hp-Version Discontinuous Galerkin Methods on Polygonal and Polyhedral Meshes

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Outline

- Background
- Construction of the FEM Meshes
- Discontinuous Galerkin FEMs on Polytopic Meshes
- Error Estimation
- Domain Decomposition Preconditioners
- Summary and Outlook

Background

Meshing Complicated Geometries

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Hackbusch & Sauter 1997→

- PDE problem: given $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ and $f \in \mathcal{H}$, find $u \in D(\mathcal{L})$ such that

$$\mathcal{L}u = f \quad \text{in } \Omega.$$

- Assume that Ω is *complicated* in the sense that it contains microstructures.

- FEM: given a mesh $\mathcal{T}_h$ of granularity h, find $u_h \in V_h(\mathcal{T}_h)$ such that

$$\mathcal{L}_h u_h = f_h.$$

- Typically $\mathcal{T}_h$ consists of (mapped) triangles/quadrilaterals (2D) or tetrahedra/hexahedra/prisms/pyramids (3D). Thereby,

$$\dim(V_h(\mathcal{T}_h)) \propto \text{ Complexity of } \Omega$$

$\Rightarrow$ Too many degrees of freedom are employed to resolve the geometry.

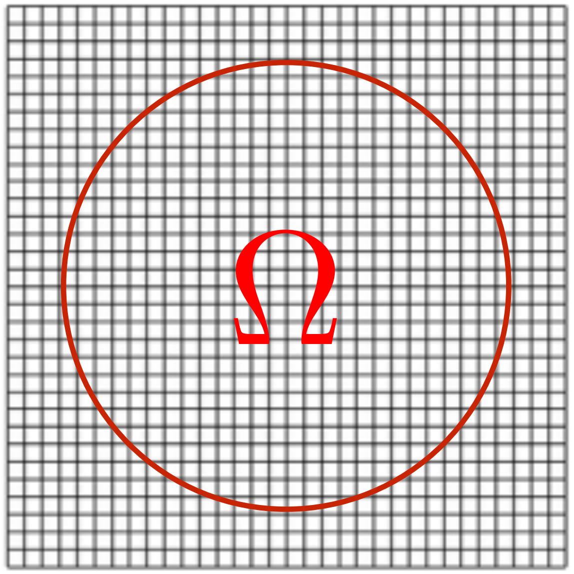
$\Rightarrow$ Limits the effectiveness of adaptive mesh refinement strategies.

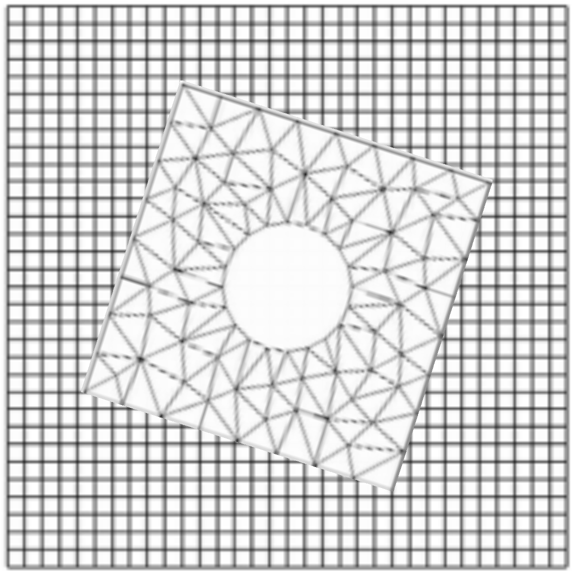
$\Rightarrow$ Design of multilevel preconditioners is difficult (Schwarz/multigrid).

Polygonal/Polyhedral Meshes

- Provide greater flexibility in mesh generation.
- Exploited as transitional elements in FEM meshes.
  - Fictitious Domain/Unfitted Methods/Overlapping Meshes [Lagrange–multiplier approaches, finite cell methods, fat boundary methods, Penalty methods, Cut–cell]

    Barrett & Elliott 1987, Dinh, Glowinski, He, Kwock, Pan, Periaux 1992, Quirk 1992, Glowinski, Pan, & Periaux 1994, 1995, Maury 2001, Baaijens 2001, Del Pino & Pironneau 2003, Hansbo & Hansbo 2002, 2004, Becker, Hansbo, & Stenberg 2003, Hansbo, Hansbo, & Larson 2003, Girault & Glowinski 2005, Glowinski & Kuznetsov 2007, Yu 2005, Vos, van Loon, & Sherwin 2007, Burman & Hansbo 2009, Engwer 2009, Bastian & Engwer 2009, Burman & Hansbo 2010, 2011, Johansson & Larson 2011, Massing 2012





- Better suited for applications (complicated and/or moving domains).

  Solid mechanics, fluid structure interaction, mathematical biology, …
- Techniques applicable to characteristic-based methods.

  Exact projection Lagrange-Galerkin method (Priestley 1994)

FEMs on Polygonal/Polyhedral Meshes

- Polygonal Finite Element Methods.

  Sukumar & Tabarraei 2004, 2007

- Extended/Generalised FEMs (Partition of Unity).

  Duarte & Oden 1996, Melenk & Babuska 1996, Moes, Dolbow, & Belytschko 1999, Daux, Moes, Dolbow, Sukumar, & Belytschko 2000, Sukumar, Moes, Moran, & Belytschko 2000, Belytschko, Moes, Usui, & Parimi 2001, Gerstenberger & Wall 2008, Bechet, Moes, & Wohlmuth 2009, Belytschko, Gracie, & Ventura 2009, Jaroslav & Renard 2009, Fries & Belytschko 2010, Shahmiri, Gerstenberger, & Wall 2011, …

- Virtual Element Method.

  Beirao da Veiga, Brezzi, Cangiani, Manzini, Marini, & Russo 2013

- Mimetic Finite Difference Method.

  Brezzi, Lipnikov, & Shashkov 2005, Brezzi, Lipnikov, & Simoncini 2005, Brezzi, Buffa, & Lipnikov 2009, Cangiani, Manzini, Russo 2009, Beirao da Veiga, Droniou, & Manzini 2011, Beirao da Veiga, Lipnikov & Manzini 2011, Beirao da Veiga & Manzini 2013,…

- Composite Finite Element Methods.

  Shortley & Weller 1938, Hackbusch & Sauter 1997→, Rech, Sauter, & Smolianski 2006, Antonietti, Giani, & H. 2012, 2013,…

- Agglomerated Finite Element Methods.

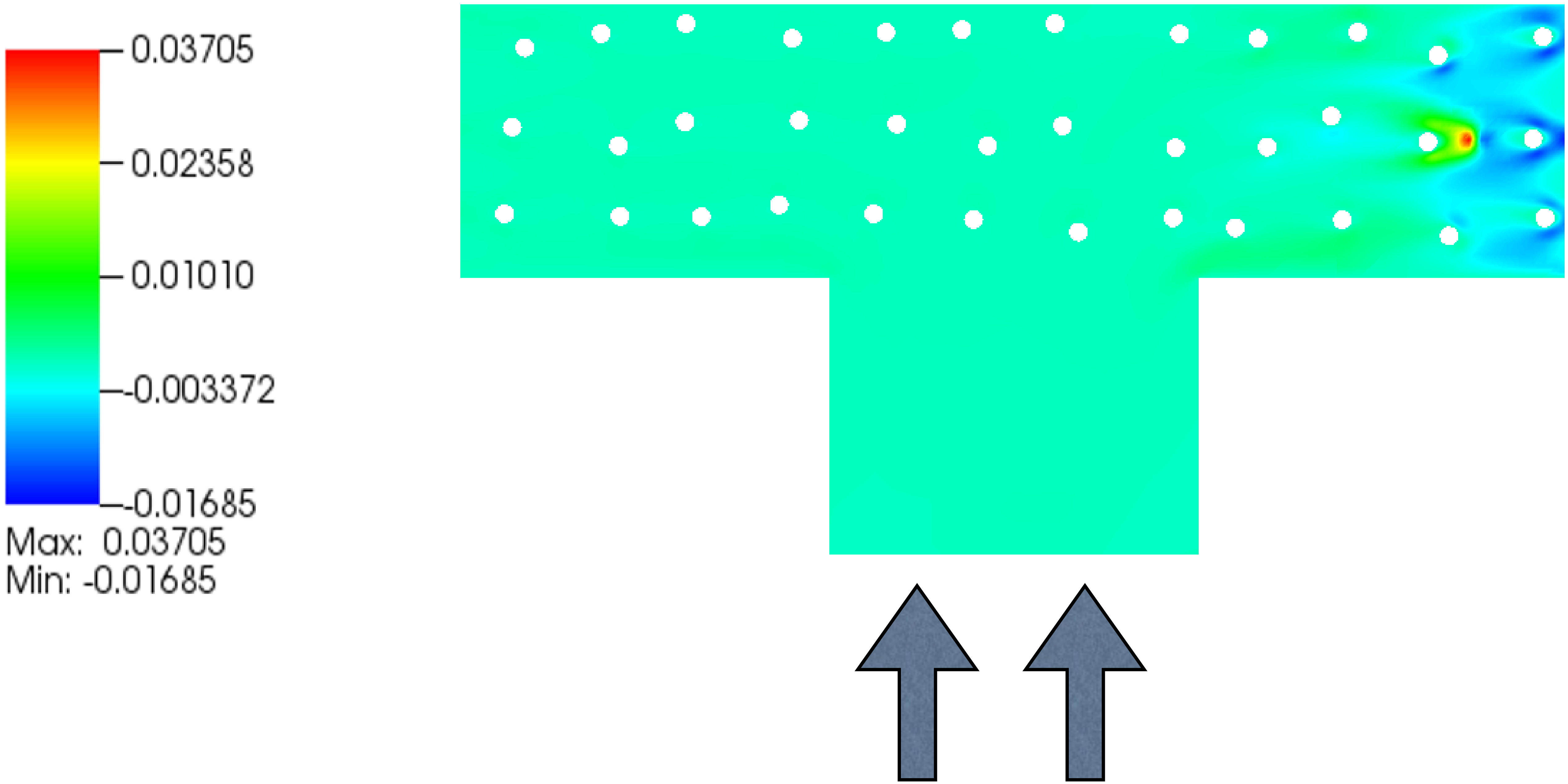
  DGFEM: Bassi, Botti, Colombo, Di Pietro, & Tesini 2012, Bassi, Botti & Colombo 2013.

Objectives:

➡ Develop (composite/agglomerated) discontinuous Galerkin finite element methods on general polytopic meshes.

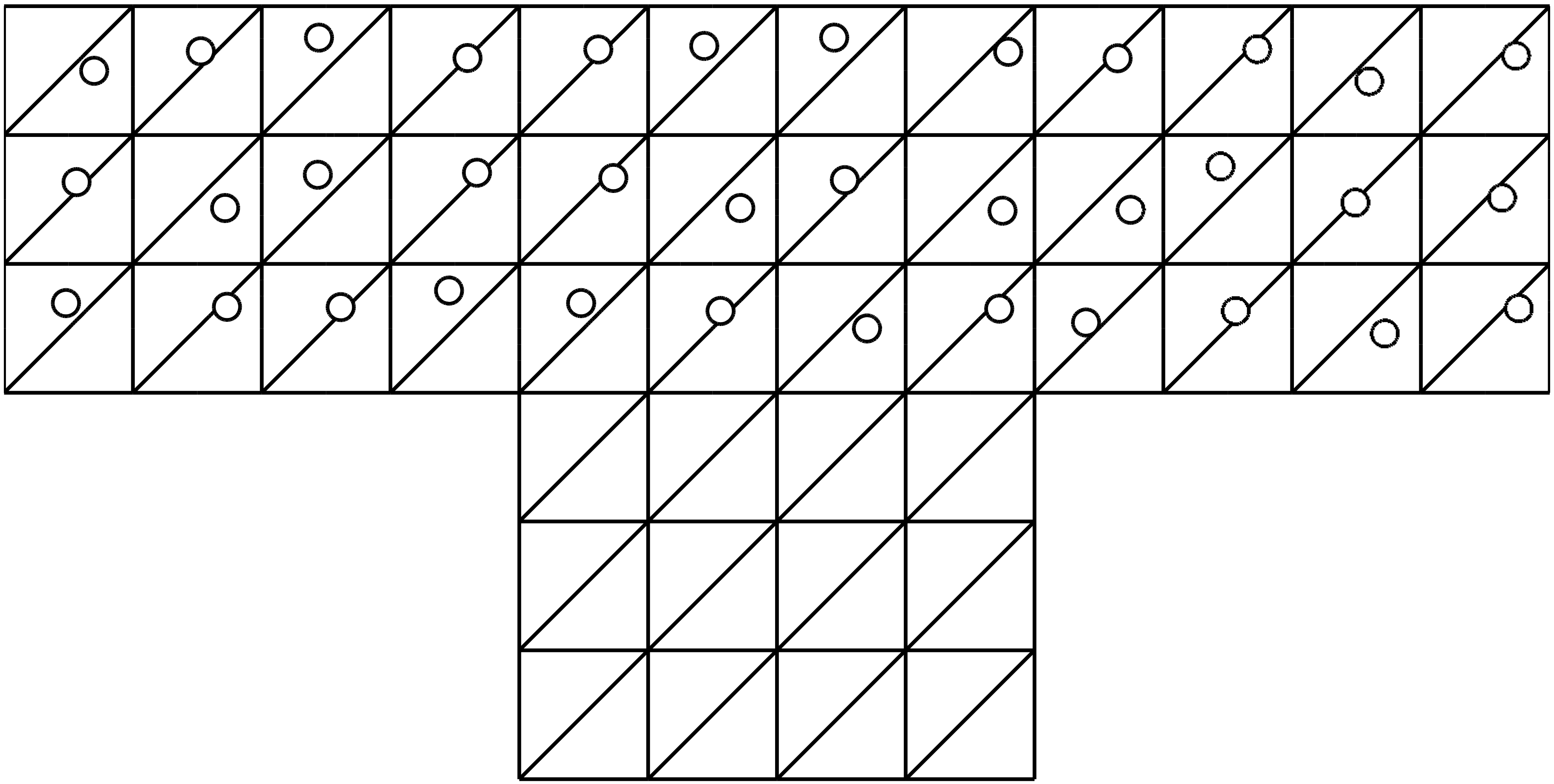
- Number of degrees of freedom is *independent* of the domain;
- Coarse approximations may be computed with engineering accuracy;
- Adaptivity is focused on resolving *important features* of the solution;
- Method naturally admits high-order polynomial orders;
- May be exploited as coarse level solvers with multilevel preconditioners.

$Re = 100$: DWR Refinement, with $J(\mathbf{u}, p) = p(11, 1.5) \approx 2.2764 \times 10^{-3}$



Giani & H. 2014

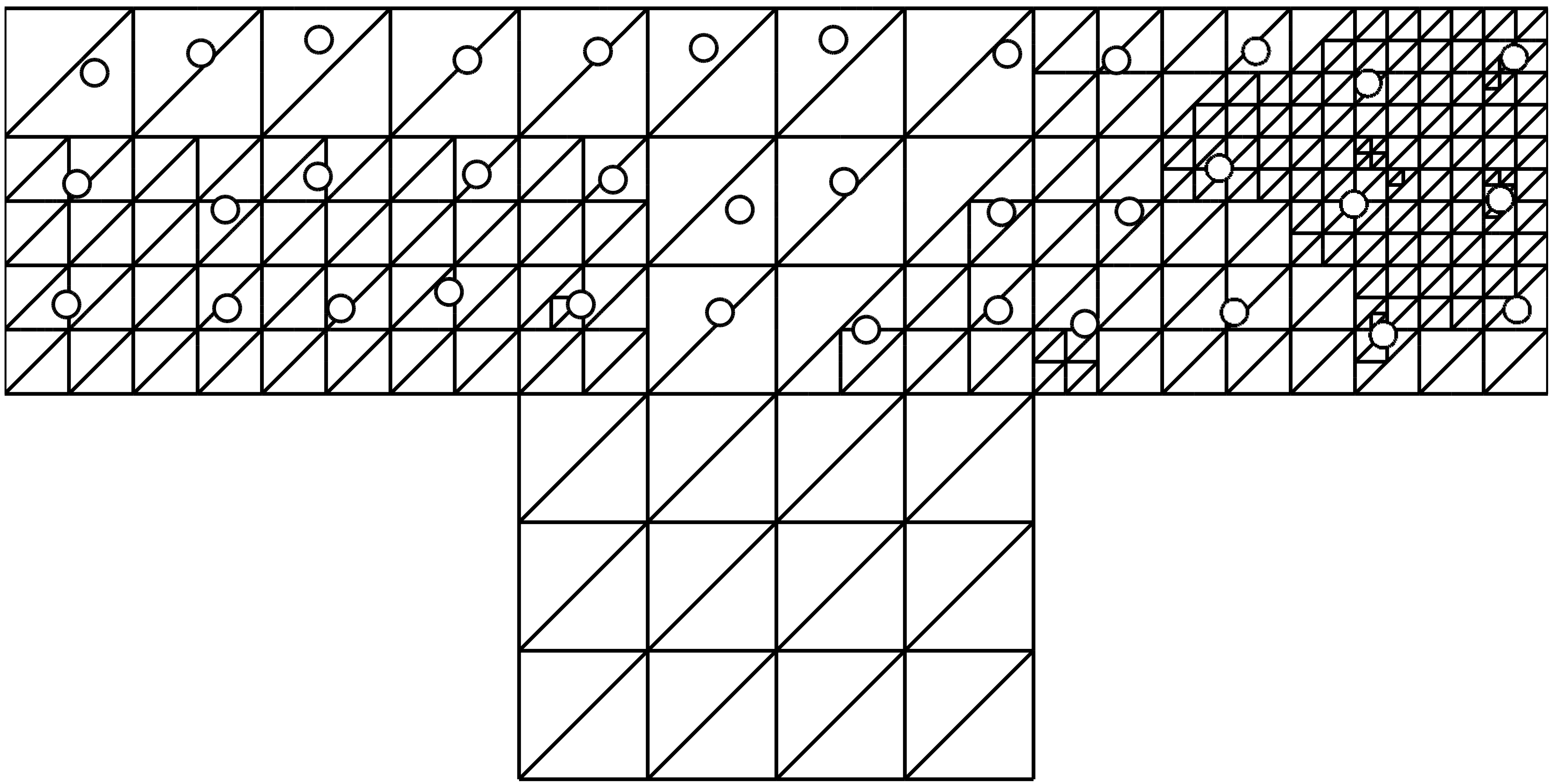
$Re = 100$: DWR Refinement, with $J(\mathbf{u}, p) = p(11, 1.5) \approx 2.2764 \times 10^{-3}$



Initial mesh consisting of 96 polygonal elements

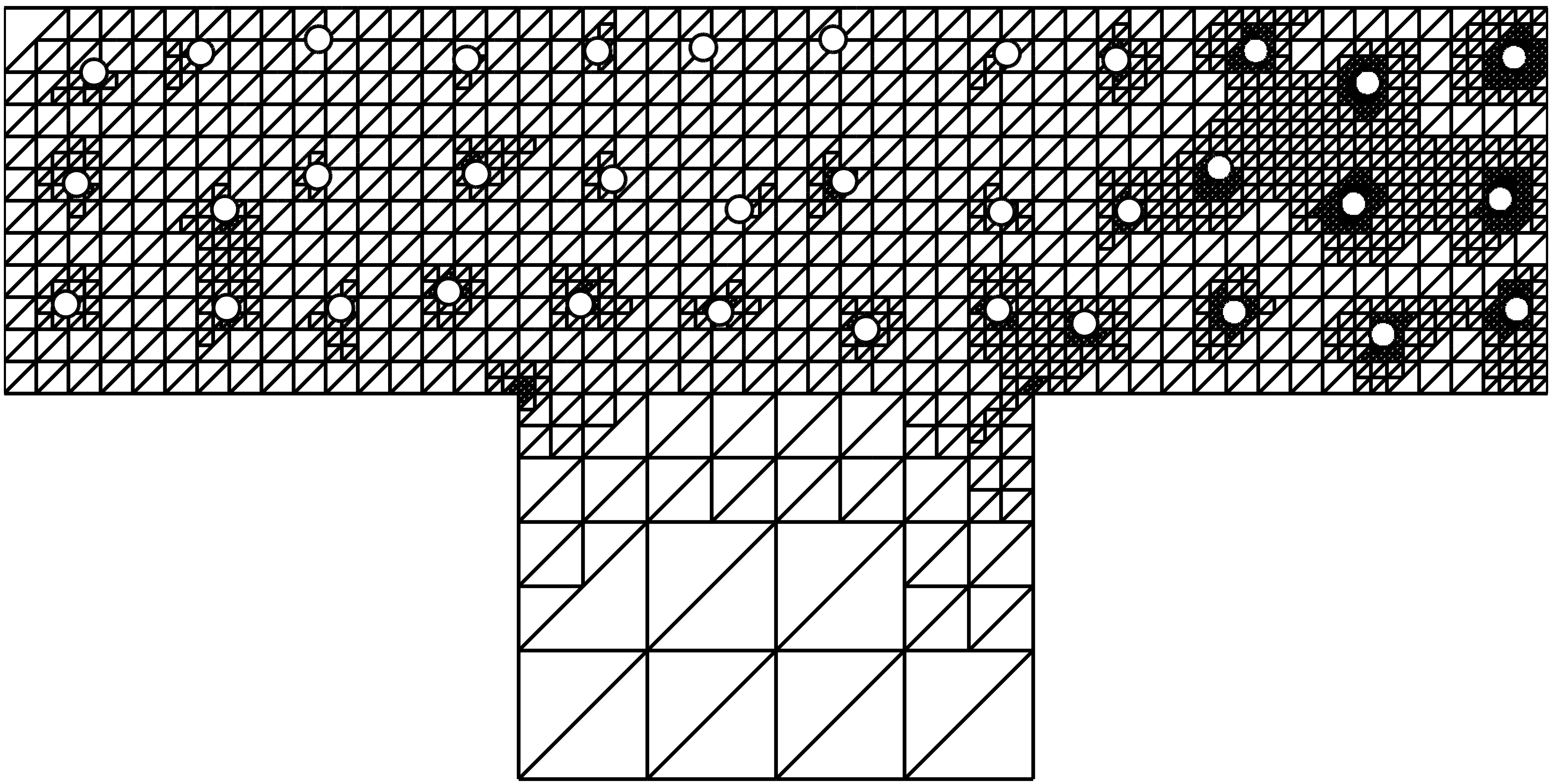
Incompressible Navier-Stokes Equations

$Re = 100$: DWR Refinement, with $J(\mathbf{u}, p) = p(11, 1.5) \approx 2.2764 \times 10^{-3}$



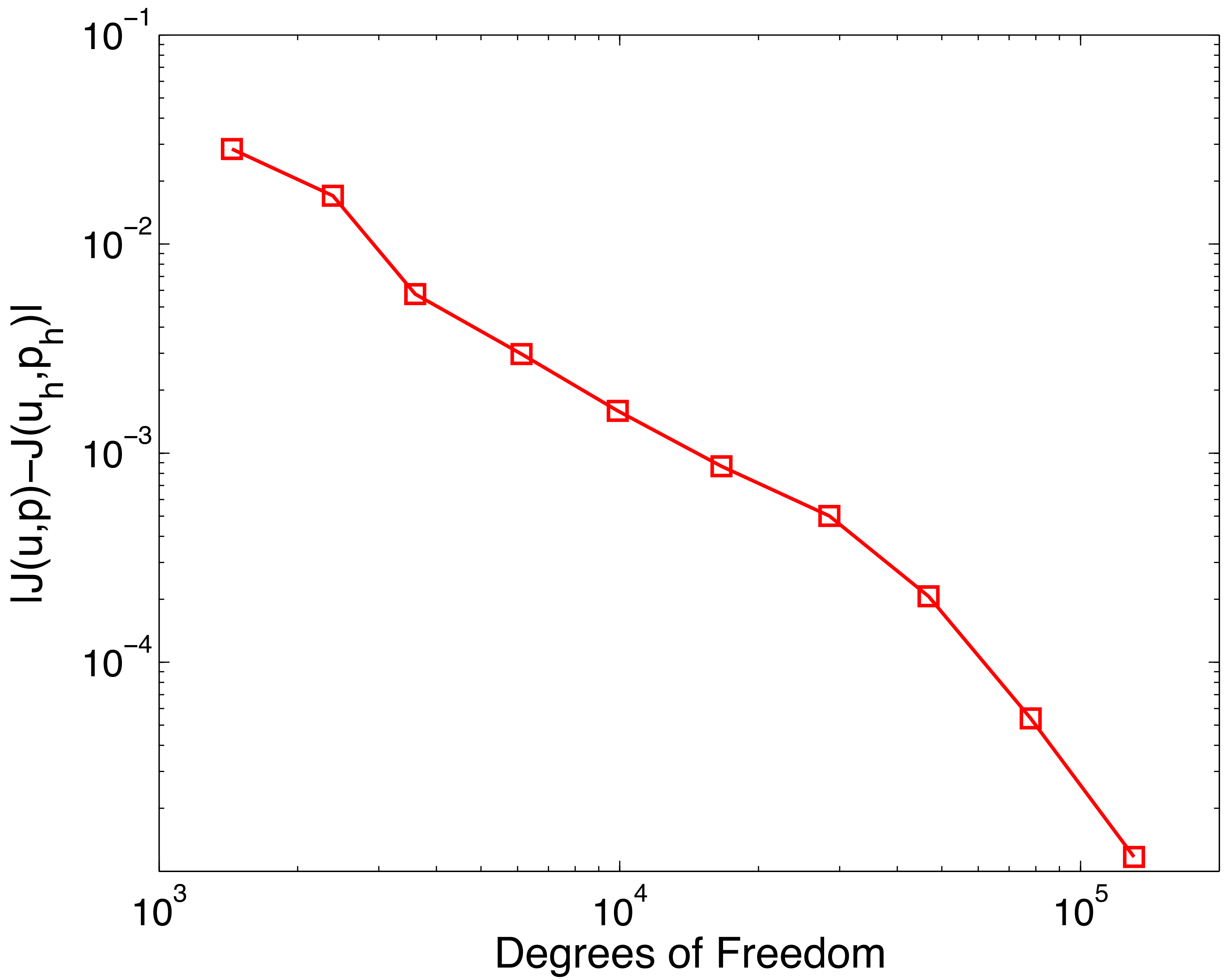
Composite mesh after 3 adaptive refinements, with 408 elements

$Re = 100$: DWR Refinement, with $J(\mathbf{u}, p) = p(11, 1.5) \approx 2.2764 \times 10^{-3}$



Composite mesh after 7 adaptive refinements, with 3118 elements

$Re = 100$: DWR Refinement, with $J(\mathbf{u}, p) = p(11, 1.5) \approx 2.2764 \times 10^{-3}$



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$Re = 100$: DWR Refinement, with $J(\mathbf{u}, p) = p(11, 1.5) \approx 2.2764 \times 10^{-3}$

No of Eles	No of Dofs	$J(\mathbf{u}, p) - J(\mathbf{u}_h, p_h)$	$\sum_{\kappa \in \mathcal{T}_{\text{CFE}}} \eta_\kappa$	θ
96	1440	-2.849E-02	-2.534E-02	**0.89**
159	2385	-1.702E-02	-1.430E-02	**0.84**
240	3600	-5.755E-03	-3.419E-03	**0.59**
408	6120	-2.974E-03	-1.554E-03	**0.52**
660	9900	-1.592E-03	-7.969E-04	**0.50**
1108	16620	-8.644E-04	-3.853E-04	**0.45**
1901	28515	-5.008E-04	-2.842E-04	**0.57**
3118	46770	-2.068E-04	-1.468E-04	**0.71**
5196	77940	-5.390E-05	-4.716E-05	**0.87**
8708	130620	-1.172E-05	-1.172E-05	**1.00**

$$\theta = \frac{\sum_{\kappa \in \mathcal{T}_{\text{CFE}}} \eta_\kappa}{J(\mathbf{u},p) - J(\mathbf{u}_h, p_h)}$$

Discontinuous Galerkin FEMs on Polytopic Meshes

Discontinuous Galerkin Methods

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- ✓ Robustness/stability;
- ✓ Locally conservative;
- ✓ *Ease of implementation;*
- ✓ Highly parallelizable;
- ✓ Flexible mesh design (hybrid grids, non-matching grids, non-uniform/anisotropic polynomial degrees);
- ✓ Wider choice of stable FE spaces for mixed problems;
- ✓ Unified treatment of a wide range of PDEs;
- ✓ Convergence of the method is *independent* of the element shape;
  - ➡ Polynomial bases may be defined in the physical frame, without the need to map from a reference element.

    (See Bassi, Botti, Colombo, Di Pietro, & Tesini 2012)
- ★ Computational overhead/efficiency (increase in DoFs);

DGFEM vs CGFEM (Cangiani, Georgoulis, & H. 2014)

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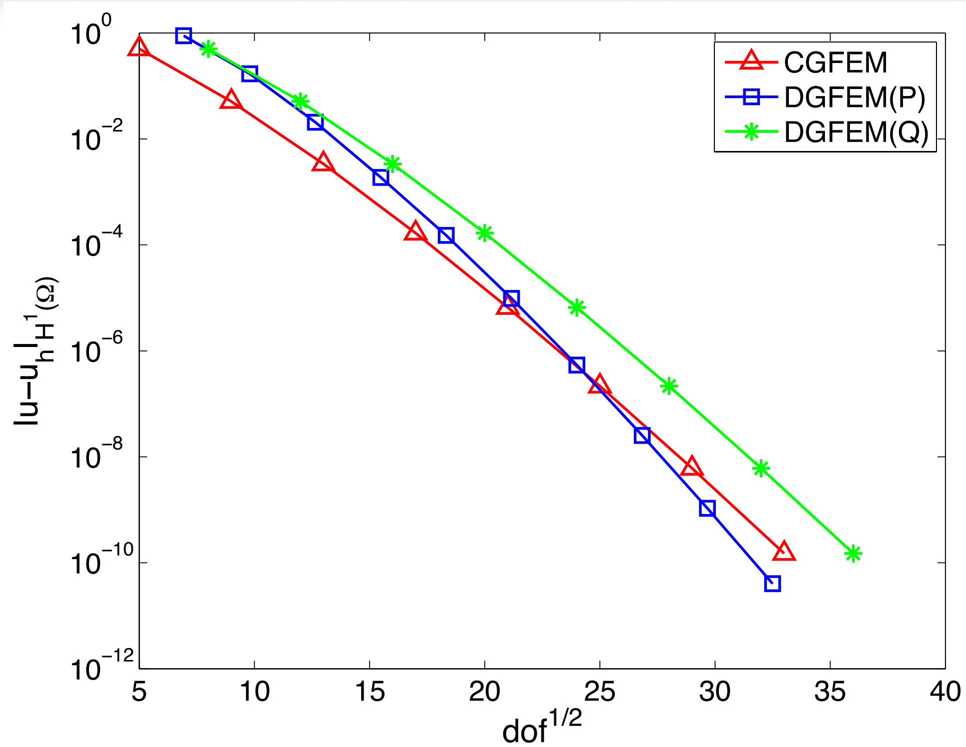
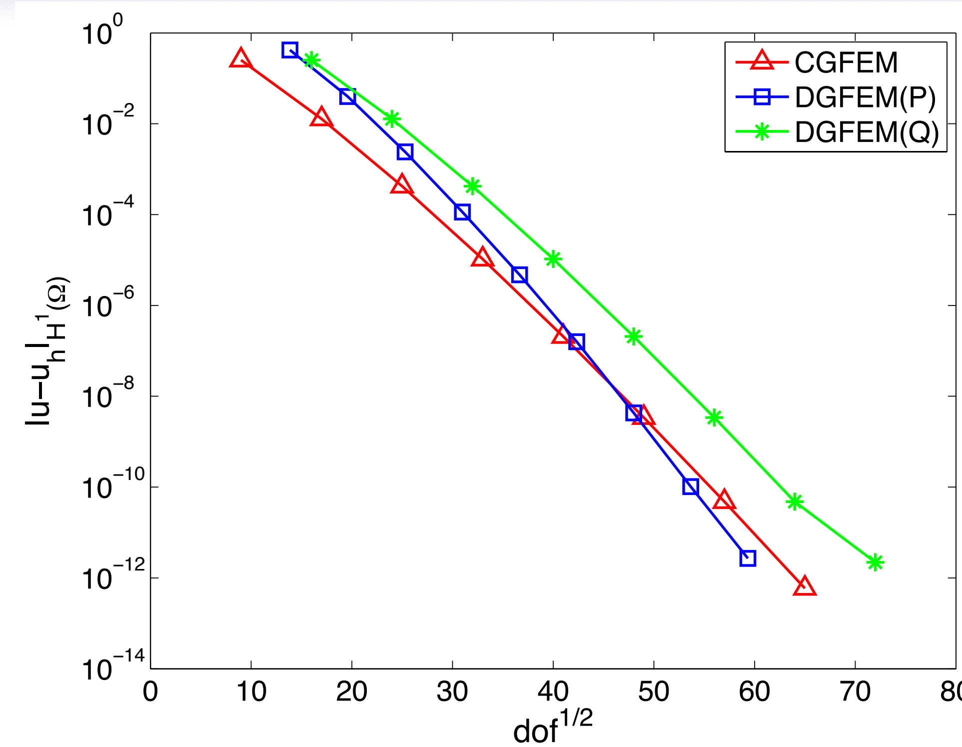
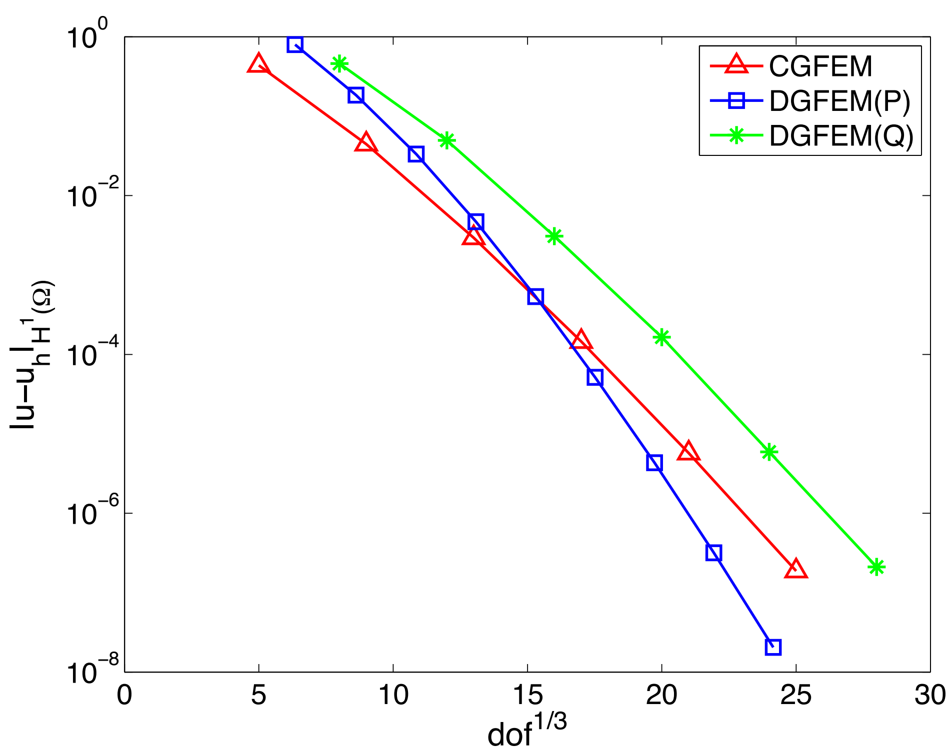


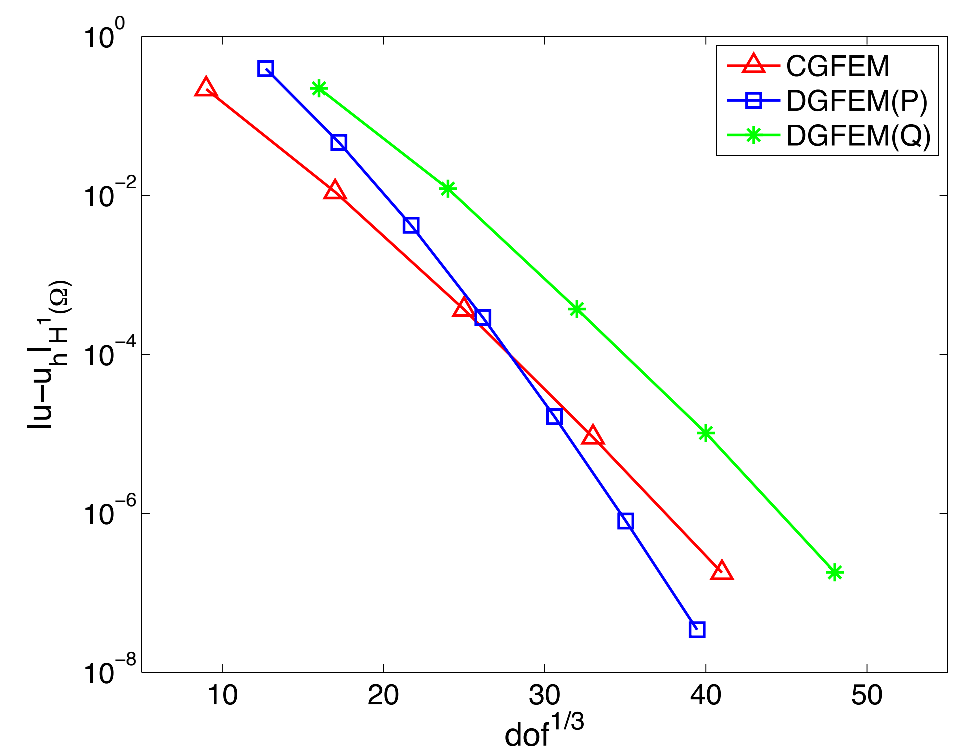
4×4 Mesh



8×8 Mesh

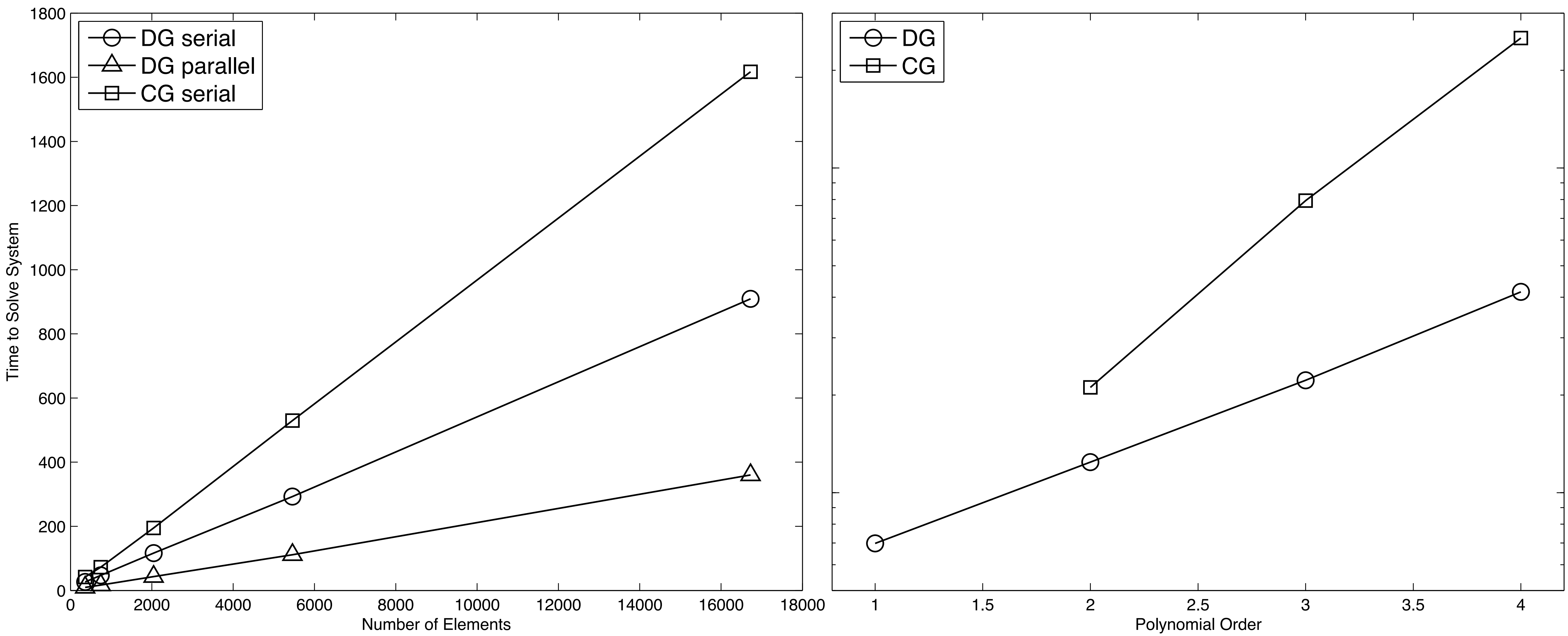


$4 \times 4 \times 4$ Mesh



$8 \times 8 \times 8$ Mesh

Chemically Reacting Flow Example



Joint work with Nathan Sime (Nottingham)

PDE Problem

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Poisson's Equation

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $f \in L_2(\Omega)$: find u such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

- Assume that Ω is *complicated* in the sense that it contains microstructures.

Theorem

There exists a linear extension operator $\mathfrak{E} : H^s(\Omega) \to H^s(\mathbb{R}^d)$, $s \in \mathbb{N}_0$, such that $\mathfrak{E}v|_\Omega = v$ and

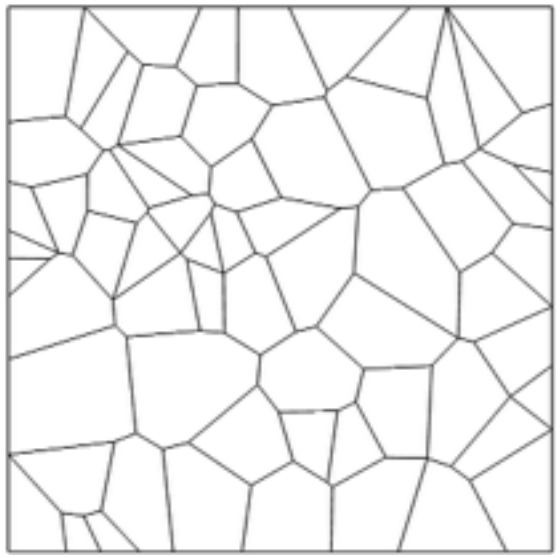
$$\|\mathfrak{E}v\|_{H^s(\mathbb{R}^d)} \le C\|v\|_{H^s(\Omega)}.$$

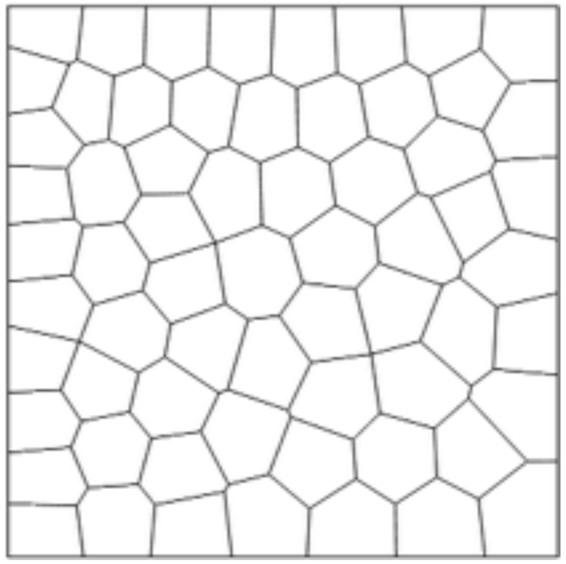
See Stein 1970, Sauter & Warnke 1999.

Construction of the FEM meshes

- Polytopic Mesh Generator

  `Polymesher`, Voronoi Mesh generator.

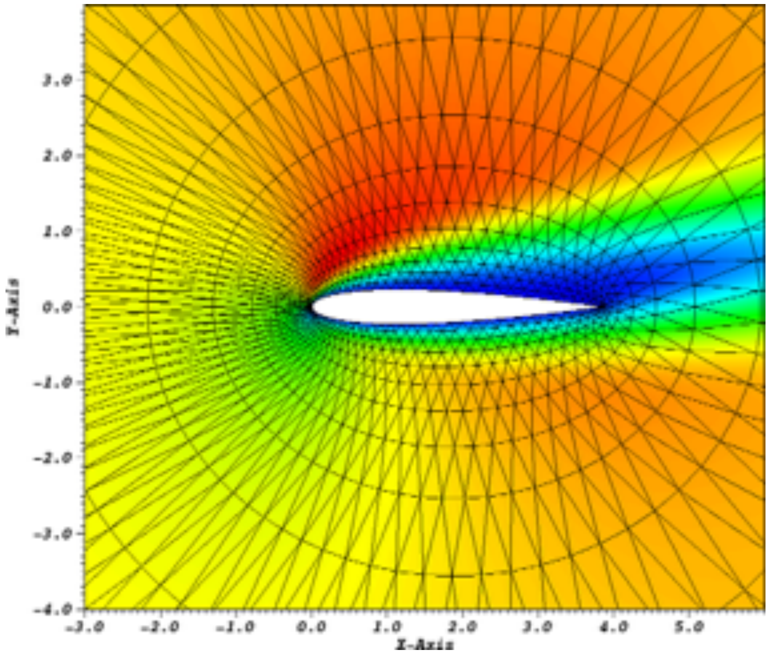


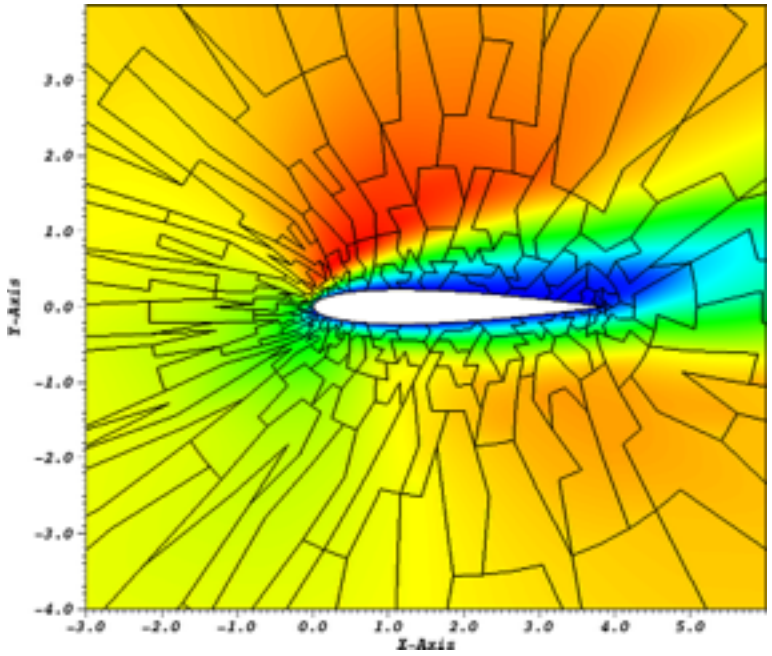


Talischi *et al.* 2012

- Agglomeration

  Graph partitioning tools, e.g., `METIS`.





Joint work with
Jochen Schuetz (Aachen)

Construction of the FEM meshes

- Composite FEM mesh construction:
  - Key Idea is to employ two meshes:
    1. $\mathcal{T}_h$ is a fine mesh which accurately represents Ω;
    2. $\mathcal{T}_{\texttt{CFE}}$ is a coarse *agglomerated* mesh consisting of polygons.
  - $\mathcal{T}_{\texttt{CFE}}$ provides an *affordable* computational mesh.

Hackbusch & Sauter 1997→

We define an *overlapping* shape-regular coarse mesh $\hat{\mathcal{T}}_H$, consisting of standard disjoint elements:

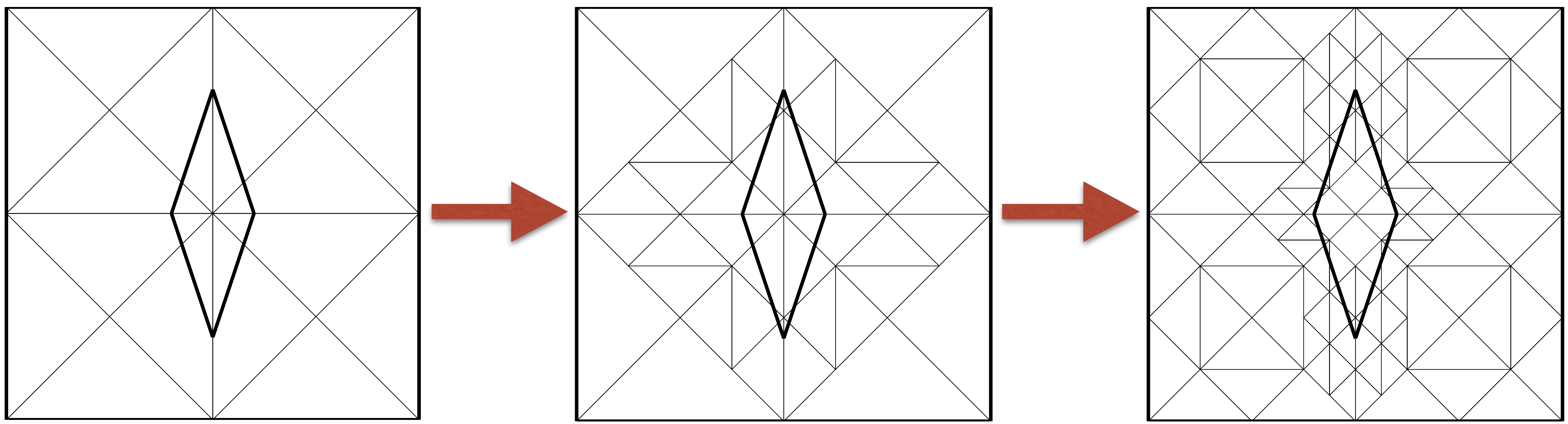
$$\Omega \subset \Omega_H = \left(\bigcup_{\hat{\kappa} \in \hat{\mathcal{T}}_H} \hat{\kappa} \right)^{\circ} \quad \text{and} \quad \hat{\kappa}^{\circ} \cap \Omega \neq \emptyset \ \ \forall \hat{\kappa} \in \hat{\mathcal{T}}_H.$$

Refinement Algorithm (Construction of the Reference Meshes):

1. Set $\hat{\mathcal{T}}_{h_1} = \hat{\mathcal{T}}_H$, and the mesh counter $\ell = 1$.
2. Set $\hat{\mathcal{T}}_{h_{\ell+1}} = \emptyset$.
3. For all $\hat{\kappa} \in \hat{\mathcal{T}}_{h_\ell}$ do
   - (a) If $\hat{\kappa} \subset \Omega$ then $\hat{\mathcal{T}}_{h_{\ell+1}} = \hat{\mathcal{T}}_{h_{\ell+1}} \bigcup \{\hat{\kappa}\}$;
   - (b) Otherwise **refine** $\hat{\kappa} = \bigcup_{i=1}^{n_{\hat{\kappa}}} \hat{\kappa}_i$. For $i = 1, \ldots, n_{\hat{\kappa}}$, if $\hat{\kappa}_i \cap \Omega \neq \emptyset$ then set $\hat{\mathcal{T}}_{h_{\ell+1}} = \hat{\mathcal{T}}_{h_{\ell+1}} \bigcup \{\hat{\kappa}_i\}$.
4. If the reference mesh $\hat{\mathcal{T}}_{h_\ell}$ is *sufficiently* fine, in the sense that it provides a good representation of the boundary of Ω, then `STOP`. Otherwise, set $\ell = \ell + 1$, and `GOTO` 2.

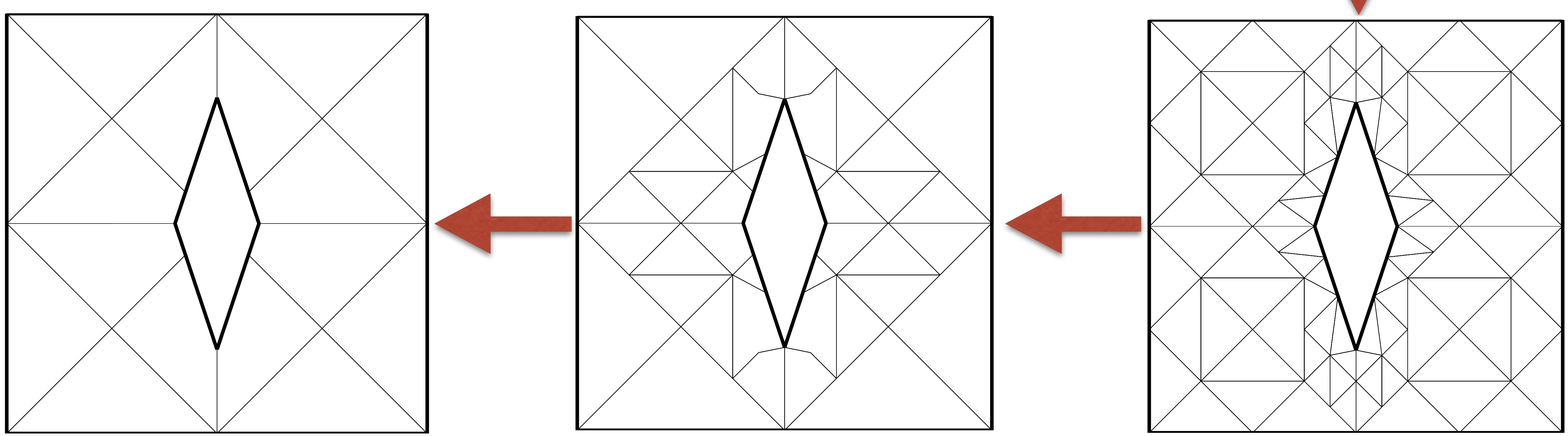
Composite FEM Meshes

Reference Meshes $\hat{\mathcal{T}}_{h_i}$, $i = 1, \ldots, \ell$.



Physical (agglomerated) Meshes $\mathcal{T}_{h_i}$, $i = 1, \ldots, \ell$.

Φ



$\mathcal{T}_{\mathrm{CFE}}$

DG Finite Element Spaces

We set

$$V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p}) = \{u \in L_2(\Omega) : u|_\kappa \in \mathcal{P}_{p_\kappa}(\kappa) \ \forall \kappa \in \mathcal{T}_{\mathtt{CFE}}\},$$

where $\mathcal{P}_p(\kappa)$ denotes the set of polynomials of degree at most $p \geq 1$ over κ.

➡ Polynomial bases are defined in the physical space, *without* any mappings.

Mesh Assumptions

- $\mathcal{F}(\mathcal{T}_{\mathtt{CFE}}) = \mathcal{F}_{\mathtt{CFE}}^{\mathcal{I}} \cup \mathcal{F}_{\mathtt{CFE}}^{\mathcal{B}}$ denotes the set of all faces in the mesh $\mathcal{T}_{\mathtt{CFE}}$.

(A1) For all elements $\kappa \in \mathcal{T}_{\mathtt{CFE}}$, we require

$$\max_{\kappa \in \mathcal{T}_{\mathtt{CFE}}} \mathrm{card}\left\{F \in \mathcal{F}_{\mathtt{CFE}}^{\mathcal{I}} \cup \mathcal{F}_{\mathtt{CFE}}^{\mathcal{B}} : F \subset \partial\kappa\right\} \leq C_F \quad \text{(uniformly)}.$$

(A2) The polynomial degree vector $\mathbf{p}$ is of bounded local variation.

hp-DGFEM (based on Symmetric Interior Penalty Method- SIPG)

Find $u_h \in V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p})$ such that

$$B_{\mathtt{DG}}(u_h, v) = F_h(v)$$

for all $v \in V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p})$, where

Stabilisation

$$\begin{aligned} B_{\mathtt{DG}}(u,v) &= \sum_{\kappa \in \mathcal{T}_{\mathtt{CFE}}} \int_\kappa \nabla u \cdot \nabla v \, dx + \sum_{F \in \mathcal{F}^{\mathcal{I}}_{\mathtt{CFE}} \cup \mathcal{F}^{\mathcal{B}}_{\mathtt{CFE}}} \int_F \sigma \, [\![u]\!] \cdot [\![v]\!] \, ds \\ &\quad - \sum_{F \in \mathcal{F}^{\mathcal{I}}_{\mathtt{CFE}} \cup \mathcal{F}^{\mathcal{B}}_{\mathtt{CFE}}} \int_F \left(\{\!\{\nabla_h v\}\!\} \cdot [\![u]\!] + \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!] \right) ds, \\ F_h(v) &= \int_\Omega f v \, dx. \end{aligned}$$

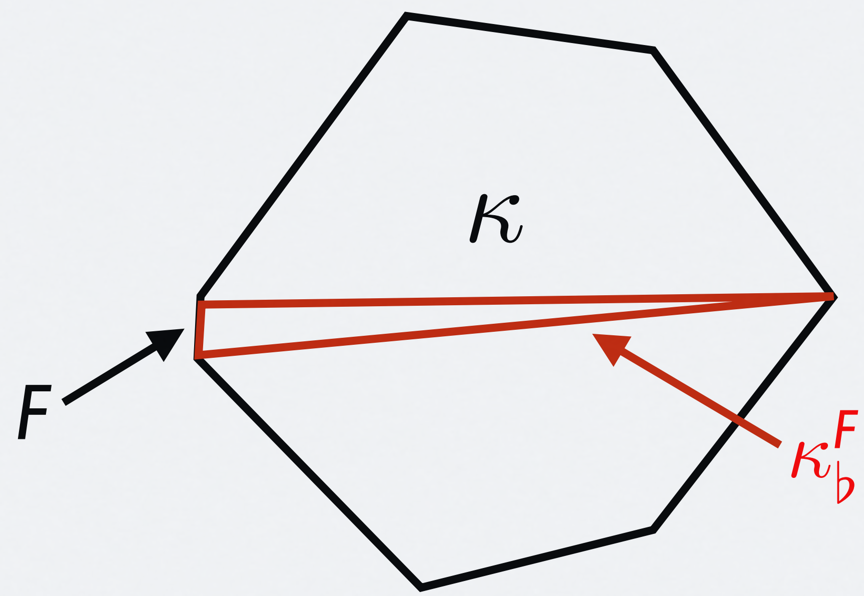
Symmetry Consistency

$\{\!\{\cdot\}\!\}$: Average Operator $\qquad [\![\cdot]\!]$: Jump Operator

Error Estimation

Face/edge Degeneration

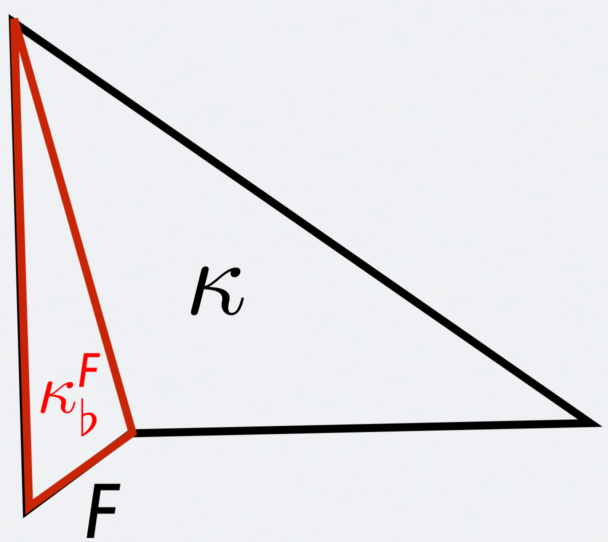
- meas$(F) \ll$ meas(κ)



Inverse Estimate

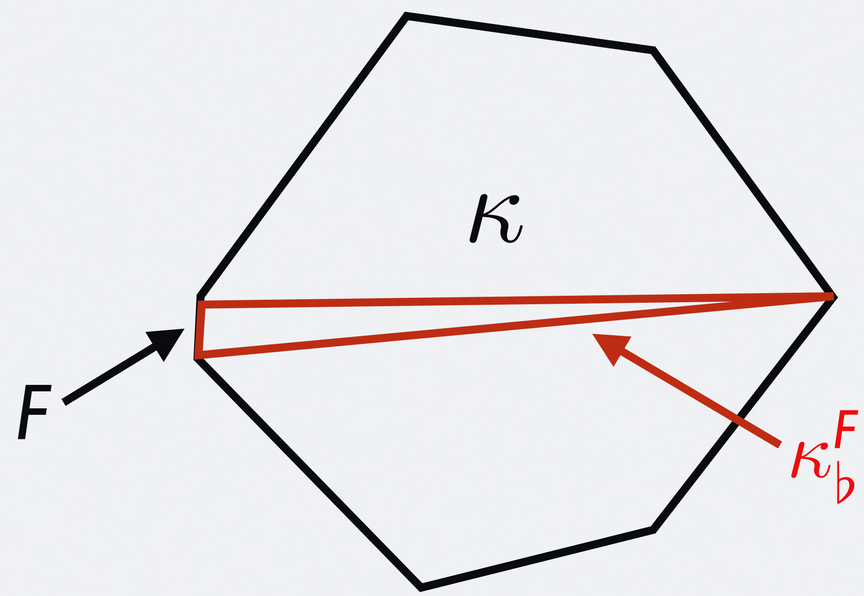
Given $v \in \mathcal{P}_p(\kappa)$, we have the inverse estimate

$$\|v\|^2_{L^2(F)} \leq C_{\mathrm{inv}} \frac{p^2|F|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|} \|v\|^2_{L^2(\kappa)}.$$



Face/edge Degeneration

- $\text{meas}(F) \ll \text{meas}(\kappa)$



Inverse Estimate

Given $v \in \mathcal{P}_p(\kappa)$, we have the inverse estimate

$$\|v\|^2_{L^2(F)} \leq C_{\text{inv}} \min\left\{ \frac{|\kappa|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, p^{2d} \right\} \frac{p^2 |F|}{|\kappa|} \|v\|^2_{L^2(\kappa)}.$$

Proof: Exploit an inverse inequality in L^∞, together with results from Georgoulis 2008.

DG-Norm

$$|||v|||_{\mathrm{DG}}^2 = \sum_{\kappa\in\mathcal{T}_{\mathrm{CFE}}} \|\nabla v\|_{L_2(\kappa)}^2 + \sum_{F\in\mathcal{F}_{\mathrm{CFE}}^{\mathcal{I}}\cup\mathcal{F}_{\mathrm{CFE}}^{\mathcal{B}}} \|\sigma^{1/2}[\![v]\!]\|_{L_2(F)}^2.$$

Interior Penalty Parameter

$$\sigma := \gamma\, C_{\mathrm{inv}} \max_{\kappa\in\{\kappa^+,\kappa^-\}} \left\{ \min\left\{ \frac{|\kappa|}{\sup_{\kappa_\flat^F\subset\kappa} |\kappa_\flat^F|}, p_\kappa^{2d} \right\} \frac{p_\kappa^2 |F|}{|\kappa|} \right\}, \quad F = \kappa^+\cap\kappa^-.$$

Lemma (Coercivity & Continuity)

For $\gamma > \gamma_{\min}$, we have

$$B_{\mathrm{DG}}(v,v) \geq C_{\mathrm{coer}} |||v|||_{\mathrm{DG}}^2 \quad \text{for all } v\in V(\mathcal{T}_{\mathrm{CFE}},\mathbf{p}),$$

and

$$B_{\mathrm{DG}}(v,w) \leq C_{\mathrm{cont}} |||v|||_{\mathrm{DG}} |||w|||_{\mathrm{DG}} \quad \text{for all } v,w\in V(\mathcal{T}_{\mathrm{CFE}},\mathbf{p}).$$

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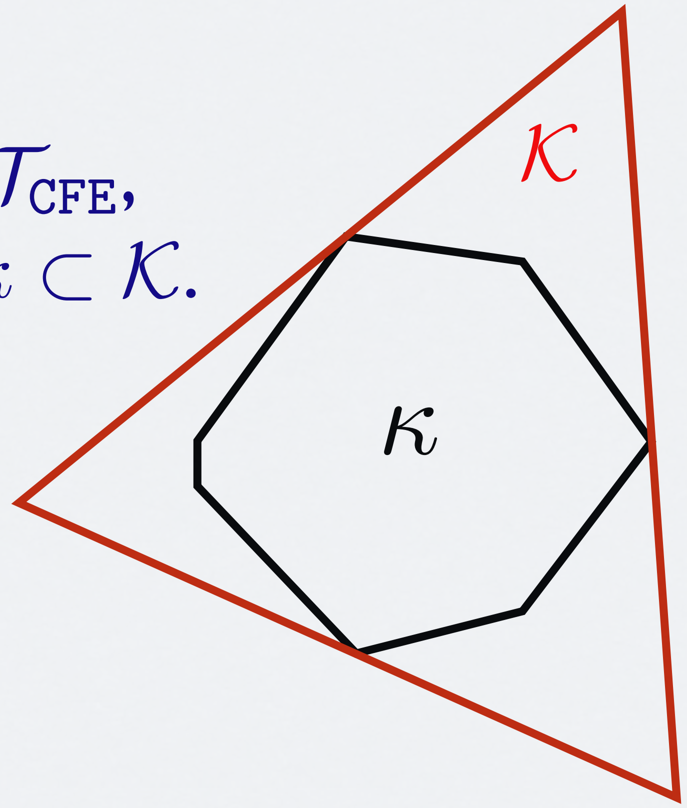
Projection Operators

Let $\mathcal{T}_\sharp = \{\mathcal{K}\}$ denote a shape-regular covering of $\mathcal{T}_{\texttt{CFE}}$, such that for each $\kappa \in \mathcal{T}_{\texttt{CFE}}$, there exists $\mathcal{K} \in \mathcal{T}_\sharp$, $\kappa \subset \mathcal{K}$.



(A3) We assume that

$$\max_{\kappa \in \mathcal{T}_{\texttt{CFE}}} \text{card}\,\{\kappa' \in \mathcal{T}_{\texttt{CFE}} : \kappa' \cap \mathcal{K} \neq \emptyset,\ \mathcal{K} \in \mathcal{T}_\sharp\ \kappa \subset \mathcal{K}\} \leq \mathcal{O}_\Omega \quad \text{(uniformly)}$$

We write $\tilde{\Pi}_p v = \Pi_p(\mathfrak{E}v|_{\mathcal{K}})|_\kappa$.

- Π_p: Projector on $\mathcal{K}$ (standard element shape).
- $\mathfrak{E}$: Extension operator.

Lemma

For $\kappa \in \mathcal{T}_{\mathrm{CFE}}$, we have

$$
\begin{aligned}
\|v - \tilde{\Pi} v\|_{H^q(\kappa)} &\leq C \frac{h_\kappa^{s_\kappa - q}}{p_\kappa^{k_\kappa - q}} \|\mathfrak{E} v\|_{H^{k_\kappa}(\mathcal{K})}, \quad 0 \leq q \leq k_\kappa, \\
\|v - \tilde{\Pi} v\|_{L^2(F)} &\leq C |F|^{1/2} \frac{h_\kappa^{s_\kappa - d/2}}{p_\kappa^{k_\kappa - 1/2}} C_m(p_\kappa, \kappa, F)^{1/2} \|\mathfrak{E} v\|_{H^{k_\kappa}(\mathcal{K})},
\end{aligned}
$$

where

$$
C_m(p_\kappa, \kappa, F) = \min \left\{ \frac{h_\kappa^d}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, \frac{1}{p_\kappa^{1-d}} \right\},
$$

and $s_\kappa = \min\{p_\kappa + 1, k_\kappa\}$, $k_\kappa > d/2$.

Theorem (Cangiani, Georgoulis, & H, 2013)

For $s_\kappa = \min\{p_\kappa + 1, k_\kappa\}$ and $p_\kappa \geq 1$, the following bound holds:

$$|||\, u - u_h \,|||^2_{\mathrm{DG}} \leq C \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(k_\kappa - 1)}} \left(1 + \mathcal{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa)\right) \|\mathfrak{E}u\|^2_{H^{k_\kappa}(\mathcal{K})}.$$

$$\mathcal{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa) = p_\kappa h_\kappa^{-d} \sum_{F \subset \partial\kappa} C_m(p_\kappa, \kappa, F)\sigma^{-1}|F|$$

$$+ p_\kappa^2 |\kappa|^{-1} \sum_{F \subset \partial\kappa} C_{\mathrm{INV}}(p_\kappa, \kappa, F)\sigma^{-1}|F| + h_\kappa^{-d+2} p_\kappa^{-1} \sum_{F \subset \partial\kappa} C_m(p_\kappa, \kappa, F)\sigma|F|,$$

$$C_{\mathrm{INV}}(p, \kappa, F) := C_{\mathrm{inv}} \min\left\{\frac{|\kappa|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, p^{2d}\right\},$$

$$C_m(p_\kappa, \kappa, F) = \min\left\{\frac{h_\kappa^d}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, \frac{1}{p_\kappa^{1-d}}\right\}.$$

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Theorem (Cangiani, Georgoulis, & H, 2013)

For $s_\kappa = \min\{p_\kappa + 1, k_\kappa\}$ and $p_\kappa \geq 1$, the following bound holds:

$$|||\, u - u_h \,|||^2_{\mathrm{DG}} \leq C \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(k_\kappa - 1)}} \left(1 + \mathcal{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa)\right) \|\mathfrak{E}u\|^2_{H^{k_\kappa}(\mathcal{K})}.$$

For uniform orders $p_\kappa = p \geq 1$, $h = \max_{\kappa \in \mathcal{T}} h_\kappa$, $s_\kappa = s$, $s = \min\{p + 1, k\}$, $k > 1 + d/2$, and $\mathrm{diam}(F) \sim h_\kappa$, $F \subset \partial\kappa$, $\kappa \in \mathcal{T}_{\mathrm{CFE}}$, we get the bound

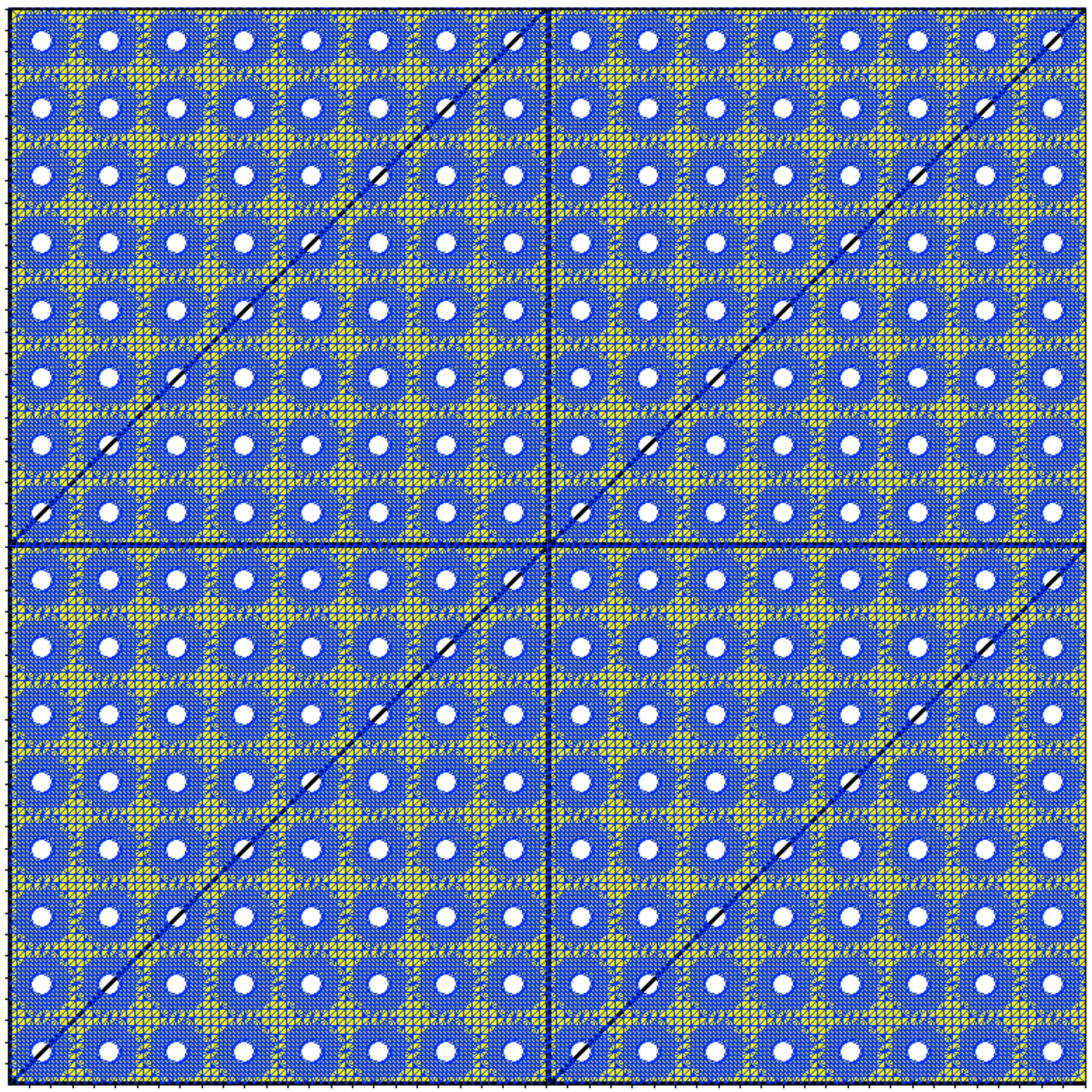
$$|||\, u - u_h \,|||_{\mathrm{DG}} \leq C \frac{h^{s-1}}{p^{k-3/2}} \|u\|_{H^k(\Omega)}.$$

cf. H., Schwab & Süli 2002.

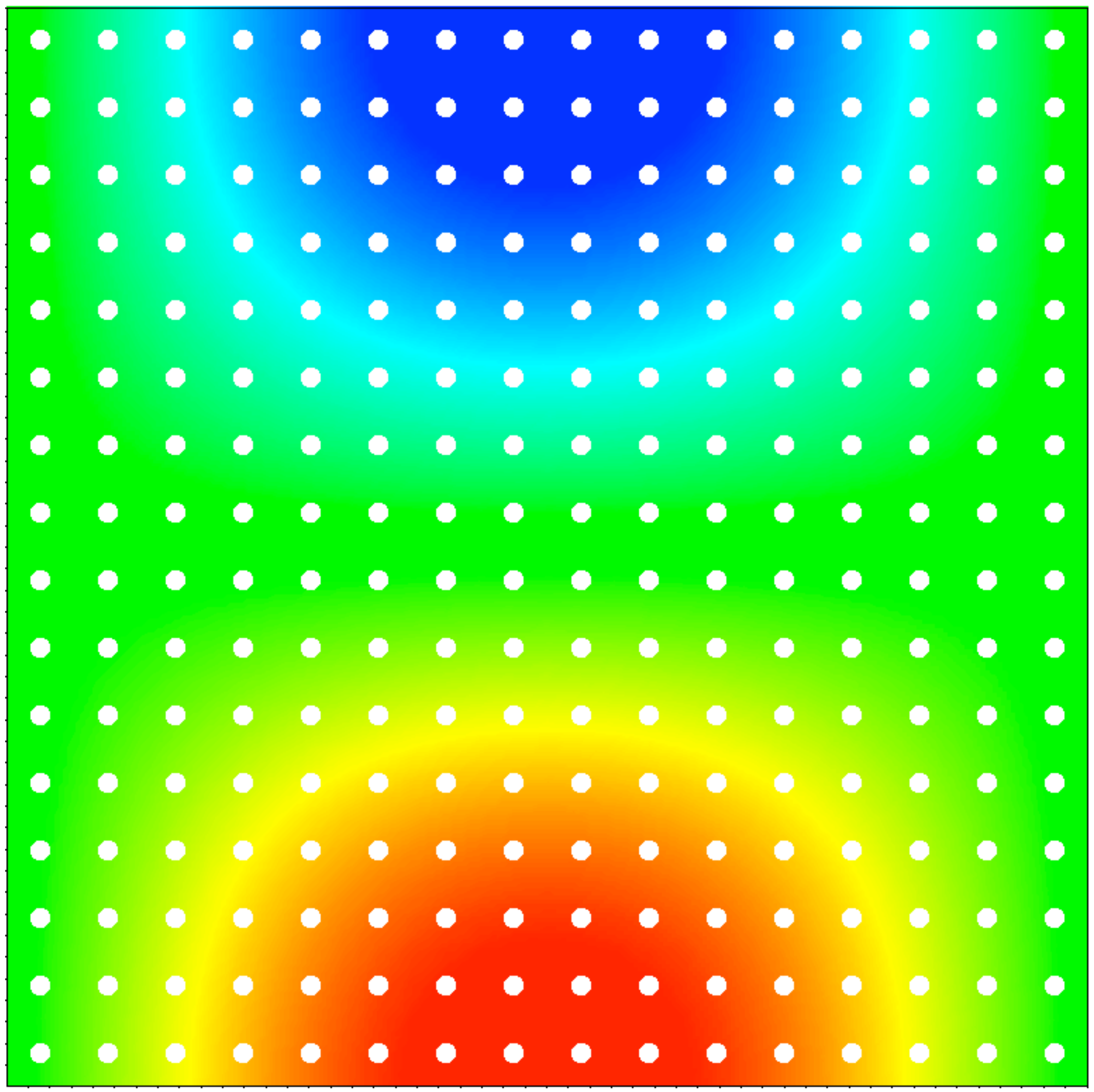
2D Example

$$-\Delta u = f \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega$$

f is selected so that $u = \sin(\pi x)\cos(\pi y)$

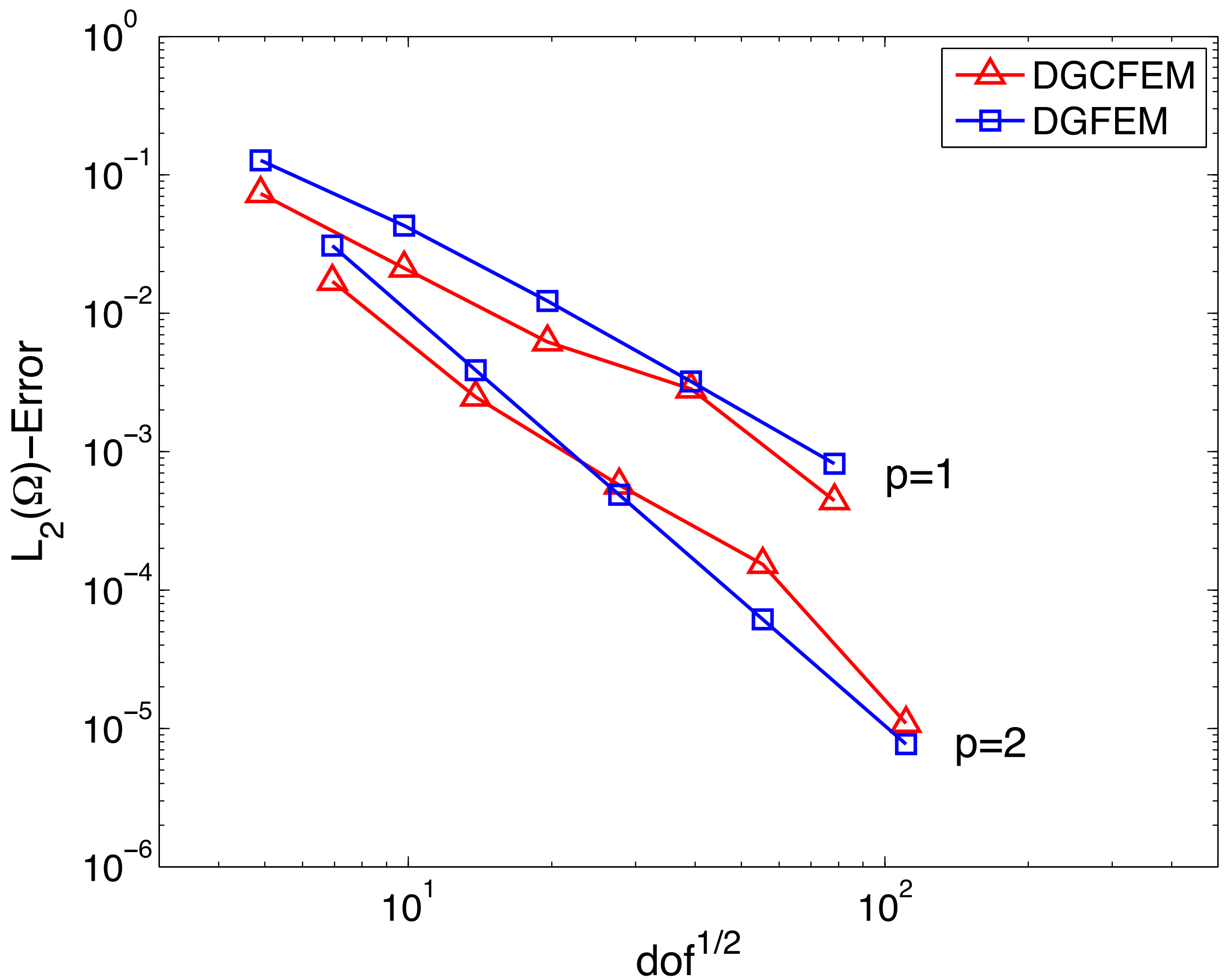


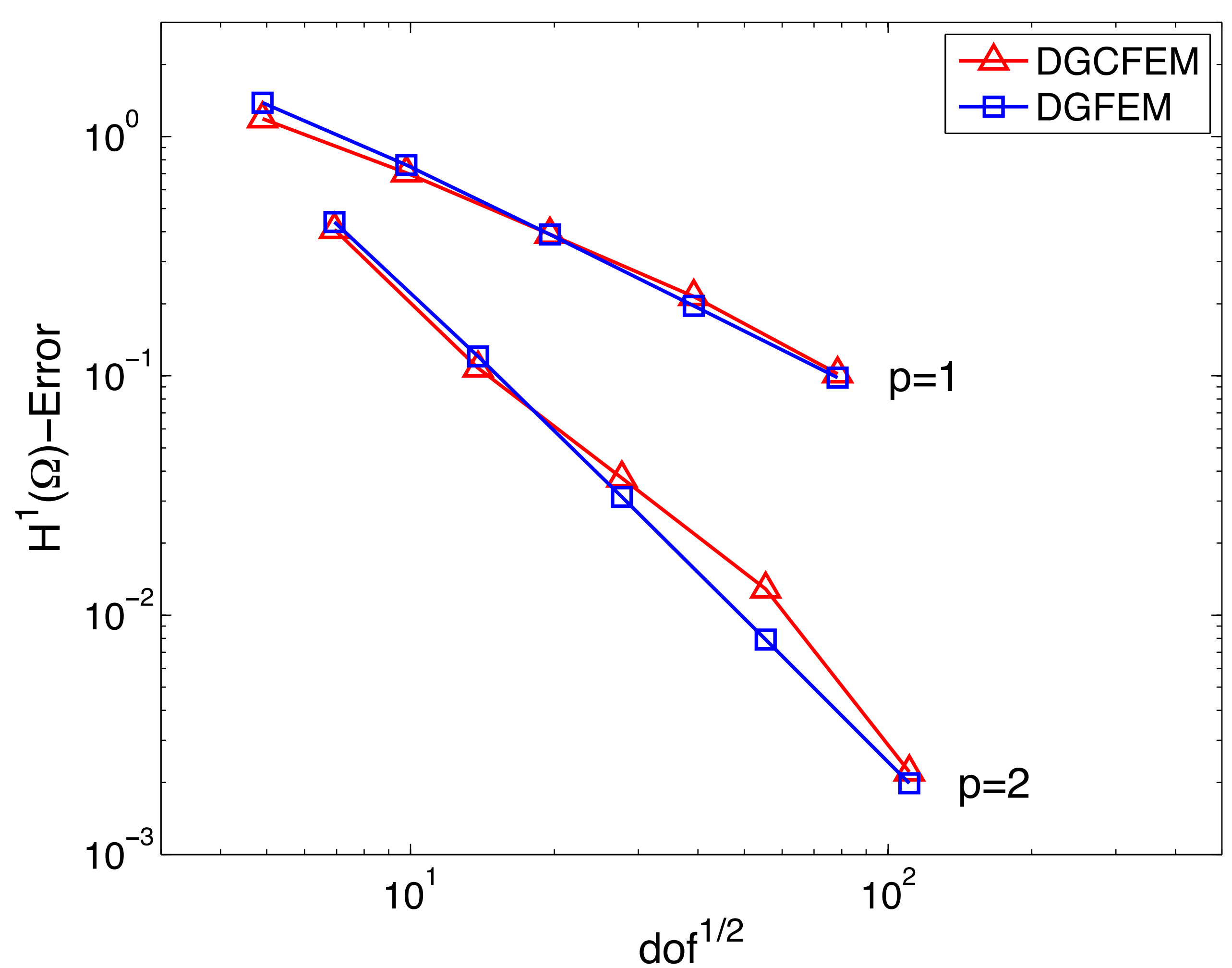
Initial Coarse/Fine meshes



Analytical Solution

2D Example



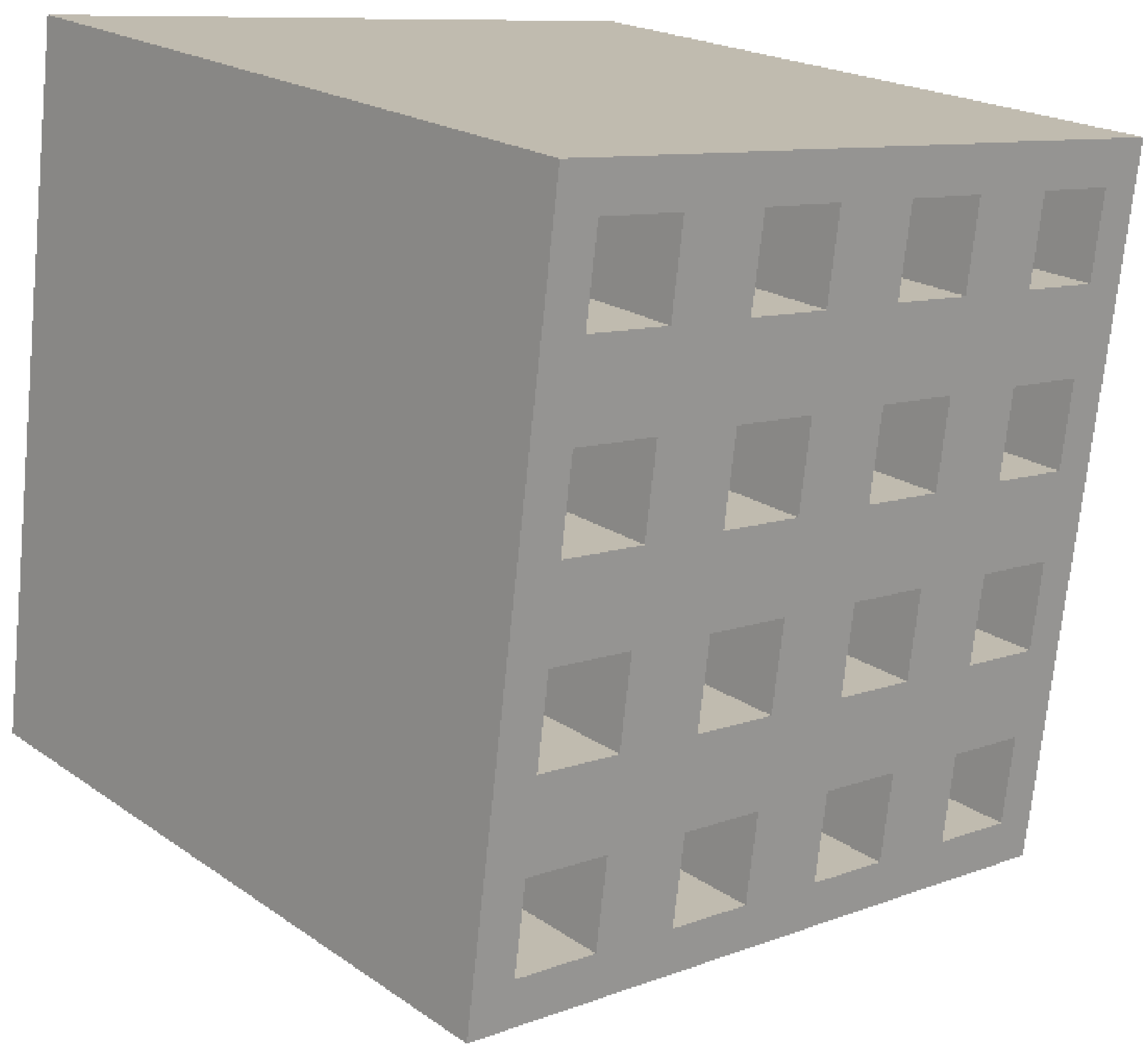


(DGFEM solution computed on domain without any holes)

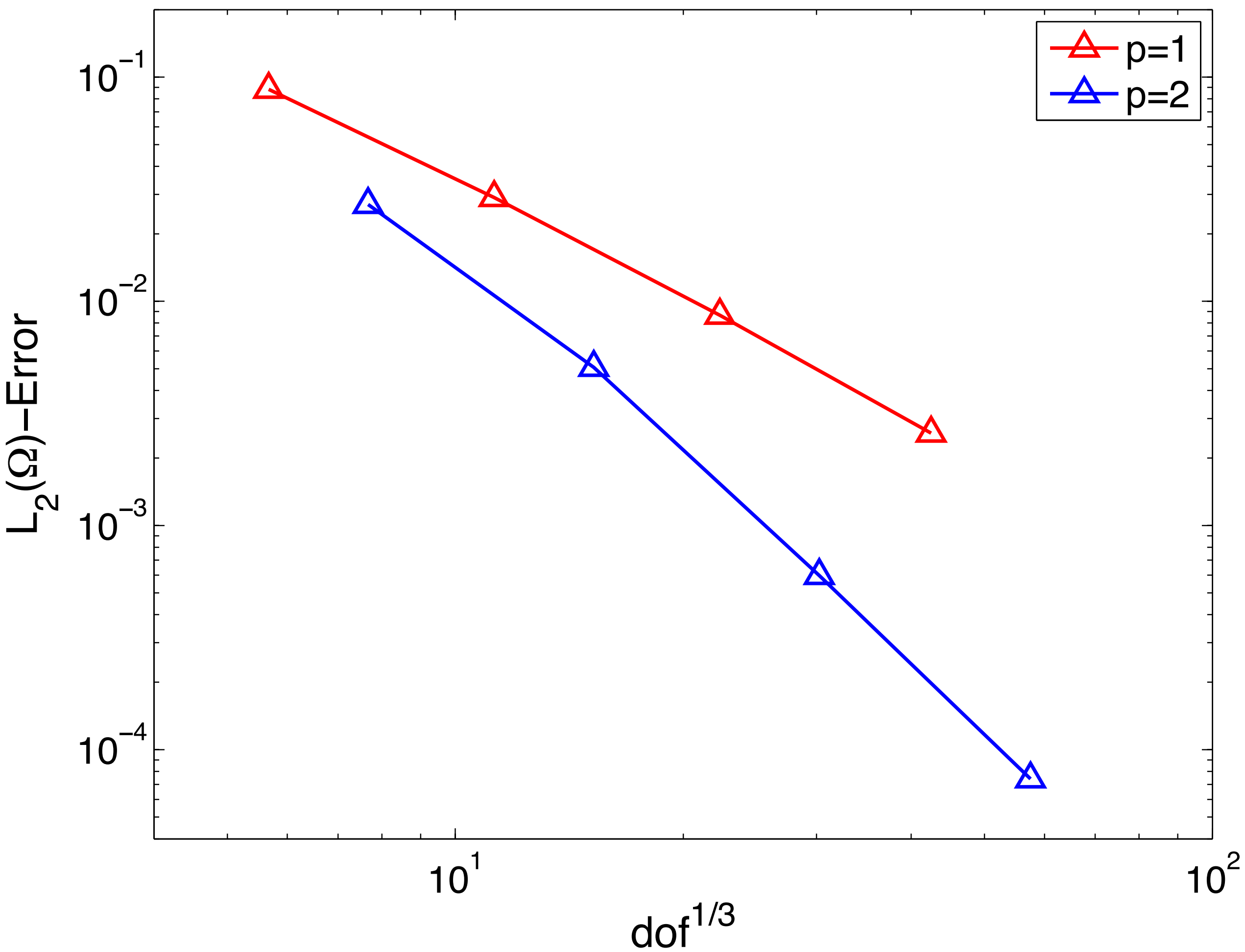
3D Example

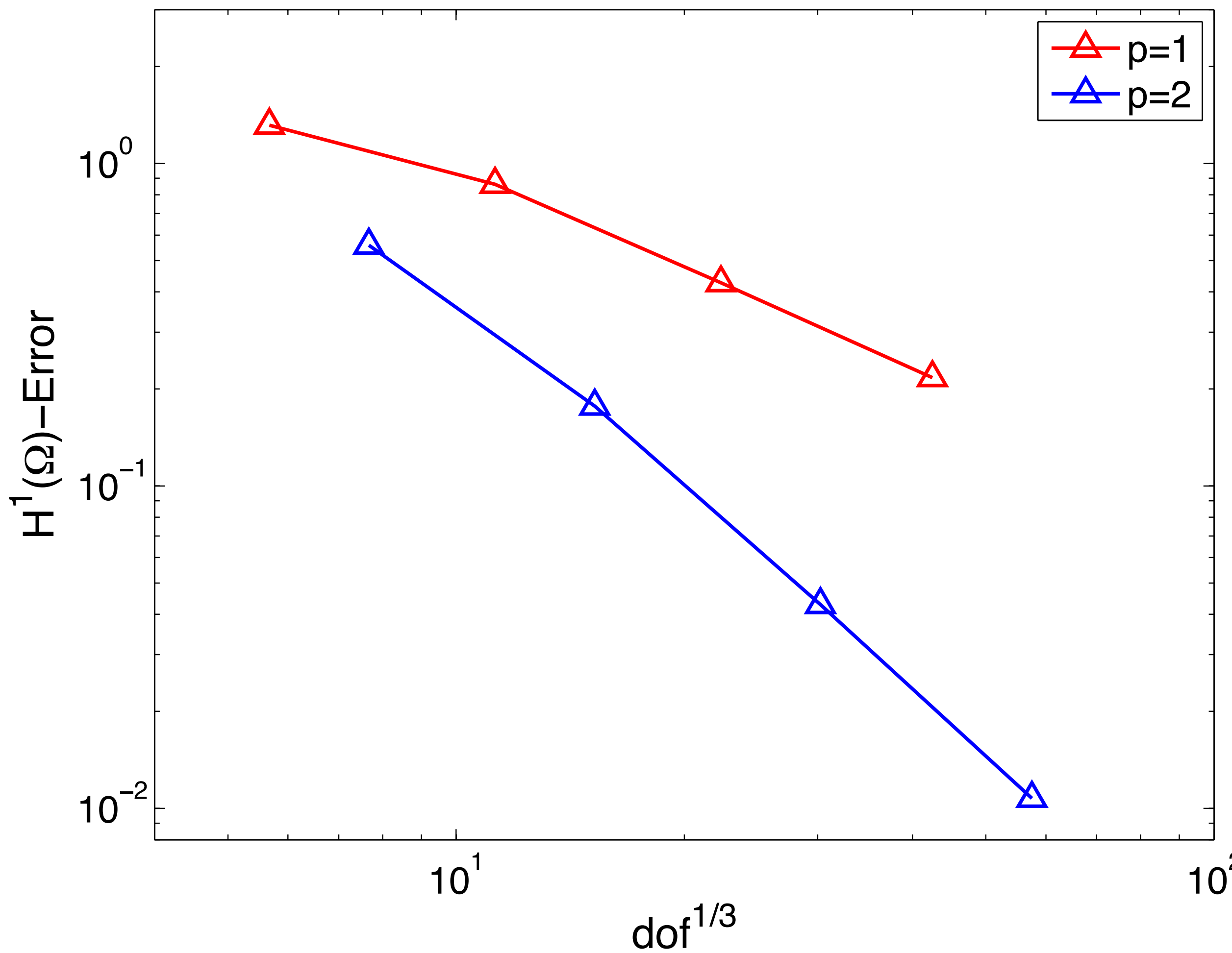
$$-\Delta u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega$$

f is selected so that $u = \sin(\pi x)\cos(\pi y)\sin(\pi z)$



3D Example


p=1
p=2
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$L_2(\Omega)$–Error
10^{1}
10^{2}
$\mathrm{dof}^{1/3}$


p=1
p=2
10^{0}
10^{-1}
10^{-2}
$H^1(\Omega)$–Error
10^{1}
10^{2}
$\mathrm{dof}^{1/3}$

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Theorem (On composite meshes)

The following *hp*-version *a posteriori* error bound holds:

$$|||\, u - u_h \,|||_{\mathrm{DG}} \leq C \left(\sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} (\eta_\kappa^2 + \mathcal{O}_\kappa^2) \right)^{\frac{1}{2}},$$

where the local error indicators η_κ, $\kappa \in \mathcal{T}_{\mathrm{CFE}}$, are defined by

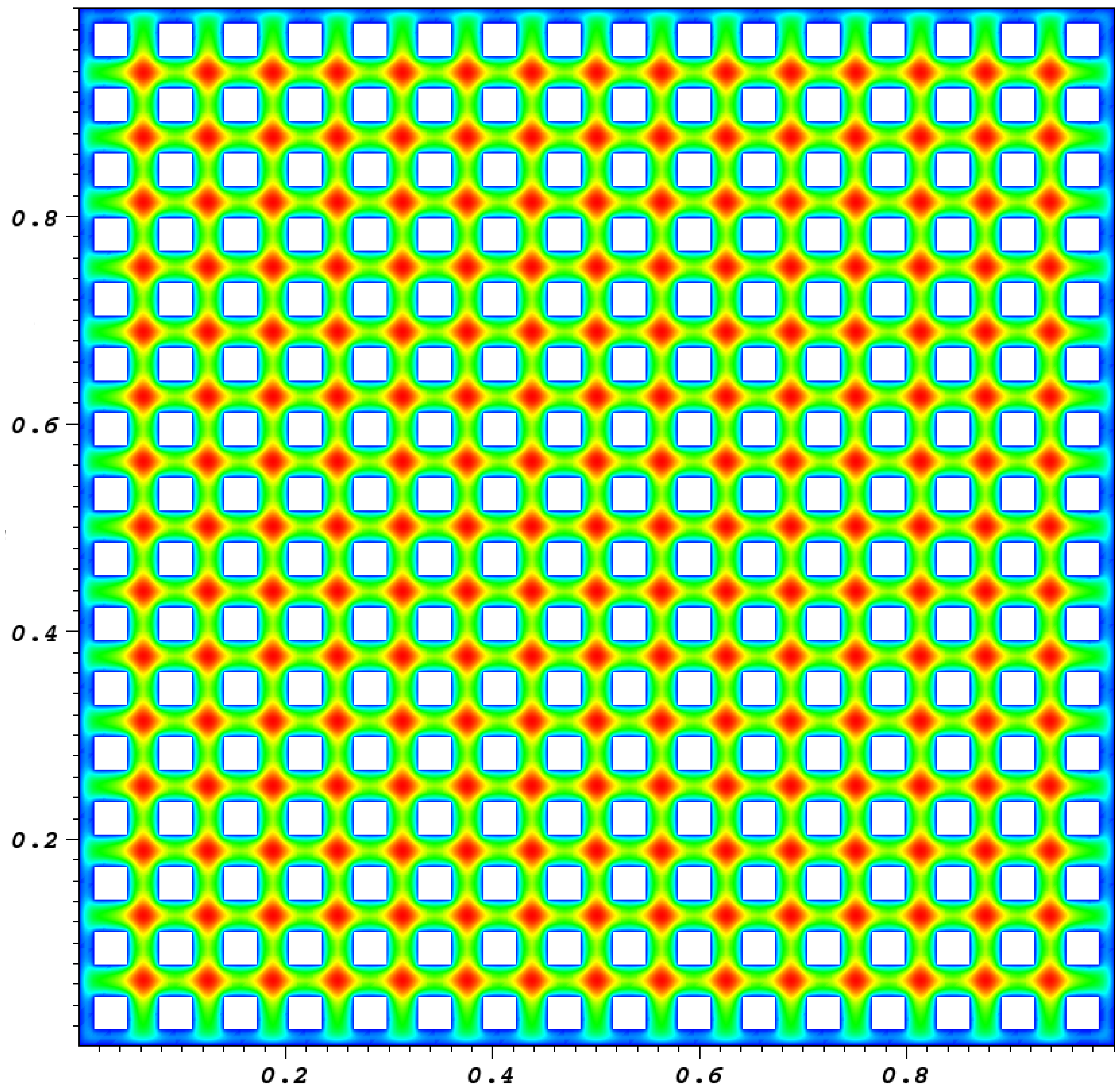
$$\begin{aligned} \eta_\kappa^2 &= h_\kappa^2 p_\kappa^{-2} \|\Pi f + \Delta u_h\|_{L_2(\kappa)}^2 + \sum_{F \subset \partial\kappa \backslash \partial\Omega} h_\kappa^2 h_F^{-1} p_\kappa^{-1} \|[\![\nabla u_h]\!]\|_{L_2(F)}^2 \\ &\quad + \sigma h_\kappa^2 h_F^{-2} p_\kappa \|[\![u_h]\!]\|_{L_2(\partial\kappa)}^2, \end{aligned}$$

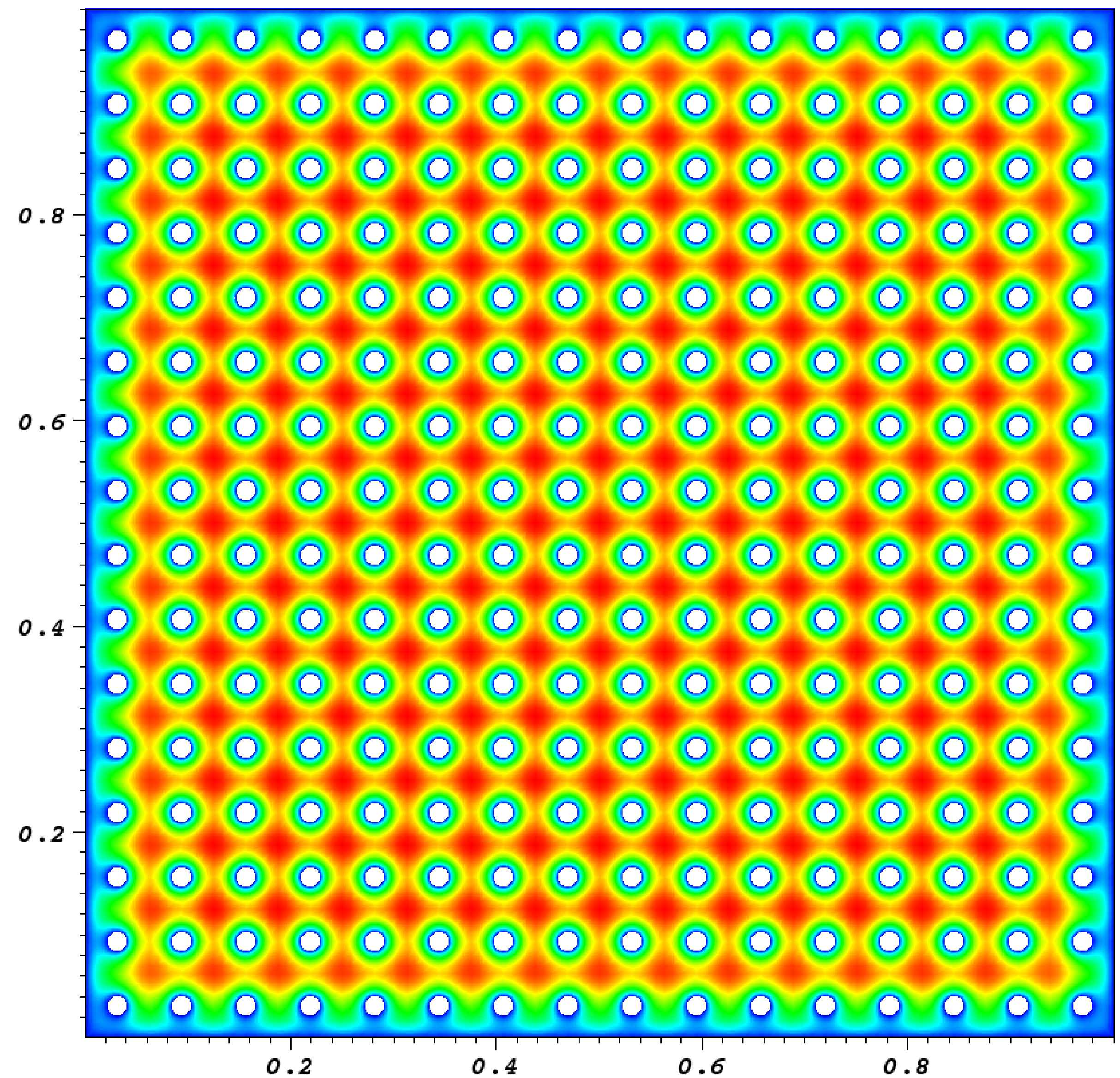
and the data oscillation term $\mathcal{O}_\kappa$ is given by

$$\mathcal{O}_\kappa = h_\kappa^2 p_\kappa^{-2} \|f - \Pi f\|_{L_2(\kappa)}^2.$$

2D Example

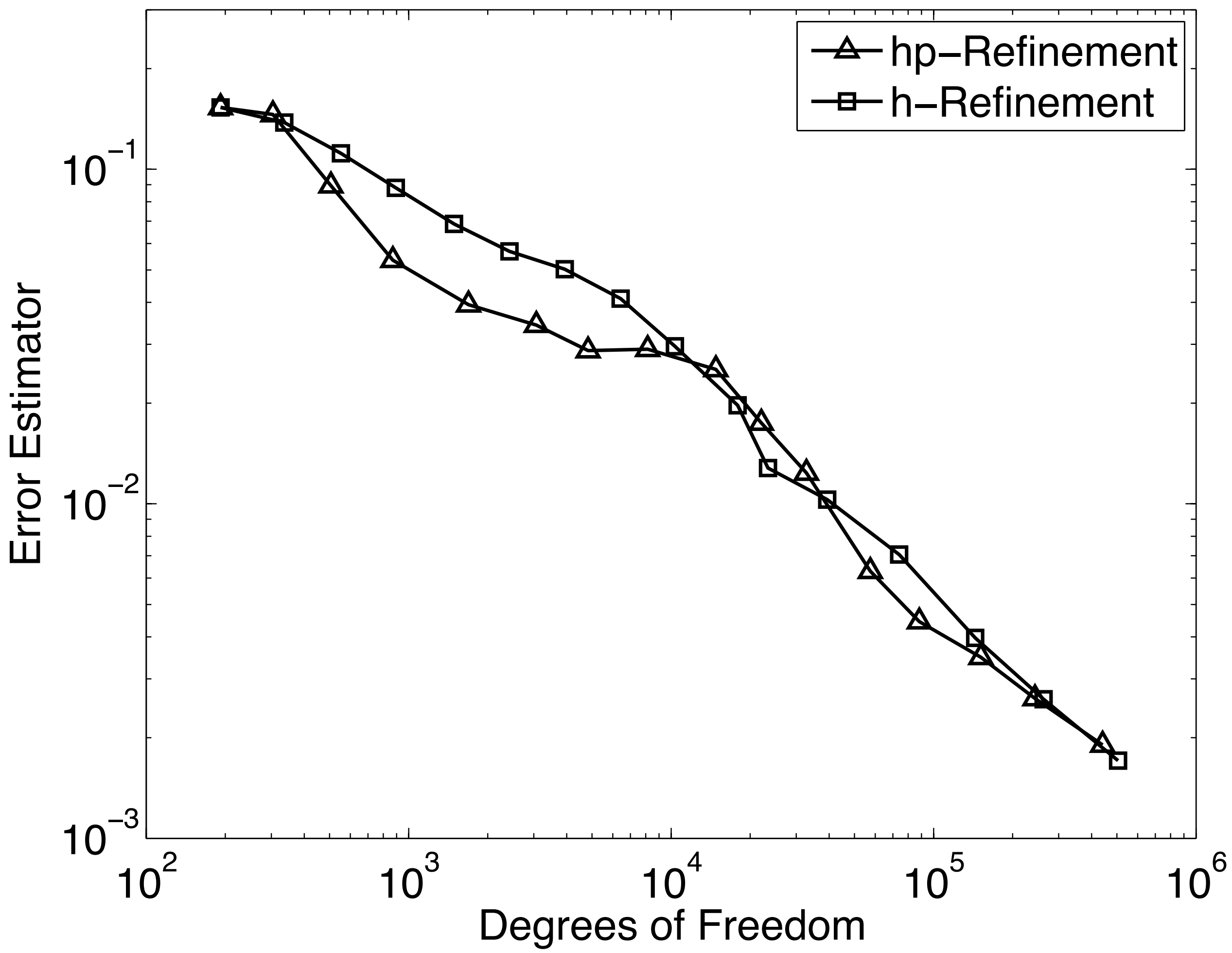
$$-\Delta u = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

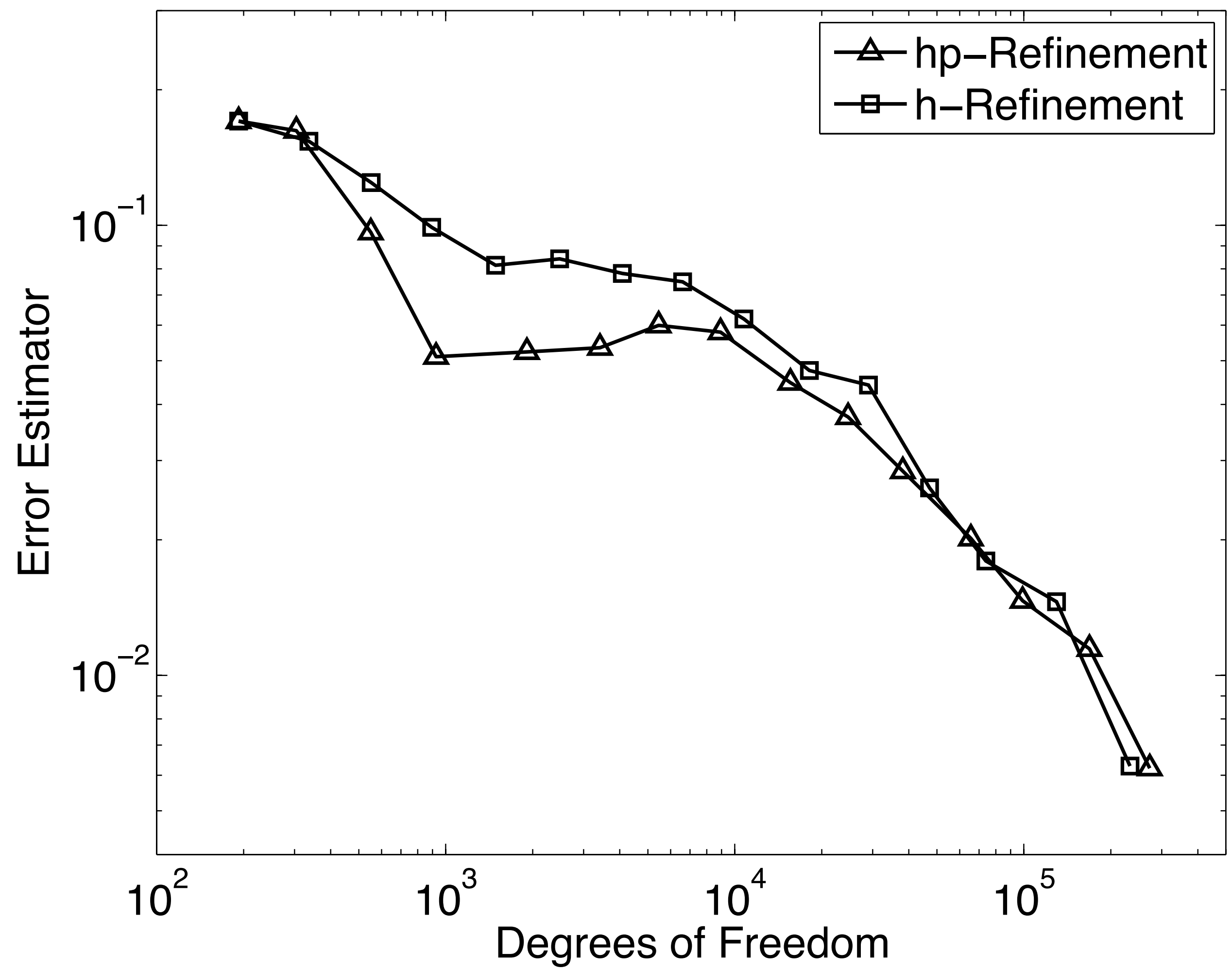




2D Example

$$-\Delta u = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$





Domain Decomposition Preconditioners

Goal

$$hp\text{–DGFEM} \longrightarrow Au = f$$

A is a large sparse, s.p.d. and ill-conditioned $\kappa(A) = \mathcal{O}(p^4 h^{-2})$

- Efficiently solve the algebraic linear system arising from the *hp*-DGFEM.
- Solver should be effective for both *h*- and *p*-version.

Domain Decomposition Preconditioning

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$$hp\text{–DGFEM} \longrightarrow Au = f$$

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- Efficiently solve the algebraic linear system arising from the hp-DGFEM.
- Solver should be effective for both *h*- and *p*-version.

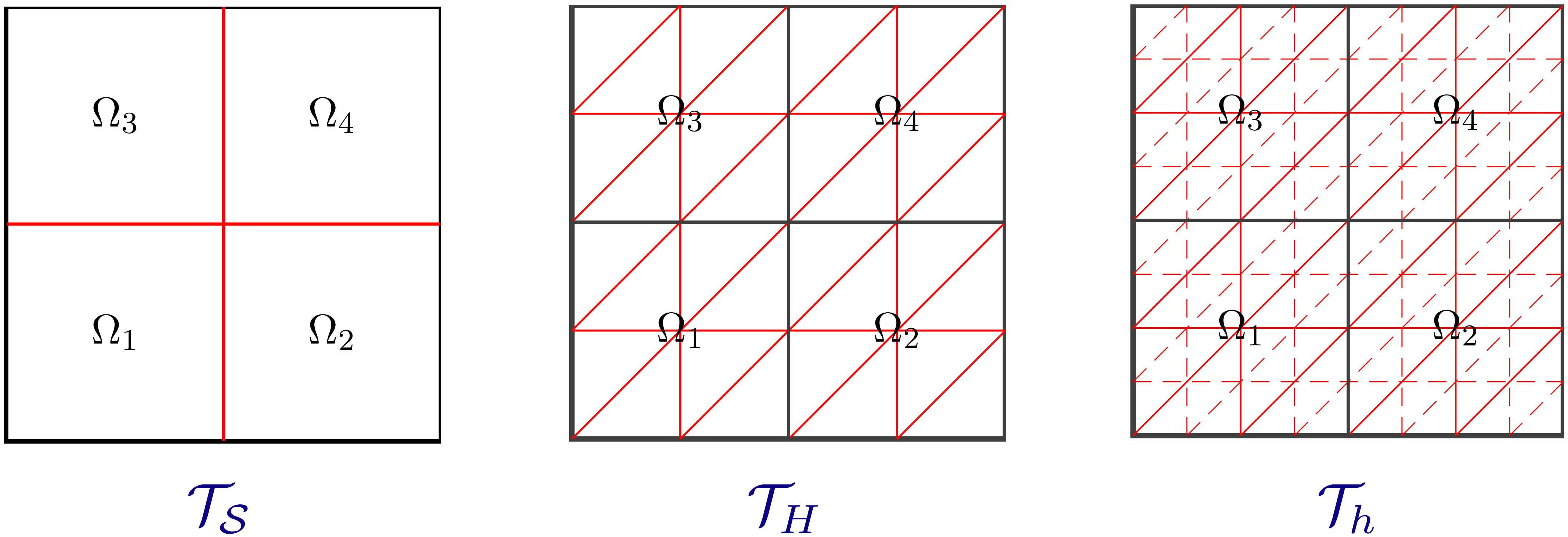
Domain Decomposition

$\Rightarrow$ Solve the PDE on $\Omega = \cup_{i=1}^{N} \Omega_i$.

$\Rightarrow$ Solve a series of local problems on each subdomain $\Omega_i, i = 1, \ldots, N$.

- Divide and Conquer: capability to treat large-scale problems.
- Parallelization: Local problems can be run on different processors.

Notation



- $\mathcal{T}_{\mathcal{S}} = \{\Omega_i\}_{i=1}^{N}$: Non-overlapping subdomain partition.
- $\mathcal{T}_h$: Fine mesh.
- $\mathcal{T}_H \equiv \mathcal{T}_{\texttt{CFE}}$: Coarse (agglomerated) mesh.

Assumption

$$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_H \subseteq \mathcal{T}_h$$

Schwarz Preconditioners for *hp*-DGFEM

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Coarse Solver (DGFEM)

$$B_{\mathtt{DG}_0}(u_0, v_0) := B_{\mathtt{CDG}}(u_0, v_0) \qquad \forall u_0, v_0 \in V(\mathcal{T}_H, q).$$

Local Solvers, *i*=1,…,*N*

Prolongation (injection) operator $R_i^\top : V(\mathcal{T}_{h_i}, p) \to V(\mathcal{T}_h, p)$, where

$$\begin{aligned} V(\mathcal{T}_{h_i}, p) &= \{v \in L_2(\Omega_i) : v|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa) \quad \forall \kappa \subset \Omega_i\}, \\ B_{\mathtt{DG}_i}(u_i, v_i) &:= B_{\mathtt{DG}}(R_i^\top u_i, R_i^\top v_i) \qquad \forall u_i, v_i \in V(\mathcal{T}_{h_i}, p). \end{aligned}$$

Local Projection Operators

$\widetilde{P}_i : V(\mathcal{T}_h, p) \to V(\mathcal{T}_{h_i}, p)$:

$$B_{\mathtt{DG}_i}(\widetilde{P}_i u, v_i) := B_{\mathtt{DG}}(u, R_i^\top v_i) \quad \forall v_i \in V(\mathcal{T}_{h_i}, p).$$

$\widetilde{P}_0 : V(\mathcal{T}_h, p) \to V(\mathcal{T}_H, q)$:

$$B_{\mathtt{DG}_0}(\widetilde{P}_0 u, v_0) := B_{\mathtt{DG}}(u, R_0^\top v_0) \quad \forall v_0 \in V(\mathcal{T}_H, q).$$

Schwarz Operators

Writing $P_i := R_i^\top \widetilde{P}_i : V(\mathcal{T}_h, p) \to V(\mathcal{T}_h, p)$, for $i = 0, 1, \ldots, N$, we have

$$P_{\text{ad}} := \sum_{i=0}^{N} P_i, \quad P_{\text{mu}} := I - (I - P_N)(I - P_{N-1}) \cdots (I - P_0).$$

Algebraic Formulation for Additive Schwarz

$$\begin{aligned} \tilde{P}_i &= A_i^{-1} R_i A, \\ P_i &:= R_i^\top \tilde{P}_i = R_i^\top A_i^{-1} R_i A, \\ P_{\text{ad}} &= \left(\sum_{i=0}^{N} R_i^\top A_i^{-1} R_i \right) A. \end{aligned}$$

- A: Full DGFEM matrix.
- A_i, $i > 1$: Local DGFEM matrix on Ω_i.
- A_0: Composite DGFEM matrix.
- $R_i : V(\mathcal{T}_h, p) \to V(\mathcal{T}_{h_i}, p)$: Restriction.
- $R_i^\top : V(\mathcal{T}_{h_i}, p) \to V(\mathcal{T}_h, p)$: Prolongation.
- P_{ad}: Preconditioned system.

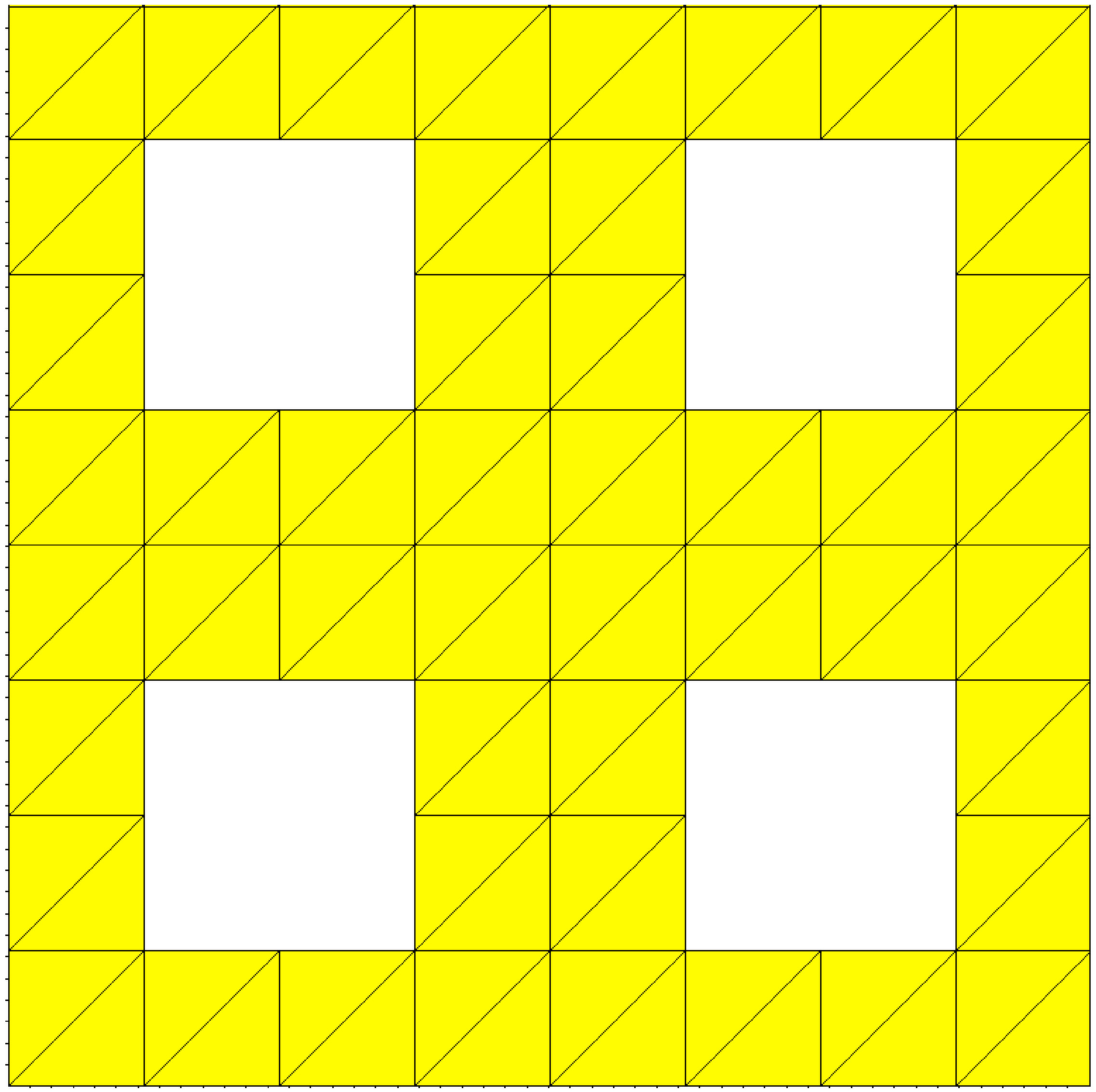
Theorem (Antonietti & H. 2011, Antonietti, Giani, & H. 2013)

The condition number $\kappa(P_{\text{ad}})$ is bounded by:

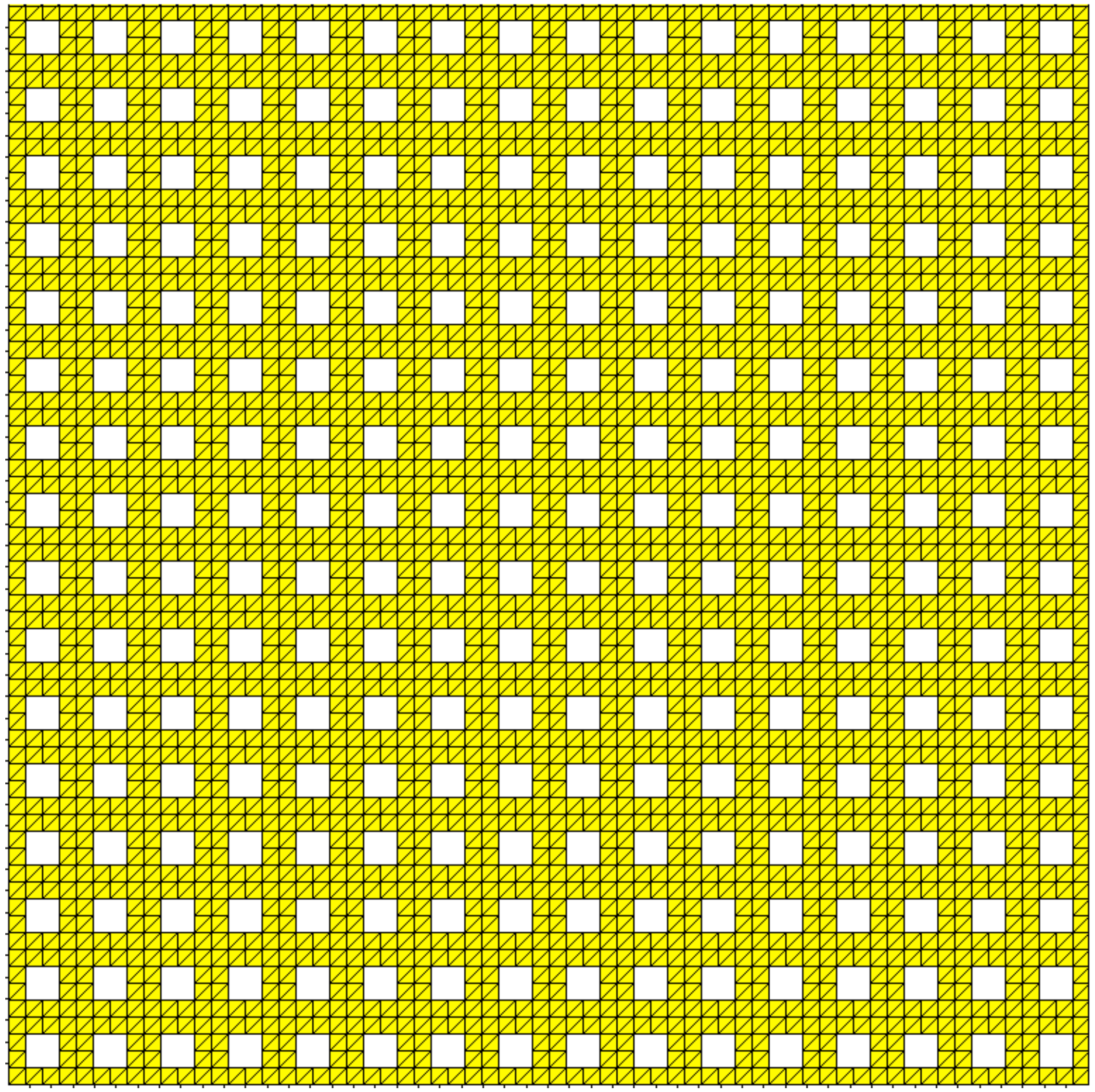
$$\kappa(P_{\text{ad}}) \leq C\gamma p^2 \frac{H}{h}.$$

- Proof is based on the abstract theory of Schwarz methods, cf. Dryja & Widlund, 1989, 1990, and standard arguments for *hp*-DGFEMs.
- Scalability (i.e., independent of the number of subdomains).
- Note: ***No overlap*** is required unlike with CGFEM
- Dependence of the condition number on the coarse space polynomial degree may be established based on the article by Smears 2013.

Poisson's Equation



Domain with 4 holes



Domain with 256 holes

Poisson's Equation

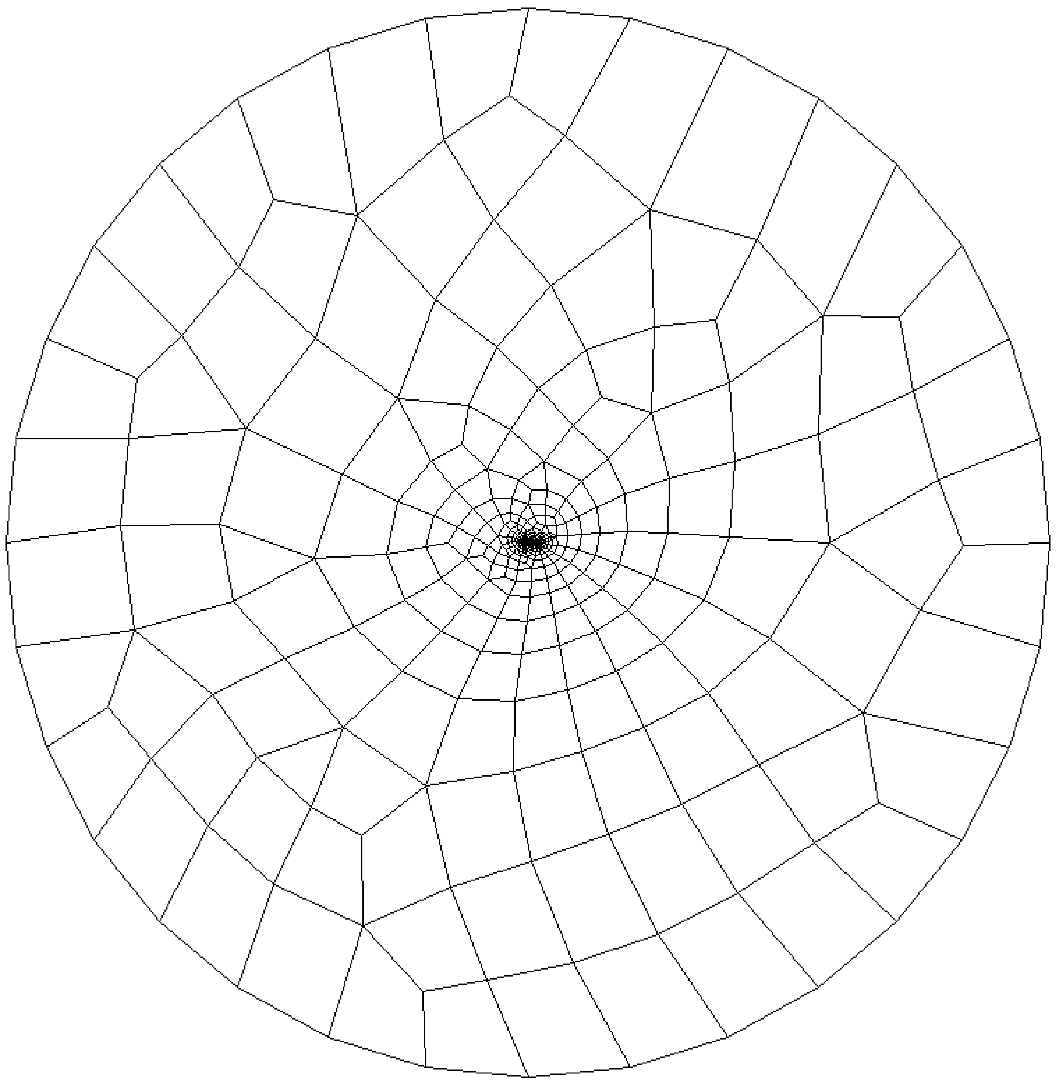
Domain with 4 holes

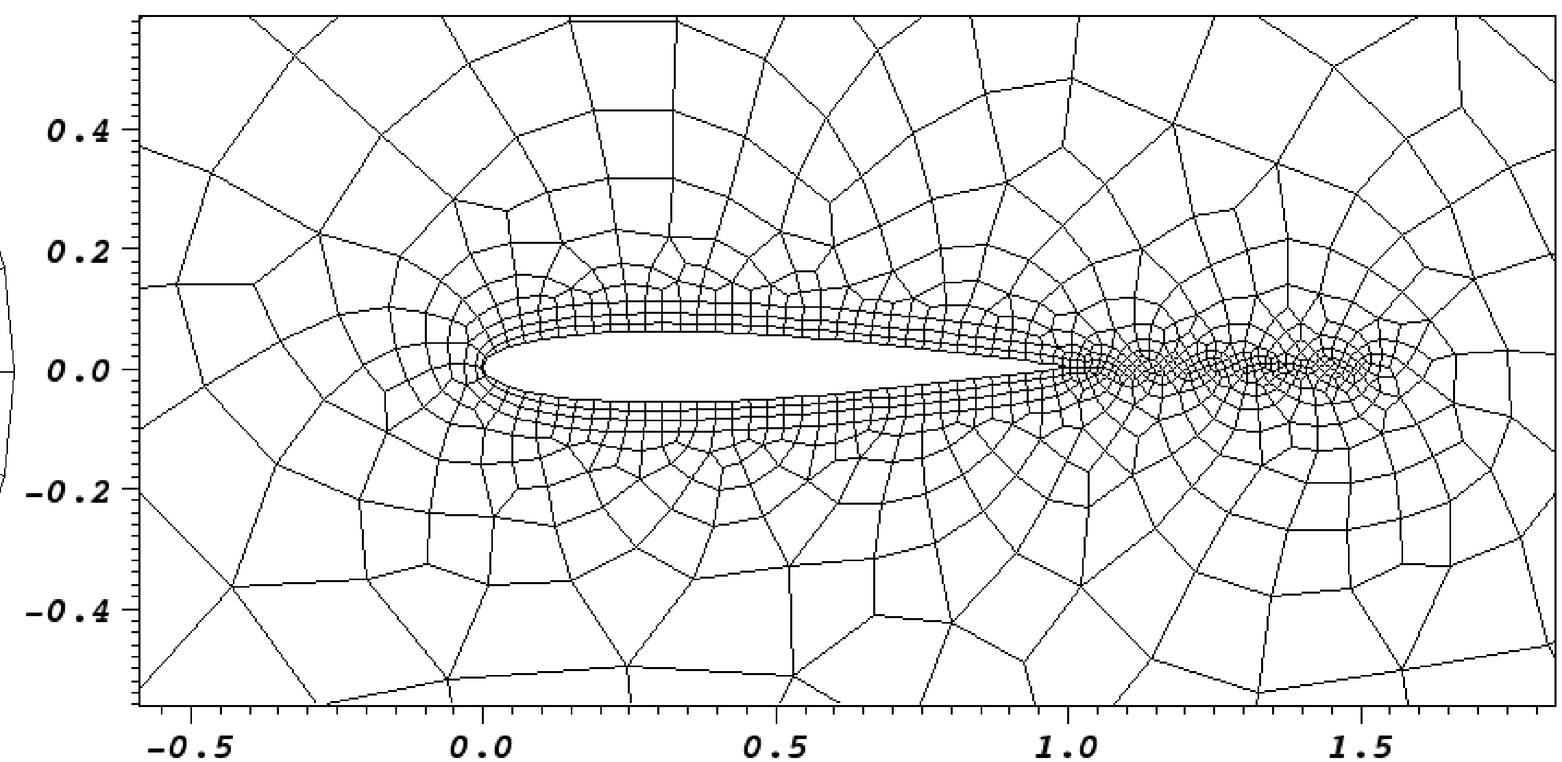
$h\backslash H$	1/2	1/4	1/8	1/16	1/32	1/64
1/8	32 (42.1)	**27 (14.5)**	-	-	-	-
1/16	**58 (96.8)**	47 (40.1)	29 (17.5)	-	-	-
1/32	**93 (203.2)**	**74 (89.8)**	48 (44.1)	31 (17.8)	-	-
1/64	**134 (411.2)**	**121 (188.3)**	80 (95.4)	50 (44.2)	31 (17.9)	-
1/128	**192 (821.9)**	**185 (369.8)**	137 (194.3)	80 (95.2)	50 (44.2)	31(17.9)

Domain with 256 holes

$h\backslash H$	1/2	1/4	1/8	1/16	1/32	1/64
1/64	**55 (83.8)**	**55 (81.3)**	**54 (69.2)**	50 (40.4)	**31 (14.7)**	-
1/128	**79 (178.6)**	**79 (174.5)**	**79 (151.4)**	**76 (93.2)**	52 (38.2)	31 (17.6)

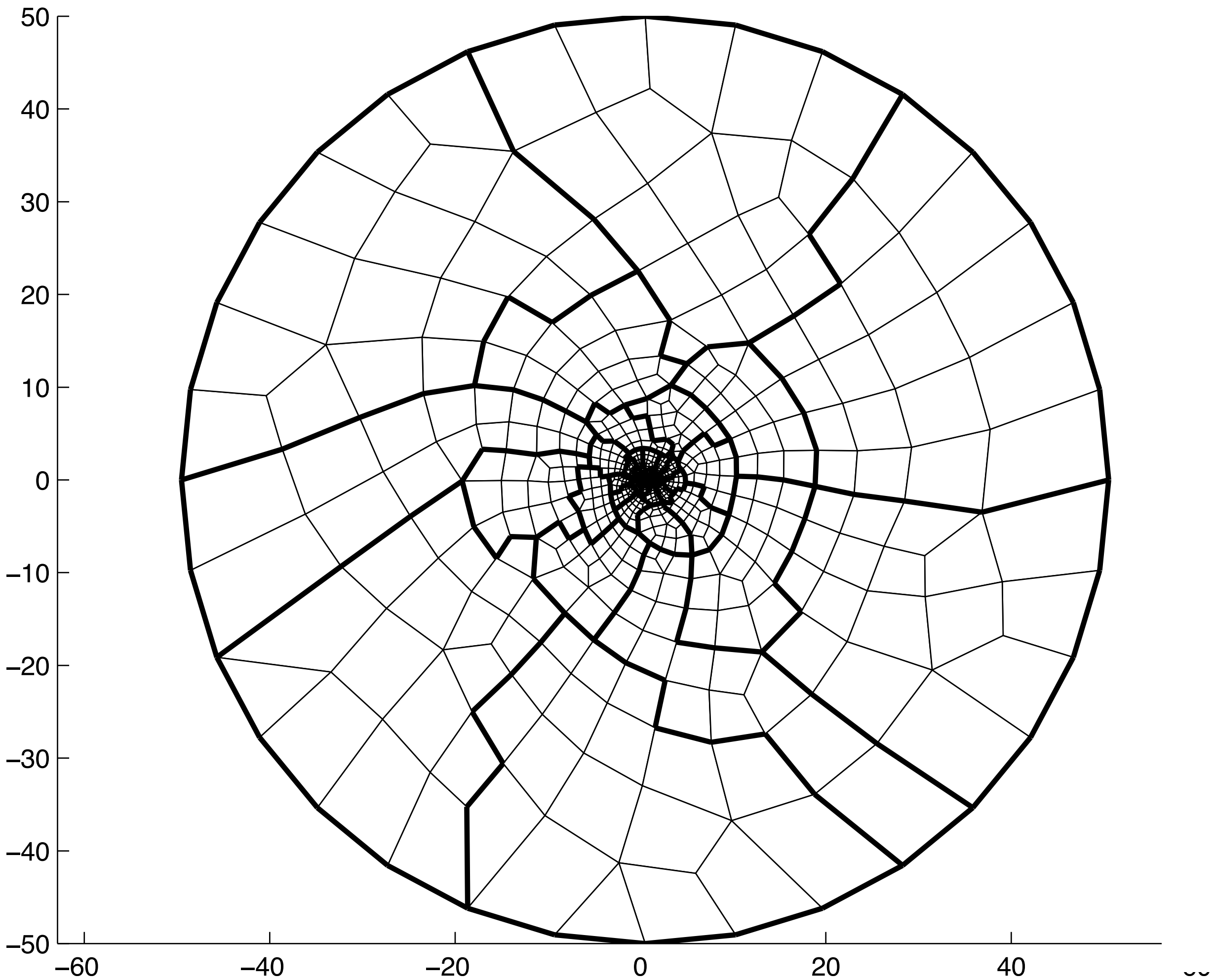
Ma=0.5, Re=5000, $\alpha = 2^\circ$ and adiabatic wall condition

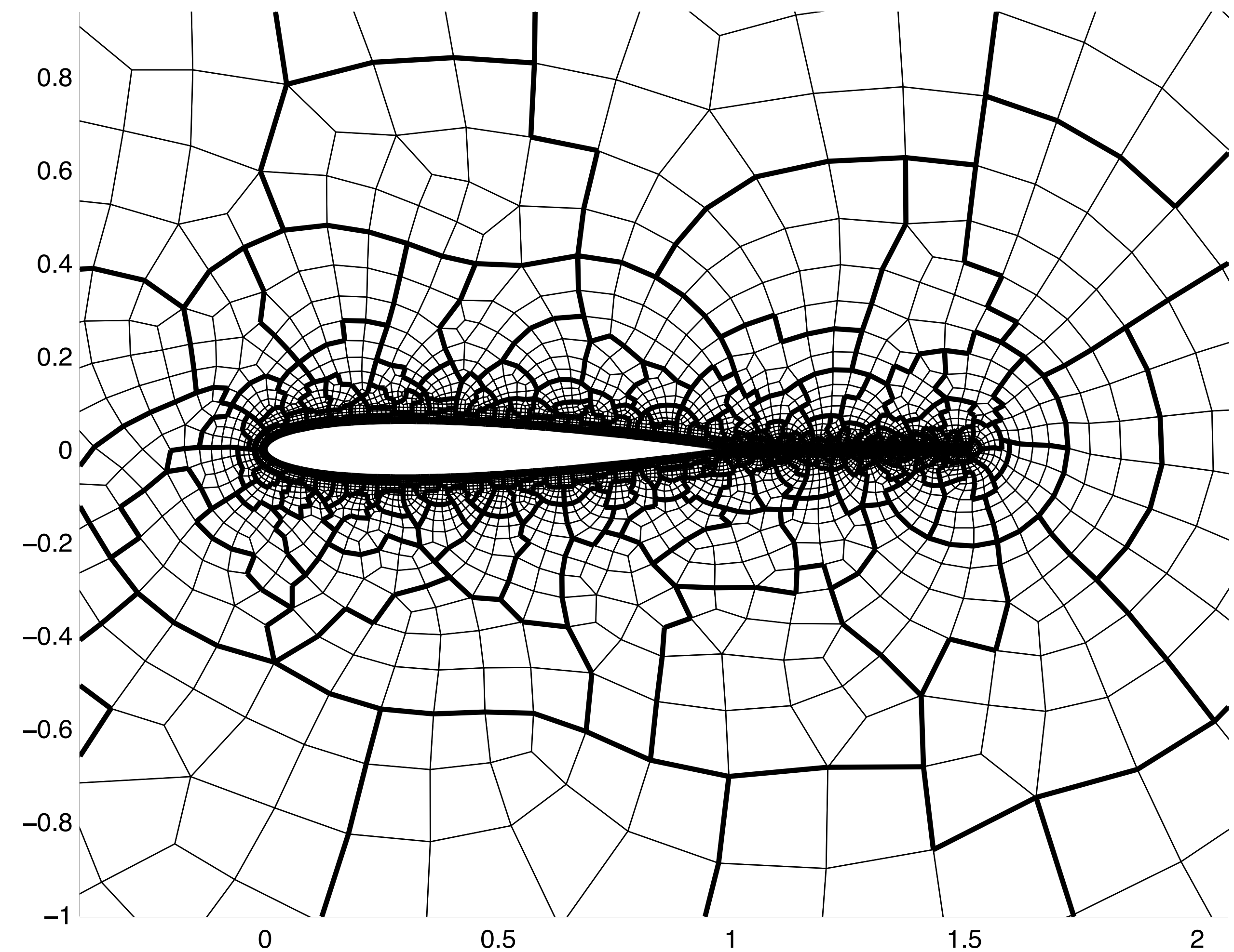




Mesh 1, consisting of 578 (hybrid) elements

METIS is employed to generate both $\mathcal{T}_S$ with $N = 250$ and $\mathcal{T}_H$.





Mesh 5 partitioned into 500 regions using METIS

2D Laminar Flow: NACA0012 Airfoil

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Ma=0.5, Re=5000, $\alpha = 2^\circ$ and adiabatic wall condition

$\mathcal{T}_h \setminus$ # Eles $\mathcal{T}_H$	**500**	**1000**	**2000**	**4000**	**8000**
Mesh 2	**124 (936,10)**	-	-	-	-
Mesh 3	186 (1303,9)	**121 (800,9)**	-	-	-
Mesh 4	310 (1957,9)	168 (1150,9)	**116 (700,9)**	-	-
Mesh 5	519 (3136,9)	278 (1796,9)	151 (1034,9)	**95 (646,9)**	-
Mesh 6	933 (5604,9)	492 (3034,9)	276 (1785,9)	162 (1090,9)	**103 (687,9)**

METIS is employed to generate both $\mathcal{T}_S$ with $N = 250$ and $\mathcal{T}_H$.

Meshes 2-6: 1134, 2113, 4246, 8946, 20229 elements, respectively.

Summary and Outlook

Summary and Outlook

- Developed the *a priori* and *a posteriori* error analysis of DGFEMs on general polytopic meshes.
- This allows for the construction of very coarse finite element meshes, even on complicated domains containing microstructures.
- Analysis of DGFEMs on general polygonal/polyhedral meshes accounts for local edge/face degeneration.
- Exploitation as coarse grid solvers for Schwarz type DD preconditioners.
- Development of multigrid preconditioners.
- Extension to problems with discontinuous coefficients.
- Application to two-grid methods for nonlinear PDEs.

  Congreve, H., & Wihler 2011, 2013, Congreve & H. 2013
- Extension to hyperbolic problems and PDEs of mixed-type.

  See Zhaonan Dong's (Peter) poster.
- Agglomeration-based adaptivity, based on exploiting METIS.