{\varepsilon}) & x \in \Omega_2 \\ \sqrt{2} + \frac{1}{2}\Big(\sin(2\pi x_2) + \sin(\frac{2\pi x_2}{\varepsilon})\Big) & x \in \Omega_3 \\ 2 & x \in \Omega_4 \end{cases}$$

- initial condition: down moving Gaussian shaped plane wave
- homogeneous Neumann boundary conditions
- $\delta = \epsilon = \frac{1}{1000}$, $h = \frac{1}{7000}$, $\Delta t = \frac{1}{1000}$
- fully resolved: almost 400 millions elements
- here only: 63'498 elements



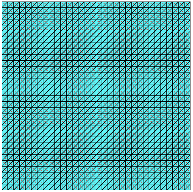
HMM
homogenized
average
$T = 0$
$T = 0.83$
$T = 1.245$
y

Outline:

I) Introduction (variations on elliptic problems)

II) Numerical homogenization of monotone parabolic problems

III) Numerical homogenization of a wave equation



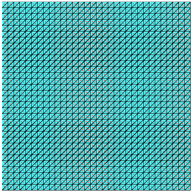
$-\nabla \cdot (a^{\varepsilon} \nabla u^{\varepsilon}) = f$

$a(x) \to a^{\varepsilon}(x)$, ε small scale.

need to resolve the fine scale ε

$\|u^{\varepsilon} - u^{h}\|_{H^1(\Omega)} \leq C \frac{h}{\varepsilon}$. Complexity: $\mathcal{O}(\varepsilon^{-d})$

Elliptic homogenization problems



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Homogenization *(Babuska, Sanchez-Palencia, Bensoussan, Lions, Papanicolaou, Tartar, Jikov, Nguetseng, Lions ...)*

Questions: as $\varepsilon \to 0$ $u^{\varepsilon} \to u^0$? Equation for u^0 ?

Assuming a^{ε} uniformly elliptic and bounded, we have:

$$u^{\varepsilon} \overset{H^1}{\rightharpoonup} u^0, \qquad (a^{\varepsilon} \nabla u^{\varepsilon}) \overset{L^2}{\rightharpoonup} \xi^0, \qquad \text{for } \varepsilon \to 0.$$

Homogenization problem: find $a^0 \in L^{\infty}(\Omega)^{d \times d}$ such that

$$-\nabla \cdot (\underbrace{a^0 \nabla u^0}_{\xi^0}) = f, \text{ on } \Omega, \ u^0 = 0 \text{ on } \partial\Omega.$$

Remark: In general a^0 is not a "simple average" (as arithmetic or harmonic avrg.).

Multiscale problem: find $u^{\varepsilon} \in H_0^1(\Omega)$ s.t.

$\int_{\Omega} a^{\varepsilon}(x)\nabla u^{\varepsilon} \cdot \nabla w dx = \int_{\Omega} f w dx, \quad \forall w \in H_0^1(\Omega).$

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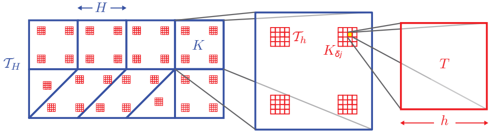
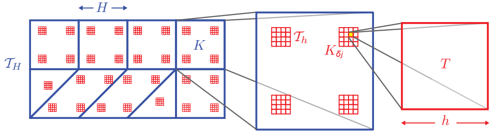


- Macro partition $\Omega = \cup_{K\in\mathcal{T}_H} K$
- Quadrature for $K \in \mathcal{T}_H$ $\{x_{K_j}, \omega_{K_j}\}_{j=1}^J$
- Sampling domain K_{δ_j} centered at x_{K_j}

Multiscale problem: find $u^\varepsilon \in H^1_0(\Omega)$ s.t.

$$\int_\Omega a^\varepsilon(x)\nabla u^\varepsilon \cdot \nabla w dx = \int_\Omega f w dx, \quad \forall w \in H^1_0(\Omega).$$



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Idea: "localize u^ε", define micro problems such that (upscaling)

$$\frac{1}{|K_\delta|}\int_{K_\delta} a^\varepsilon(x)\nabla u^h dx = \frac{1}{|K|}\int_K a^0_K \nabla u^H dx$$

Let u^h be the solution of the micro problems $(u^h - u^H) \in S^1(K_\delta, \mathcal{T}_h)$

$$\int_{K_\delta} a^\varepsilon(x)\nabla u^h \cdot \nabla z^h dx = 0,\ \forall z^h \in S^1(K_\delta, \mathcal{T}_h).$$

Then

$$\sum_{K\in\mathcal{T}} \frac{|K|}{|K_\delta|} \int_{K_\delta} a^\varepsilon(x)\nabla u^h dx \cdot \nabla w^H(x_K) = \sum_{K\in\mathcal{T}} |K| a_K^0 \nabla u^H(x_K) \cdot \nabla w^H(x_K)$$

Equivalently

$$B_H(u^H, w^H) = \sum_{K\in\mathcal{T}} \frac{|K|}{|K_\delta|} \int_{K_\delta} a^\varepsilon(x)\nabla u^h \cdot \nabla w^h dx = \sum_{K\in\mathcal{T}} \int_K a_K^0 \nabla u^H \cdot \nabla w^H dx$$

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Macro problem: find $u^H \in S_0^1(\Omega, \mathcal{T}_H)$ s.t. $B_H(u^H, w^H) = \int_\Omega f w^H dx,$

"looks like" a FEM with numerical quadrature for a homogenized problem

$$\int_\Omega a^0(x)\nabla u^0 \cdot \nabla w dx = \int_\Omega f w dx \ \forall w \in H_0^1(\Omega).$$

FE-HMM: find $u^H \in S_0^\ell(\Omega, \mathcal{T}_h)$ s.t.

$$B_H(u^H, w^H) := \sum_{K\in\mathcal{T}_H} \sum_{j=1}^{J} \omega_{K_j} a_K^0(x_{K_j}) \nabla u^H(x_{K_j}) \cdot \nabla w^H(x_{K_j}) = \int_\Omega f w^H dx$$

where $a_K^0(x_{K_j})$ is an average involving $a^\varepsilon(x)$ and micro problems $u^h \in S^q(K_{\delta_j}, \mathcal{T}_h)$.

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Thm (a priori error estimates).

$$\|u^0 - u^H\|_{H^1(\Omega)} \leq C\left(H^\ell + \left(\tfrac{h}{\varepsilon}\right)^{2q} + e_{MOD}\right)$$

$$\|u^0 - u^H\|_{L^2(\Omega)} \leq C\left(H^{\ell+1} + \left(\tfrac{h}{\varepsilon}\right)^{2q} + e_{MOD}\right)$$

Reconstruction:

$$\|u^\varepsilon - u^{rec,\varepsilon}\|_{\bar{H}^1(\Omega)} \leq C(H^\ell + \left(\tfrac{h}{\varepsilon}\right)^q + \sqrt{\varepsilon} + e_{MOD})$$

(locally periodic pblms: $e_{MOD} = 0$ (sampling domain match period) or $e_{MOD} = \delta + \varepsilon/\delta$).

1. Generalized FEM (Babuska and Osborn 1983): modify the FE space to take into account explicitly the microstructure of the problem.
   - **Increasing FE space by adding functions with right microstructure** (residual-free bubble FEM (...; Brezzi,Marini 2002; Sangalli 2003;...), variational multiscale method (Hughes 1995; Hughes et al.; Ramm et al. 2006; Nolen,Papanicolaou,Pironneau 2008;...)
   - **Changing the basis function of the FE space** (MsFEM (Hou,Wu,Cai 1999;...;Allaire,Brizzi 2005;...;Efendiev,Hou 2009;...) two-scale FEM (Matache,Schwab 2002;...), corrector techniques (Gloria 2006, 2008), harmonic coordinates (Owhadi, Zhang, Berlyand 2008, ...), domain dec. techniques (Graham, Scheichl 2007, ...), LOD (Petersheim,Malqvist, Henning 2012,2013, ...)
2. Constrained macro simulations (macro-to-micro methods)
   - **Representative volume (averaging) elements, iterative coupling** (Fish et al. 1997;...; Feyel,Chaboche 2000; Terada,Kikuchi 2001; Miehe,Schröder,Bayreuther 2002;...; Geers,Kouznetsova,Brekelmans 2010;...)
   - **Equation free framework** (Kevrekidis, Gear et al. 2003, Samaey, Dirk, Kevrekidis 2005, ... Gear, Kevrekidis 2010)
   - **Heterogenenous multiscale method** (E,Engquist 2003; A.A,E 2003; A.A 2005; A.A,Schwab 2005; E,Ming,Zhang 2005; Ming,Yue 2007, Ohlberger 2005,...; A.A,E,Engquist,Vanden-Eijnden 2012, A.A, Huber, Vilmart, Bai 2011,2012,2013)

HMM - numerical homogenisation strategy - relies on sale separation, but

- not necessarily periodic;
- computational cost independent of the smallest scale;
- can be coupled with fine scale solver on regions without scale separation (Oden, Vemaganti "Goal oriented" 2000, Babuska, Lipton "Local-global projection" 2011, A.A., Jecker "Optimization based coupling" 2014, ...).

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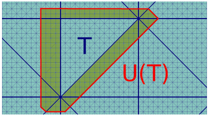
PDEs without scale separation, L^∞ coefficients.
Example: local orthogonal decomposition (LOD) (Malquist and Petersheim 2011).

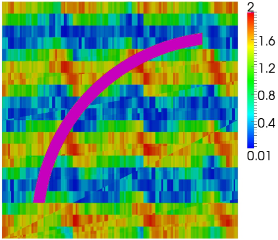
Let V_H coarse grid, V_h fine grid (obtained by refinement of V_H) consider

$$V_h = V_H^{ms} \oplus V_h^f,$$



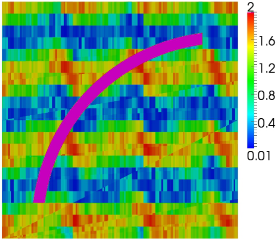
where

- $V_h^f = kern(I_H)$, $I_H : V_h \longrightarrow V_H$ L^2 projection;
- $V_H^{ms} = \{w^H + Q_h(w^H), w^H \in V_H\}$, where $Q_h(w^H) \in V_h^f$ solution of localized elliptic problem.



Pink conductivity channel, high contrast 10^4

LOD method for the wave equation with a continuum of scales (A.A, Henning, 2014)



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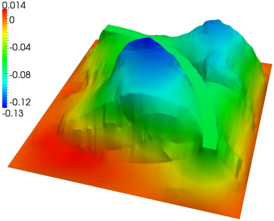
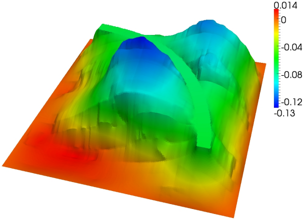
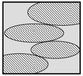


reference solution u_h, $(h = 2^{-8})$, $T = 1$.

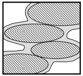


multiscale solution u_H^{ms} $(h = 2^{-4})$ $T = 1$.

Variations on elliptic problems ...



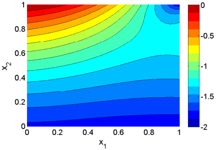
Saturated medium



Unsaturated medium

$$v(x) = -a(p(x))(\nabla p(x) - g(x))$$

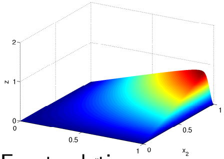
Numerical homogenization for nonlinear PDEs (A.A, G. Vilmart Numer. Math. 2012 and Math. Comp. 2014)



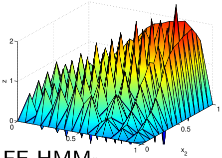
$$-\nabla \cdot (a^{\varepsilon}(x, u^{\varepsilon})\nabla u^{\varepsilon}) = f \quad \text{in } \Omega$$
$$u^{\varepsilon} = 0 \quad \text{on } \partial\Omega.$$

Discontinuous Galerkin FE-HMM for advection-diffusion pblm (A.A., M. Huber Numer. Math. 2014)

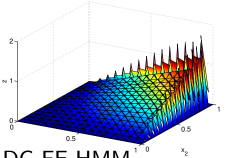
$$-\nabla \cdot (a^{\varepsilon}\nabla u^{\varepsilon}) + b^{\varepsilon} \cdot \nabla u^{\varepsilon} = f \qquad \text{in } \Omega,$$
$$u^{\varepsilon} = 0 \qquad \text{on } \partial\Omega.$$



Exact solution



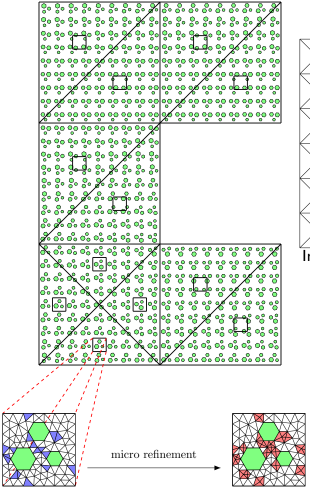
FE-HMM

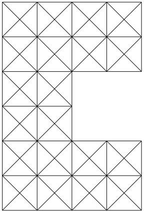


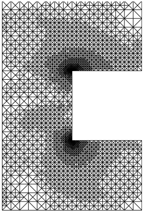
DG-FE-HMM

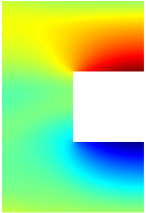
Darcy-Stokes coupling *(A.A.,O. Budac, Adaptive multiscale method for Stokes flow in porous media 2013, preprint)*

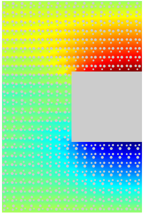
Elliptic problem




Initial mesh


Iteration 15


FE-HMM: p^H

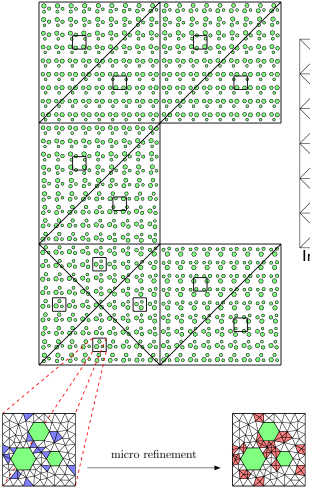

FE: p^{ε}

Stokes problem

P_2 / P_1

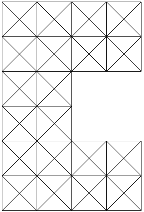
Darcy-Stokes coupling *(A.A.,O. Budac, Adaptive multiscale method for Stokes flow in porous media 2013, preprint)*

Elliptic problem



Stokes problem

P_2 / P_1



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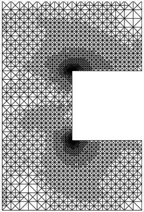

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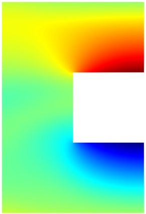
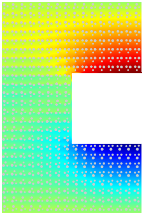
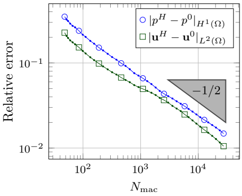
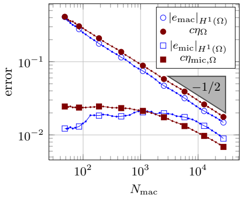


FE-HMM: p^H



FE: p^ε





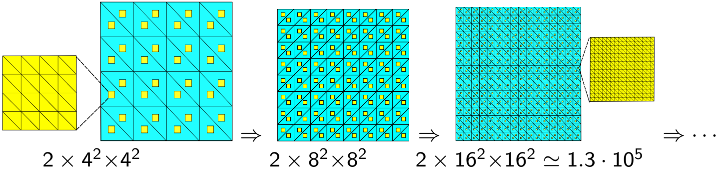
Theorem. $|p^H - p^0|^2_{H^1(\Omega)} \leq C_1 \left(\eta^2_\Omega + \eta^2_{mic,\Omega} + \xi^2_{data,\Omega} \right)$

Do we need accurate micro-computations in each coarse element ?

Idea: construct reduced basis space of micro functions following the reduced basis methods (C. Prud'homme, ..., Maday, Rozza, Patera 2002- ...)

Reduced basis FE-HMM *A.A and Y. Bai JCP 2012, A.A and Y. Bai CMAME 2013 (related work: Boyaval SIAM MMS 08)*

Issues: 1) repeated computation of cell problems 2) simultaneous micro and macro refinement



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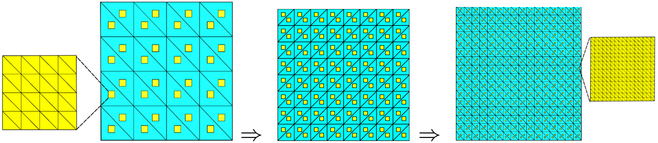
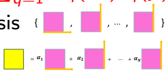
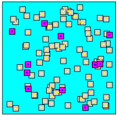


$2 \times 4^2 \times 4^2$ $\Rightarrow$ $2 \times 8^2 \times 8^2$ $\Rightarrow$ $2 \times 16^2 \times 16^2 \simeq 1.3 \cdot 10^5$ $\Rightarrow \cdots$

Reduced basis FE-HMM. Assume: $a^{\varepsilon}(G_{\tau}(y)) = \sum_{q=1}^{Q} \Theta_q(x_{\tau}) a_q(y)$

- Offline stage: Select the $N-$dim reduced basis
- Online stage: Solve standard FEM





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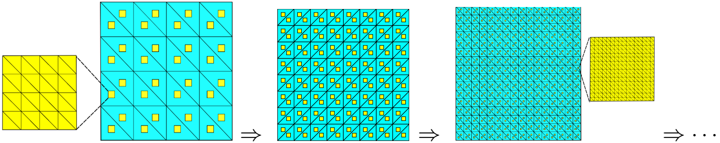

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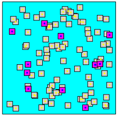

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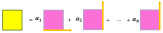
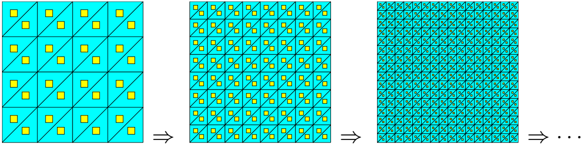
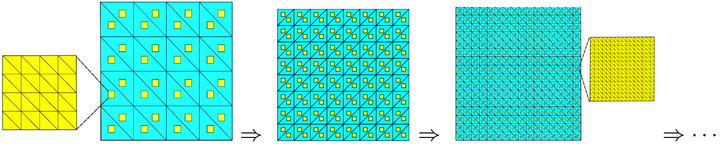
- Online stage: Solve standard FEM 



2×4^2 2×8^2 $2 \times 16^2 = 512$

Reduced basis FE-HMM *A.A and Y. Bai JCP 2012, A.A and Y. Bai CMAME 2013 (related work: Boyaval SIAM MMS 08)*

Issues: 1) repeated computation of cell problems 2) simultaneous micro and macro refinement



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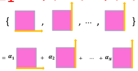

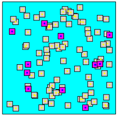

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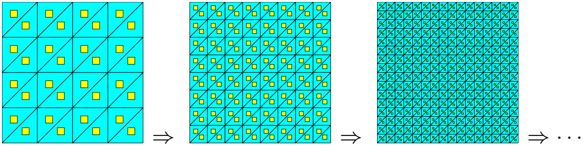

2×4^2 $\quad$ 2×8^2 $\quad$ $2\times 16^2 = 512$

Theorem: $\|u^0 - u^H\|_{H^1(\Omega)} \leq C\left(H^\ell + \left(\frac{h}{\varepsilon}\right)^{2q} + e^{-\beta N} + e_{MOD}\right)$

using results from Buffa, Maday et al.,M2AN 2012

RB-FE-HMM versus FE-HMM (non uniformly periodic 2D problem)

	RB-FE-HMM	offline:193 (424)(s)	FE-HMM	
DOF	L^2 error	online time (s)	L^2 error	time cost (s)
8×8	0.0161	0.03	0.0176	0.14
16×16	0.0040	0.10	0.0044	0.98
32×32	0.0010	0.28	0.0011	109
64×64	2.5347e-04	1.21	2.7702e-04	1760
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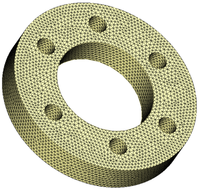
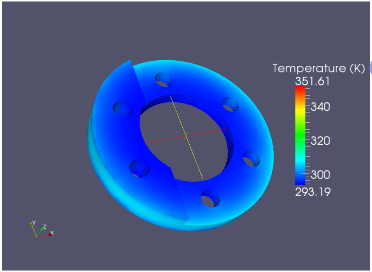
Nonlinear RB-FE-HMM (A.A, Y. Bai, G. Vilmart DCDS 2014, IJNME 2014) $\frac{\partial u^\varepsilon}{\partial t} - \nabla \cdot (a^\varepsilon(u^\varepsilon)\nabla u^\varepsilon) = f$

$\Rightarrow$ new a posteriori error estimator to guarantee convergence Newton method.





b

Reduced basis FE-HMM, a priori and a posteriori analysis *(A.A & Y. Bai JCP 2012, CMAME 2013)*

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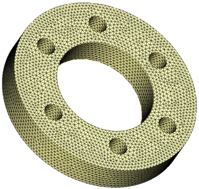

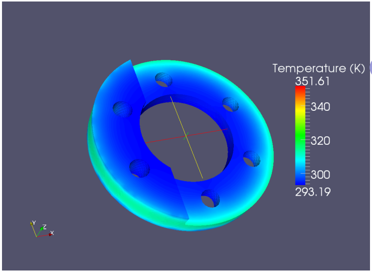

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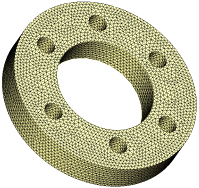

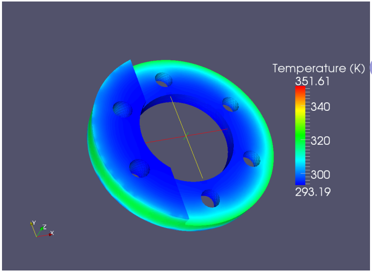

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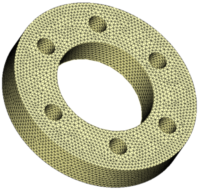

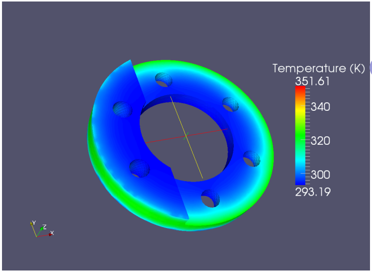

Reduced basis FE-HMM, a priori and a posteriori analysis *(A.A & Y. Bai JCP 2012, CMAME 2013)*

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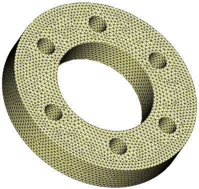

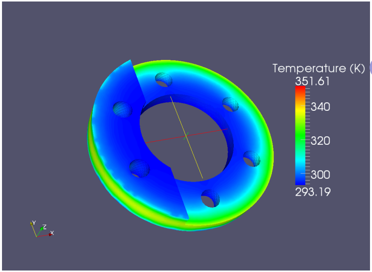

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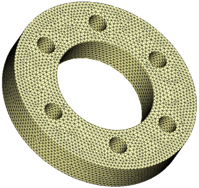

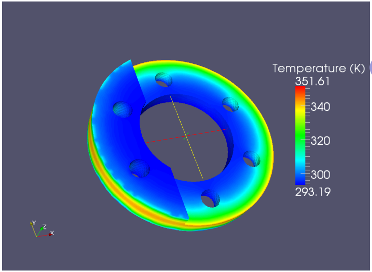

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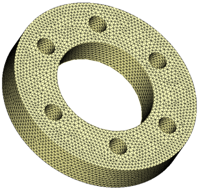

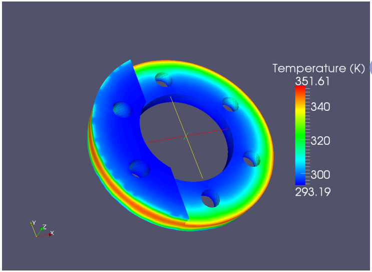

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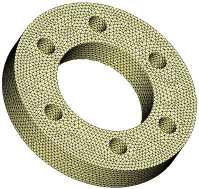
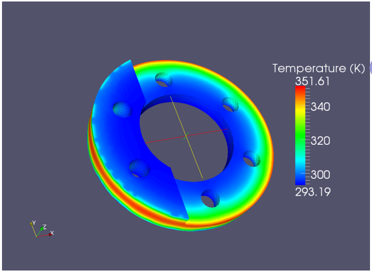

Reduced basis FE-HMM, a priori and a posteriori analysis *(A.A & Y. Bai JCP 2012, CMAME 2013)*

RB-FE-HMM versus FE-HMM (non uniformly periodic 2D problem)

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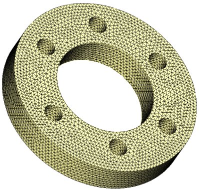

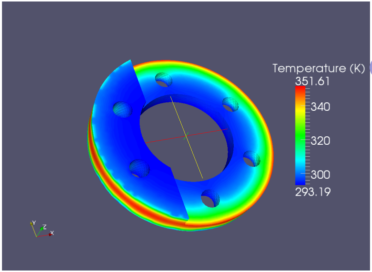

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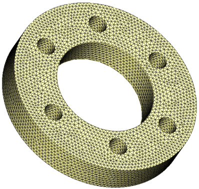

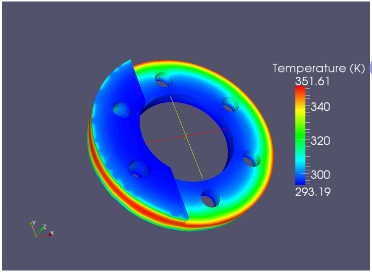

b

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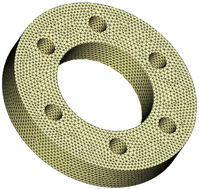

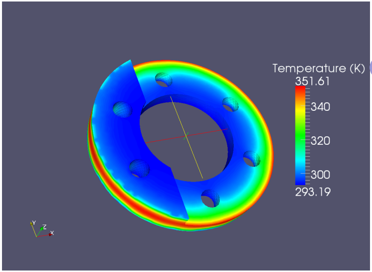

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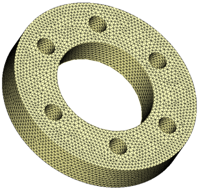

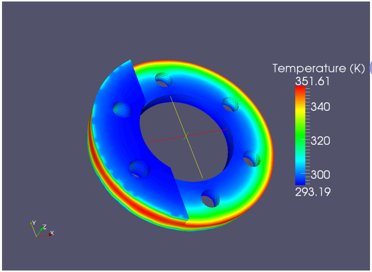

b

RB-FE-HMM versus FE-HMM (non uniformly periodic 2D problem)

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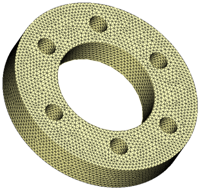

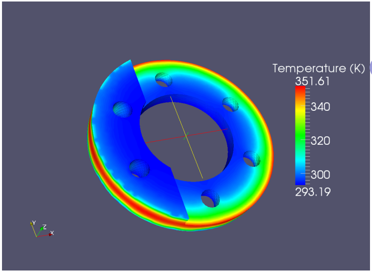

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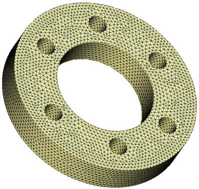

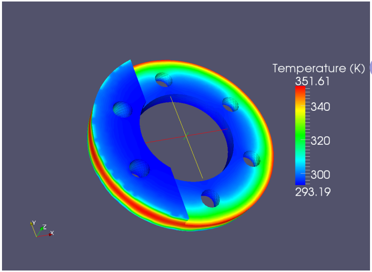

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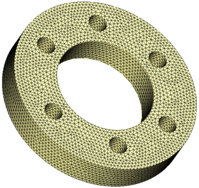

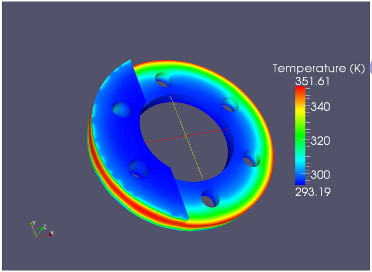

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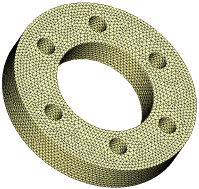

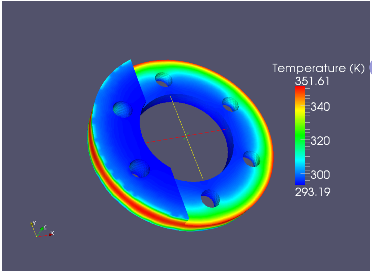

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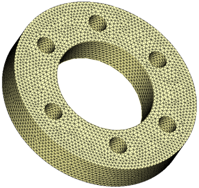

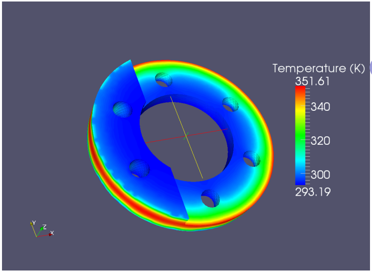

▸ b

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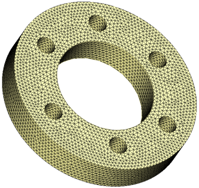

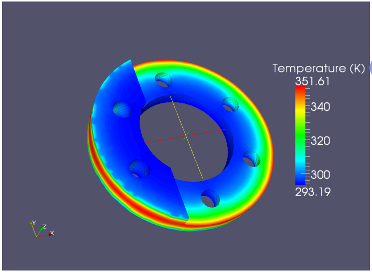

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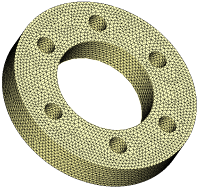

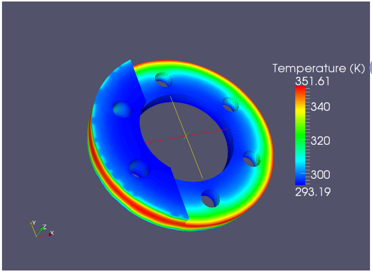

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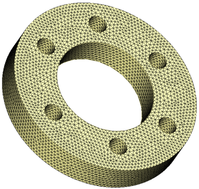

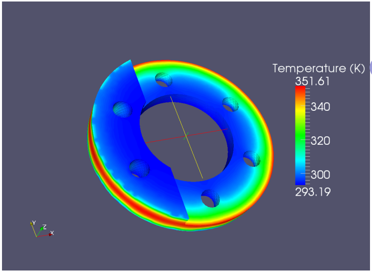

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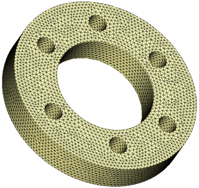

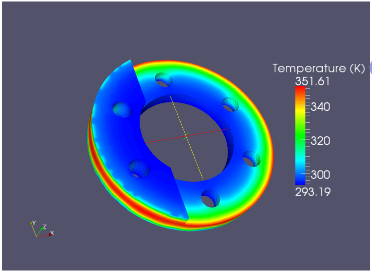

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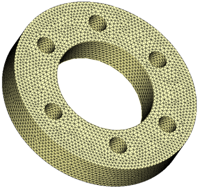

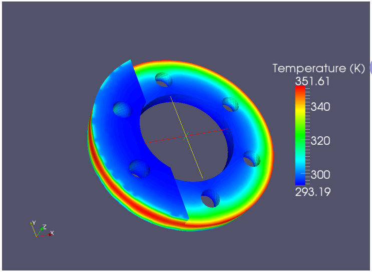

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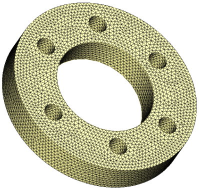

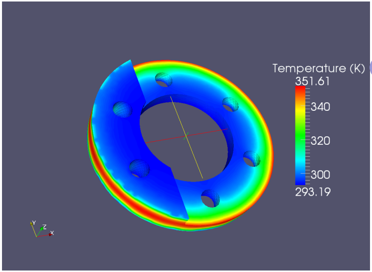

Outline:

I) Introduction (variations on elliptic problems)

II) Numerical homogenization of monotone parabolic problems

III) Numerical homogenization of a wave equation

$\Omega \subset \mathbb{R}^d$, $d \leq 3$, convex polygonal domain, $T > 0$.

$$\partial_t u^\varepsilon - div(\mathcal{A}^\varepsilon(x, \nabla u^\varepsilon)) = f \qquad \text{in } \Omega \times (0, T),$$
$$u^\varepsilon|_{\partial\Omega\times(0,T)} = 0, \qquad u^\varepsilon|_{t=0} = g \qquad \text{in } \Omega.$$

Setting:

- $\mathcal{A}^\varepsilon : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ with $\mathcal{A}^\varepsilon(x, \xi)$ *nonlinear* in 2nd argument;
- Lipschitz continuous: $|\mathcal{A}^\varepsilon(x, \xi) - \mathcal{A}^\varepsilon(x, \eta)| \leq L|\xi - \eta|$;
- Strongly monotone: $[\mathcal{A}^\varepsilon(x, \xi) - \mathcal{A}^\varepsilon(x, \eta)] \cdot (\xi - \eta) \geq \lambda|\xi - \eta|^2$.

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Challenges for numerical methods:

- $\mathcal{A}^\varepsilon(x, \xi)$ varies rapidly in space (at scale ε);
- large systems of nonlinear equations.

Current work: FEHMM for nonlinear monotone parabolic problems

- Nonlinear method: A. A., M.Huber, preprint. 2014 ;
- Linearized method: A. A., M.Huber, G. Vilmart, preprint, 2014.

Goal: approximate macroscopic behavior of u^ε.
Question: $u^\varepsilon \to u^0$ for $\varepsilon \to 0$? characterization of u^0?

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Tools of homogenization theory (De Giorgi, Spagnolo, Murat, Tartar...)

Parabolic G-convergence: Pankov ('97), Svanstedt ('99)

- multiscale problem: $\partial_t u^\varepsilon - div(\mathcal{A}^\varepsilon(x, \nabla u^\varepsilon)) = f$;
- effective problem: $\partial_t u^0 - div(\mathcal{A}^0(x, \nabla u^0)) = f$.

With: weak convergence of a subsequence $u^\varepsilon \rightharpoonup u^0$.

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Existence of **effective model** (small scales averaged out). . .
. . . but *not* known explicitely.

Strategy: Heterogeneous multiscale method

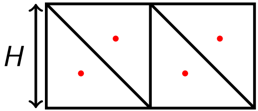
- solve effective problem using macro solver;
- recover effective map $\mathcal{A}^0$ "on the fly" by local sampling.

Effective equation: $\partial_t u^0 - div(\mathcal{A}^0(x, \nabla u^0)) = f$ in $\Omega \times (0, T)$.

I) Macro solver: (time) implicit Euler, timestep size Δt,

(space) FEM, $S_H^1 = S^1(\Omega, \mathcal{T}_H)$, macro mesh $\mathcal{T}_H$, $H \gg \varepsilon$,

$$\int_\Omega \frac{u_{n+1}^H - u_n^H}{\Delta t} w^H dx + B_H(u_{n+1}^H; w^H) = \int_\Omega f\, w^H dx, \quad \forall\, w^H \in S_H^1.$$



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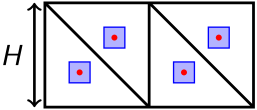
II) Modified macro map: K_δ sampling domain, $\delta \sim \varepsilon$,

$$B_H(v^H; w^H) = \sum_{K \in \mathcal{T}_H} \frac{|K|}{|K_\delta|} \int_{K_\delta} \mathcal{A}^\varepsilon(x, \nabla v^h_K) \cdot \nabla w^H dx.$$



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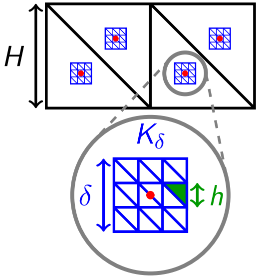
I) Macro solver: **(time)** implicit Euler, timestep size Δt,
(space) FEM, $S_H^1 = S^1(\Omega, \mathcal{T}_H)$, macro mesh $\mathcal{T}_H$, $H \gg \varepsilon$,

$$\int_\Omega \frac{u_{n+1}^H - u_n^H}{\Delta t} w^H dx + B_H(u_{n+1}^H; w^H) = \int_\Omega f\, w^H dx, \quad \forall\, w^H \in S_H^1.$$

II) Modified macro map: K_δ sampling domain, $\delta \sim \varepsilon$,

$$B_H(v^H; w^H) = \sum_{K \in \mathcal{T}_H} \frac{|K|}{|K_\delta|} \int_{K_\delta} \mathcal{A}^\varepsilon(x, \nabla v_K^h) \cdot \nabla w^H dx.$$



III) Local sampling: find $v_K^h - v^H \in S_h^1$

$$\int_{K_\delta} \mathcal{A}^\varepsilon(x, \nabla v_K^h) \cdot \nabla q^h dx = 0, \quad \forall\, q^h \in S_h^1.$$

- $S_h^1 = S^1(K_\delta, \mathcal{T}_h)$ micro FE space;
- Dirichlet/periodic boundary cond. in S_h^1.

Theorem. If $u^0, \partial_t u^0 \in C^0(H^2), \partial_t^2 u^0 \in C^0(L^2)$, micro sol. $W^{1,\infty}(K_\delta) \cap H^2(K_\delta)$

$$(\Delta t \sum_{n=1}^{N} \left\| \nabla u_n^H - \nabla u^0(\cdot, t_n) \right\|_{L^2(\Omega)}^2)^{1/2} \leq C(\Delta t + H + (\tfrac{h}{\varepsilon})^2 + e_{mod}).$$

If in add. $u^0 \in C^0(W^{2,\infty})$, elliptic regularity, quasi-uniform meshes

$$\max_{1 \leq n \leq N} \left\| u_n^H - u^0(\cdot, t_n) \right\|_{L^2(\Omega)} \leq C(\Delta t + H^2 + (\tfrac{h}{\varepsilon})^2 + e_{mod}).$$

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Literature (single scale).

- Douglas & Dupont SINUM 70, optimal H^1
- Wheeler SIAM 73, optimal L^2
- Dendy SIAM 77, nonlin. monotone, not optimal L^2 for $\mathcal{P}^1$ (rates $2 - d/2$)
- Dobrowolski SIAM 80, nonlin. monotone optimal L^2 (weighted norm Frehse,Rannacher 78)

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Main difficulties.

- variational crimes (quadrature, upscaling);
- sharp $\mathcal{C}^0(L^2)$ error without weighted norm techniques;
- fully discrete error estimates.
- Multiscale literature: Efendiev, Pankov MMS 04 (random monotone parabolic, no rates), Gloria MMS 06 (elliptic monotone operators, convergence rates in some situations).

Theorem. If $u^0, \partial_t u^0 \in C^0(H^2), \partial_t^2 u^0 \in C^0(L^2)$, micro sol. $W^{1,\infty}(K_\delta) \cap H^2(K_\delta)$

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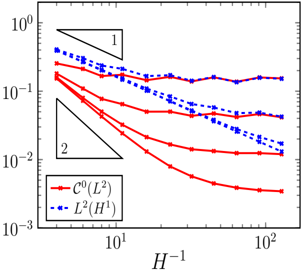
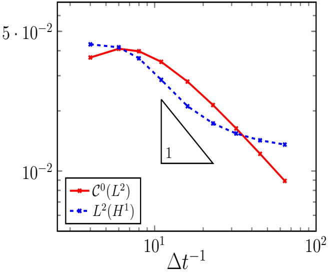
Test. Loc. periodic $\mathcal{A}^\varepsilon$, opt. coupling $\to e_{mod} = 0$.

Spatial errors. Δt



Temporal error. H, h small



I) Error propagation: error $\theta_n^H = u_n^H - \mathcal{U}_n^H$ with $\mathcal{U}_n^H$ elliptic projection of u^0

$$\begin{aligned}\frac{1}{2}\left\|\bar{\partial}_t\theta_n^H\right\|^2_{L^2(\Omega)} + \lambda\left\|\nabla\theta_{n+1}^H\right\|^2_{L^2(\Omega)} &\leq \int_\Omega \bar{\partial}_t\theta_n^H\,\theta_{n+1}^H dx \\ &+ \; err_{time} + err_{FEM} + err_{quadrature} + err_{upscaling}\end{aligned}$$

II) Linear elliptic projection: $B_\pi(t; v, w) = \int_\Omega \underbrace{\mathscr{A}^0(x,t)}_{D_\xi\mathcal{A}^0(x,\nabla u^0(x,t))} \nabla v\cdot\nabla w\,dx$

$$B_\pi(t;\mathcal{U}^H(\cdot,t), w^H) = B_\pi(t; u^0(\cdot,t), w^H), \qquad \forall\, w^H \in S_0^1(\Omega,\mathcal{T}_H),$$

L^2 estimate. Use max norm estimates (linear pblm.) $\|u^0(\cdot,t) - \mathcal{U}^H(\cdot,t)\|_{W^{1,\infty}(\Omega)}$.

III) Quadrature error: Error estimates for FEM with numerical quadrature for single scale monotone parabolic problems (generalized results of Raviart 1973 for linear problems).

IV) Upscaling error:

- micro error: $(\frac{h}{\varepsilon})^2$ (adjoint of micro problem for non-symmetric tensors following Du & Ming C. Math. Sci. 10, A.A & Vilmart Math.Comp. 14);
- modeling error: $(\frac{\varepsilon}{\delta})^{1/2}$ under assumption $\mathcal{A}^\varepsilon(x,\xi) = \mathcal{A}(x, x/\varepsilon, \xi)$ where $\mathcal{A}(x,y,\xi)$ is Y-periodic in y (generalized results for linear problems E, Ming, Zhang AMS 2005)

Nonlin. meth.: Old macro state u_n^H. **Unknown:** New macro state u_{n+1}^H.

(macro eq.) $\int_\Omega \frac{u_{n+1}^H - u_n^H}{\Delta t} w^H dx + B_H(u_{n+1}^H; w^H) = \int_\Omega f\, w^H dx.$

(micro eqs.) $\int_{K_\delta} \mathcal{A}^\varepsilon(x, \nabla u_{n+1,K}^h) \cdot \nabla q^h dx = 0, \quad \text{for every } K_\delta.$

$\Rightarrow$ Coupled nonlinear problems on macro *and* micro scale;

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$\Rightarrow$ Coupled nonlinear problems on macro *and* micro scale;

Linearized meth.: Assume $\mathcal{A}^\varepsilon(x,\xi) = a^\varepsilon(x,\xi)\xi$. Consider

$\mathcal{S}^{H,h} = S_0^1(\Omega, \mathcal{T}_H) \times \prod_{K\in\mathcal{T}_H} S^1(K_\delta, \mathcal{T}_h),\ \hat{u}_n \mapsto \hat{u}_{n+1} = (u_{n+1}^H, \{u_{n+1,K}^h\}) \in \mathcal{S}^{H,h}$

1. *evolution of the macroscopic state:* find $u_{n+1}^H \in S_0^1(\Omega, \mathcal{T}_H)$ solution of linear problem

$$\int_\Omega \frac{1}{\Delta t}(u_{n+1}^H - u_n^H)\, w^H dx + B_H(\hat{u}_n; u_{n+1}^H, w^H) = \int_\Omega f w^H dx;$$

2. *update micro states:* $\hat{u}_{n+1}$. For $K \in \mathcal{T}_H$, compute $u_{n+1,K}^h$ solution of linear micro problem s.t. $u_{n+1,K}^h - u_{n+1}^H \in S_h^1$ with parameter $\hat{u}_n$.

Nonlin. meth.: Old macro state u_n^H. **Unknown:** New macro state u_{n+1}^H.

(macro eq.) $\int_\Omega \frac{u_{n+1}^H - u_n^H}{\Delta t} w^H dx + B_H(u_{n+1}^H; w^H) = \int_\Omega f\, w^H dx.$

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(macro map)

$B_H(\hat{u}_n, u_{n+1}^H, w^H) = \sum_K \frac{|K|}{|K_\delta|} \int_{K_\delta} a^\varepsilon(\nabla \hat{u}_{n,K}^h) \nabla \hat{u}_{n+1,K}^h \cdot \nabla w^H dx.$

(micro eqs.) $\int_{K_\delta} a^\varepsilon(\nabla \hat{u}_{n,K}^h) \nabla \hat{u}_{n+1,K}^h \cdot \nabla q^h dx = 0.$

where $\hat{u}_{n+1,K}^h = u_{n+1}^H + u_{n+1,K}^h$.

$\Rightarrow$ **Simultaneous** linearization of macro and micro equations.

Theorem. In addition to assumptions for the fully nonlinear method, assume $CL_a|u^0|_{W^{1,\infty}(\Omega)} < \lambda_a$ where λ_a ellipticity and L_a Lipschitz cst. (w.r.t. ξ) of $a^\varepsilon(x,\xi)$, then

$$\Big(\Delta t \sum_{n=1}^{N} \big\|\nabla u_n^H - \nabla u^0(\cdot, t_n)\big\|^2_{L^2(\Omega)}\Big)^{1/2} \leq C(\Delta t + H + (\tfrac{h}{\varepsilon})^2 + e_{mod}),$$

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$$\max_{1\leq n\leq N} \big\|u_n^H - u^0(\cdot,t_n)\big\|_{L^2(\Omega)} \leq C\big(\Delta t + H^2 + (\tfrac{h}{\varepsilon})^2 + e_{mod}\big).$$

Results.:

- well-defined scheme;
- rigorous a priori estimates;
- simple implementation.

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$$\max_{1\leq n\leq N} \left\|u_n^H - u^0(\cdot,t_n)\right\|_{L^2(\Omega)} \leq C(\Delta t + H^2 + (\tfrac{h}{\varepsilon})^2 + e_{mod}).$$

Efficiency:
$a^\varepsilon(\cdot,\xi) \sim 1 + 1/(1+|\xi|^2)^{0.5}$, optimal refinement Δt, H, h. Error $L^\infty(L^2)$ norm.

Δt	lin.	nonlin.	speed-up
$1/16$	0.0697	0.0646	7
$1/64$	0.0174	0.0159	9.5
$1/256$	0.0043	0.004	10

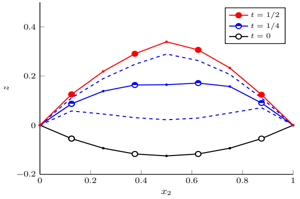
Results.:

- well-defined scheme;
- rigorous a priori estimates;
- simple implementation.

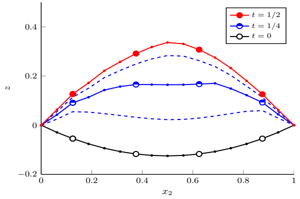
Example with a multiscale $p-$Laplacian *(A. A., M. Huber, G. Vilmart, 2014)*

$\Omega = (0,1)^2$, profiles at $x_1 = 1$.

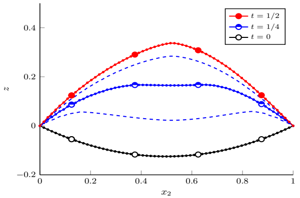
$$a^\varepsilon(x,\xi) = \frac{11}{10} + \sin(2\pi(x_1+x_2)) + \frac{\frac{9}{8}+\sin(2\pi\frac{x_1}{\varepsilon})}{\frac{9}{8}+\cos(2\pi\frac{x_2}{\varepsilon})} + \frac{\frac{9}{8}+\sin(2\pi\frac{x_2}{\varepsilon})}{\frac{9}{8}+\sin(2\pi\frac{x_1}{\varepsilon})}|\xi|^{p-2} Id.$$



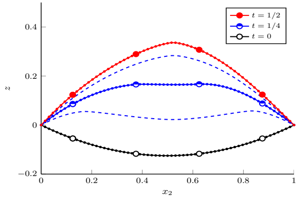
(a) Linearized scheme. $N_{mac} = N_{mic} = N = 8$.



(b) Linearized scheme. $N_{mac} = N_{mic} = N = 16$.



(c) Linearized scheme. $N_{mac} = N_{mic} = N = 64$.



(d) Nonlinear scheme as reference. $N_{mac} = N_{mic} = 64$ and $N = 256$.

Outline:

I) Introduction (variations on elliptic problems)

II) Numerical homogenization of monotone parabolic problems

III) Numerical homogenization of a wave equation

Problem.

$$\partial_{tt} u_\varepsilon - \nabla \cdot (a^\varepsilon(x) \nabla u_\varepsilon) = f \quad \text{in } \Omega \times]0, T[$$
$$u_\varepsilon = 0 \quad \text{on }]0, T[\times \partial\Omega,$$
$$u_\varepsilon(x, 0) = g_1(x), \partial_t u_\varepsilon(x, 0) = g_2(x).$$

Homogenization: Bensoussan,Lions,Papanicolaou 78, Brahim-Ostmane,Francfort,Murat 92, ...

$u^\varepsilon \rightharpoonup u^0$ weakly* to the solution of $\partial_{tt} u^0 - \nabla \cdot \left(a^0(x) \nabla u^0\right) = f$

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Numerical homogenization method (FE-HMM)

Find $u^H \in [0, T] \times S_0^\ell(\Omega, \mathcal{T}_h) \to \mathbb{R}$ such that

$$(\partial_{tt} u^H, v^H) + B_H(u^H, v^H) = (f(t), v^H) \quad \forall v^H \in S_0^1(\Omega, \mathcal{T}_H)$$
$$u^H = 0 \quad \text{on } \partial\Omega \times]0, T[,$$
$$u^H(x, 0) = \Pi_H g_1(x), \; \partial_t u^H(x, 0) = \Pi_H g_2(x)$$

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$$u^H = 0 \quad \text{on } \partial\Omega \times]0, T[,$$
$$u^H(x, 0) = \Pi_H g_1(x), \;\; \partial_t u^H(x, 0) = \Pi_H g_2(x)$$

Bilinear form: $B_H(u^H, w^H) := \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} a^\varepsilon(x) \nabla u^h_{K_j} \cdot \nabla w^h_{K_j} dx$

Theorem. (A.A, M. Grote 2011)

Under suitable regularity assumptions on u^0, a^0 we have

$$\|\partial_t(u^0 - u^H)\|_{L^\infty(0,T;L^2(\Omega))} + \|u^0 - u^H\|_{L^\infty(0,T;H^1(\Omega))} \leq C(H^\ell + (\tfrac{h}{\varepsilon})^{2q} + e_{mod})$$

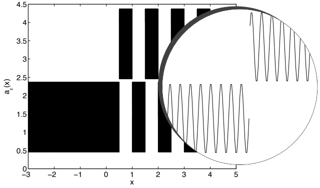
$$\|u^0 - u^H\|_{L^\infty(0,T;L^2(\Omega))} \leq C(H^{\ell+1} + (\tfrac{h}{\varepsilon})^{2q} + e_{mod})$$

Analysis:

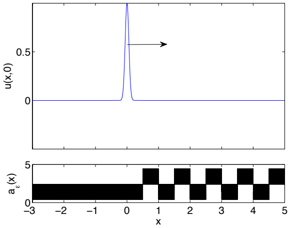
- results on FEM for wave with num. quadrature (Baker and Dougalis 76);
- control propagation of micro/modeling error;
- Strang-type Lemma for the propagation of consistency error (variational crime).

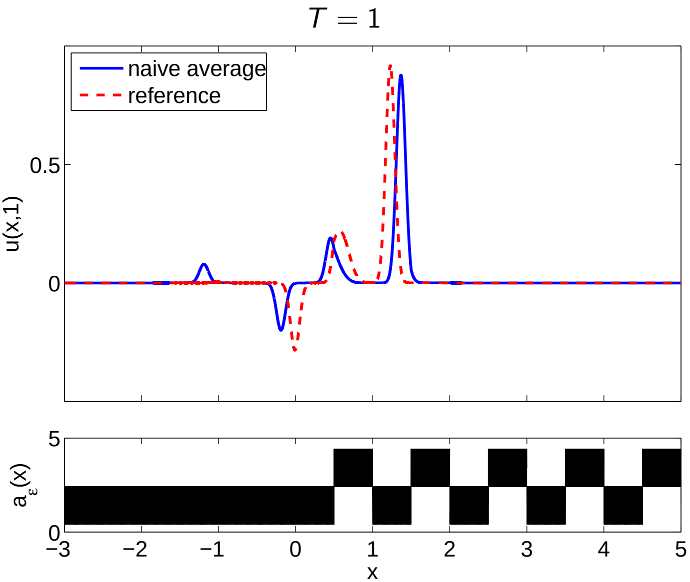
$$a^{\varepsilon}(x) = \begin{cases} \sqrt{2} + \sin 2\pi \frac{x}{\varepsilon} & , x < 0 \text{ or } x \in (k, k+0.5), k \in \mathbb{N}_0 \\ \sqrt{2} + \sin 2\pi \frac{x}{\varepsilon} + 2 & , x \in (k+0.5, k+1), k \in \mathbb{N}_0 \end{cases}$$

Tensor for the computational domain, with zoom about $x = 4.5$ ($\varepsilon = \delta = 1/100$)

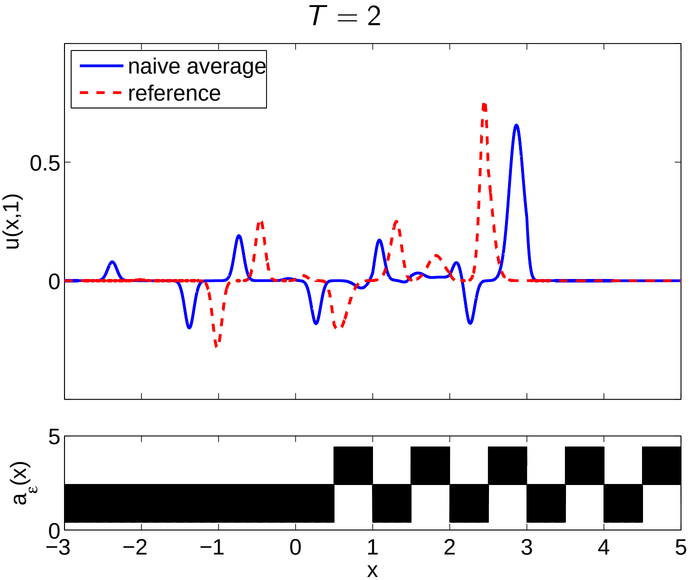


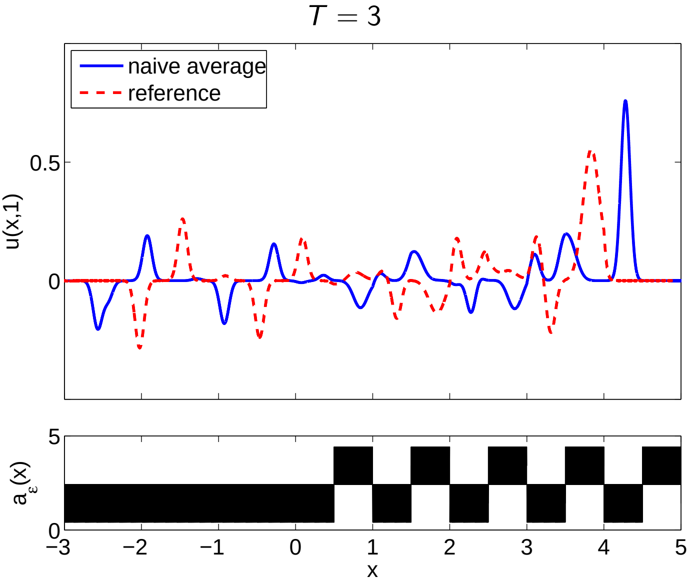
Initial conditions such that we have a right-moving Gaussian pulse:





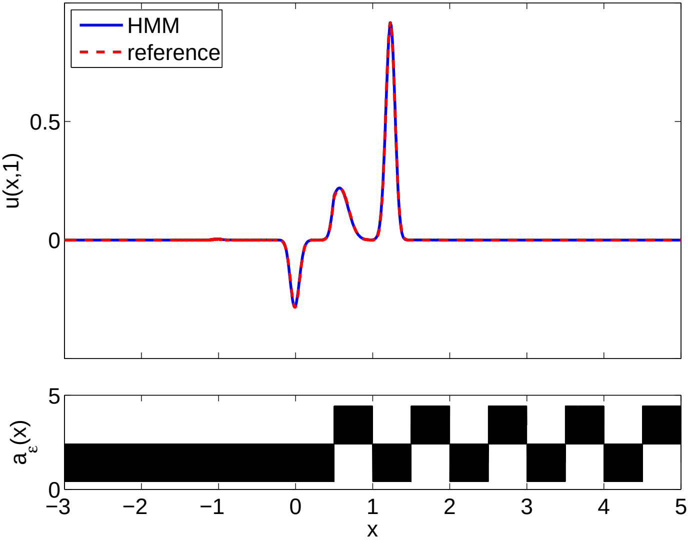
$T = 1$
naive average
reference
u(x,1)
0.5
0
$a_\varepsilon(x)$
5
0
−3
−2
−1
0
1
2
3
4
5
x


$T = 2$
naive average
reference
u(x,1)
0.5
0
$a_\varepsilon(x)$
5
0
−3
−2
−1
0
1
2
3
4
5
x



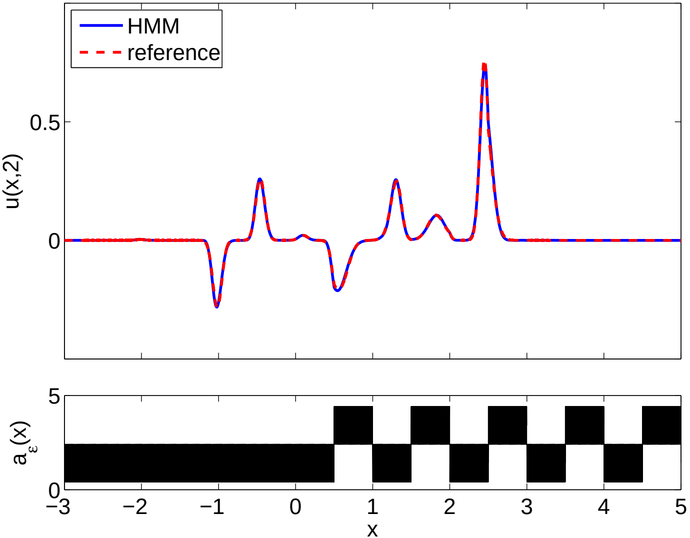
T = 3
naive average
reference
u(x,1)
0.5
0
5
0
a_ε(x)
−3
−2
−1
0
1
2
3
4
5
x

CFL condition for FE-HMM: $\Delta t \leq C \cdot H$, here $H = 10^{-2}, \Delta t \simeq 10^{-3}$
CFL condition for fine scale: $\Delta t \leq C \cdot h$, here $h = 10^{-4}, \Delta t \simeq 10^{-5}$



CFL condition for FE-HMM: $\Delta t \leq C \cdot H$, here $H = 10^{-2}, \Delta t \simeq 10^{-3}$

CFL condition for fine scale: $\Delta t \leq C \cdot h$, here $h = 10^{-4}, \Delta t \simeq 10^{-5}$



CFL condition for FE-HMM: $\Delta t \leq C \cdot H$, here $H = 10^{-2}, \Delta t \simeq 10^{-3}$
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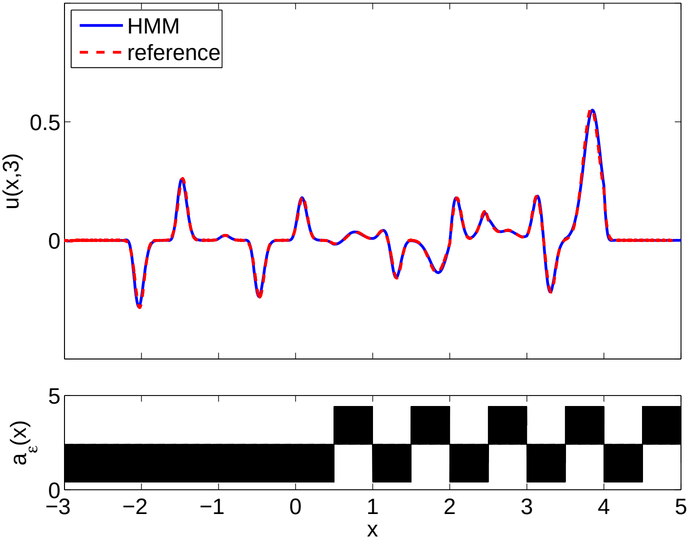
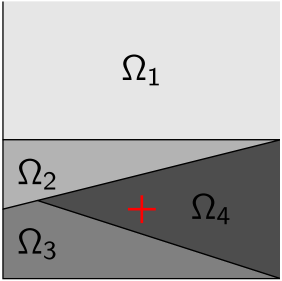
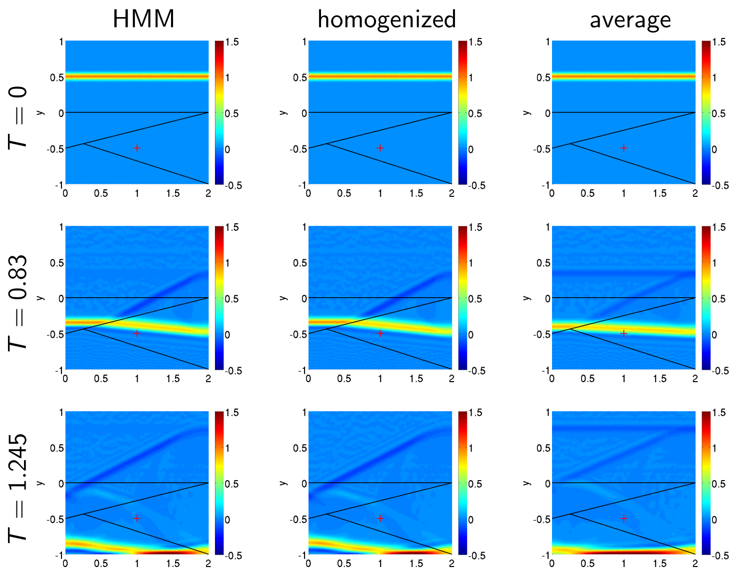


$\Omega = [0,2] \times [-1,1] \subset \mathbb{R}^2$ divided into four distinct subdomains.



$$a^\varepsilon(x) = \begin{cases} 1 & x \in \Omega_1 \\ \sqrt{2} + \sin(\frac{2\pi x_2}{\varepsilon}) & x \in \Omega_2 \\ \sqrt{2} + \frac{1}{2}\Big(\sin(2\pi x_2) + \sin(\frac{2\pi x_2}{\varepsilon})\Big) & x \in \Omega_3 \\ 2 & x \in \Omega_4 \end{cases}$$

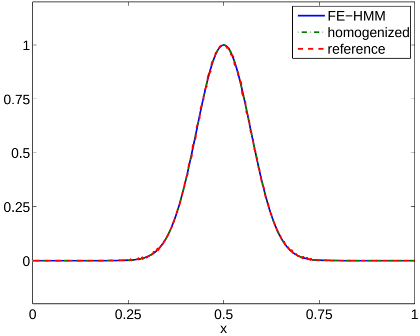
- initial condition: down moving Gaussian shaped plane wave
- homogeneous Neumann boundary conditions
- $\delta = \epsilon = \frac{1}{1000}$, $h = \frac{1}{7000}$, $\Delta t = \frac{1}{1000}$
- fully resolved: almost 400 millions elements
- here only: 63'498 elements



HMM
homogenized
average
T = 0
T = 0.83
T = 1.245
y

Long-time behavior (periodic medium)

$t = 1.47722$ (1 rev.)



Long-time behavior (periodic medium)

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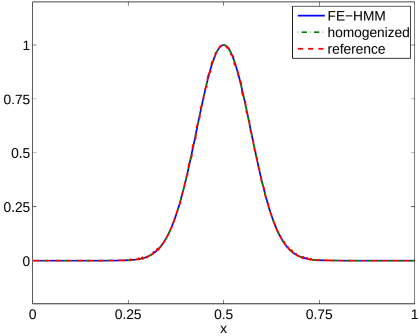
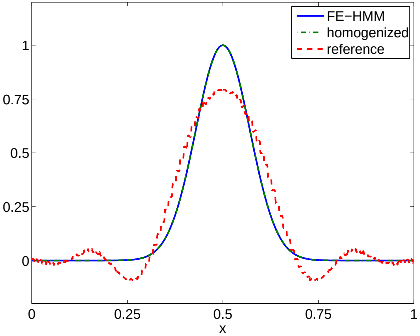


$t = 147.722$ (100 rev.)



Dispersive limits: formal homogenization (Bloch waves) (Santosa & Symes 1991,...)

$$\partial_{tt}\bar{u} - a^0\partial_{xx}\bar{u} - \varepsilon^2 b^0 \partial_{xxxx}\bar{u} = f$$

Effective model for the FD-HMM (Engquist, Holst, Runborg, LNCS 2012) but:

- requires increasingly large space-time sampling domain
- requires high-order macro scheme
- correction to initial data, regularization techniques

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Another dispersive limits: "good Boussinesq equation" (Lamacz M3AS 2011)

$$\partial_{tt}\bar{u} - a^0\partial_{xx}\bar{u} - \varepsilon^2\frac{b^0}{a^0}\partial_{ttxx}\bar{u} = f$$

Rigorous convergence proof: u^ε can be approximated with error $\mathcal{O}(\varepsilon)$ in a L^∞ norm on the time interval $\mathcal{O}(1/\varepsilon^2)$.

Remark: b^0 is given by a cascade of cell problems.

Numerical homogenization method (FE-HMM-L)

Find $u^H \in [0, T] \times S_0^\ell(\Omega, \mathcal{T}_h) \to \mathbb{R}$ such that

$$
\begin{aligned}
&(\partial_{tt} u^H, v^H)_Q + B_H(u^H, v^H) = (f(t), v^H) \quad \forall v^H \in S_0^\ell(\Omega, \mathcal{T}_h) \\
&u^H = 0 \ \text{ on } \partial\Omega \times]0, T[, \\
&u^H(x, 0) = \Pi_H g_1(x), \ \partial_t u^H(x, 0) = \Pi_H g_2(x)
\end{aligned}
$$

$$
B_H(v^H, w^H) := \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} a^\varepsilon(x) \nabla v^h_{K_j} \cdot \nabla w^h_{K_j} dx
$$

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L^2 scalar product correction:

$$(v^H, w^H)_Q = (v^H, w^H) + \underbrace{\sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} (v^h_{K_j} - v^H_{lin})(w^h_{K_j} - w^H_{lin}) dx}_{(v^H, w^H)_M}$$

L^2 scalar product correction:

$$(v^H, w^H)_Q = (v^H, w^H) + \underbrace{\sum_{K\in\mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} (v^h_{K_j} - v^H_{lin})(w^h_{K_j} - w^H_{lin}) dx}_{(v^H, w^H)_M}$$

Why does it work ?

Lemma: For all $v^H, w^H \in S_0^\ell(\Omega, \mathcal{T}_h)$ we have

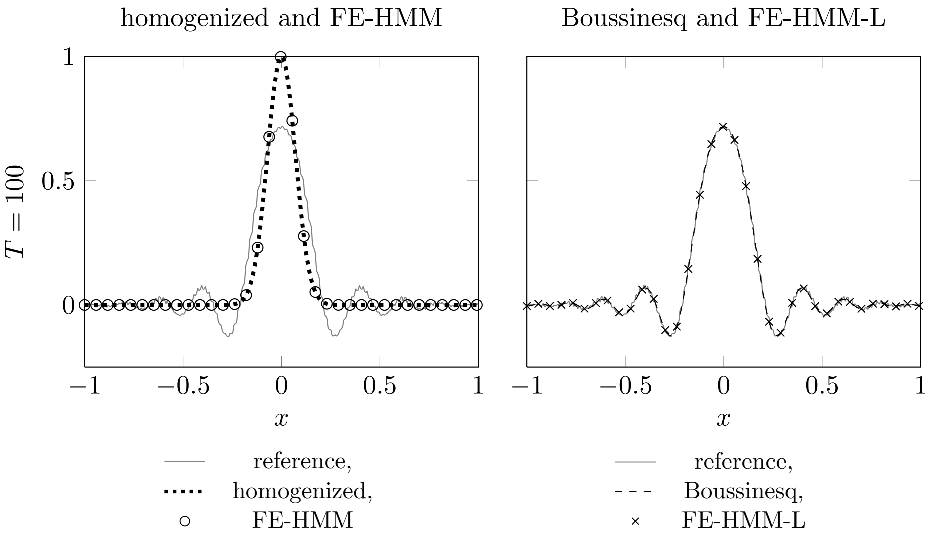
$$(v^H, v^H)_M \geq 0, \ |(v^H, w^H)_M| \leq C\varepsilon^2 \|\nabla v^H\|_{L^2(\Omega)} \|\nabla w^H\|_{L^2(\Omega)}.$$

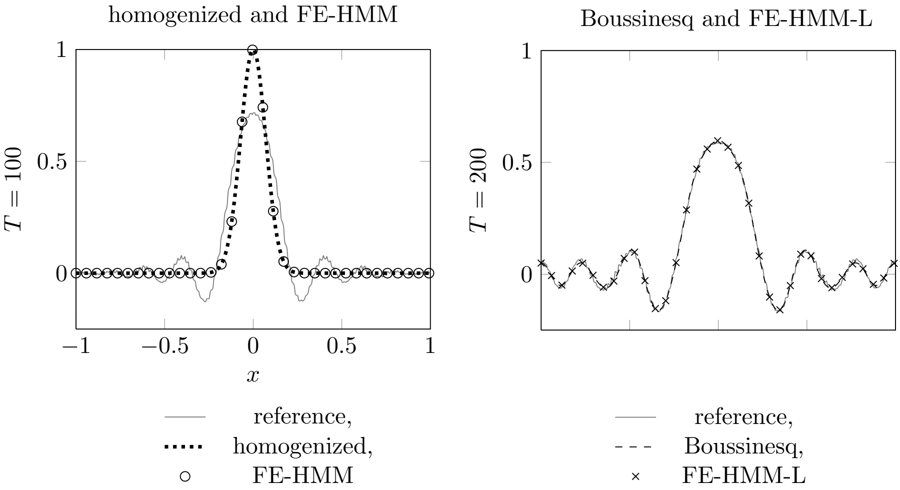
Theorem. (A.A, M. Grote, C, Stohrer 2013)

Under suitable regularity assumptions on u^0, a^0 we have

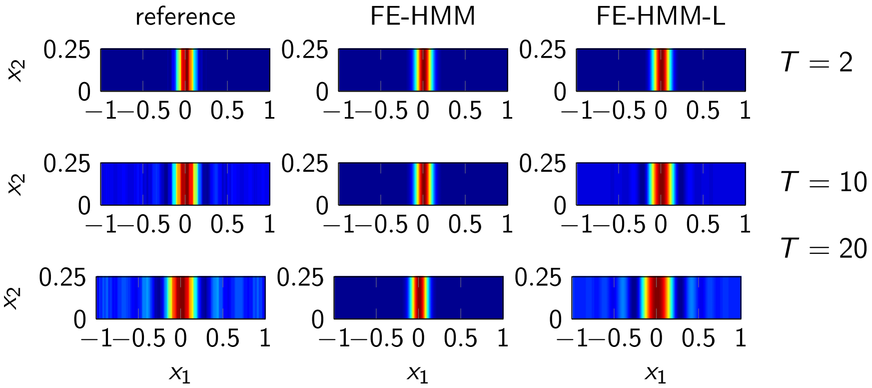
$$\|\partial_t(u^0 - u^H)\|_{L^\infty(L^2)} + \|u^0 - u^H\|_{L^\infty(H^1)} \leq C(H^\ell + \varepsilon^2 + \underbrace{(\frac{h}{\varepsilon})^{2q} + m_e}_{r_{HMM}})$$

$$\|u^0 - u^H\|_{L^\infty(L^2)} \leq C(H^{\ell+1} + \varepsilon^2 + \underbrace{(\frac{h}{\varepsilon})^{2q} + m_e}_{r_{HMM}})$$


homogenized and FE-HMM
Boussinesq and FE-HMM-L
$T = 100$
1
0.5
0
−1
−0.5
0
0.5
1
x
reference,
homogenized,
FE-HMM
reference,
Boussinesq,
FE-HMM-L


homogenized and FE-HMM
Boussinesq and FE-HMM-L
$T = 100$
$T = 200$
1
0.5
0
−1
−0.5
0
0.5
1
x
reference,
homogenized,
FE-HMM
reference,
Boussinesq,
FE-HMM-L

Two-dimensional wave guide



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