Isogeometric analysis for nearly incompressible materials

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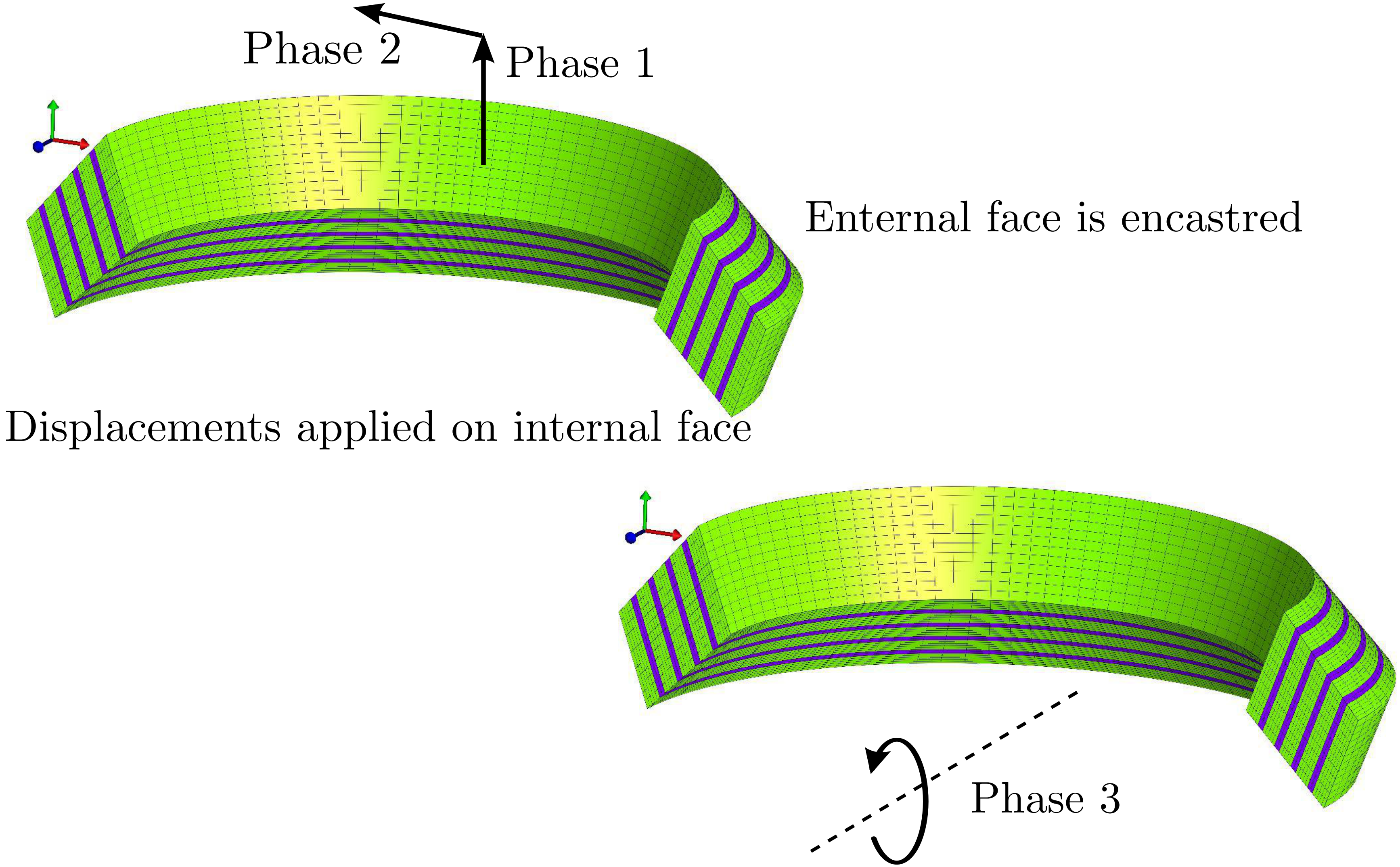
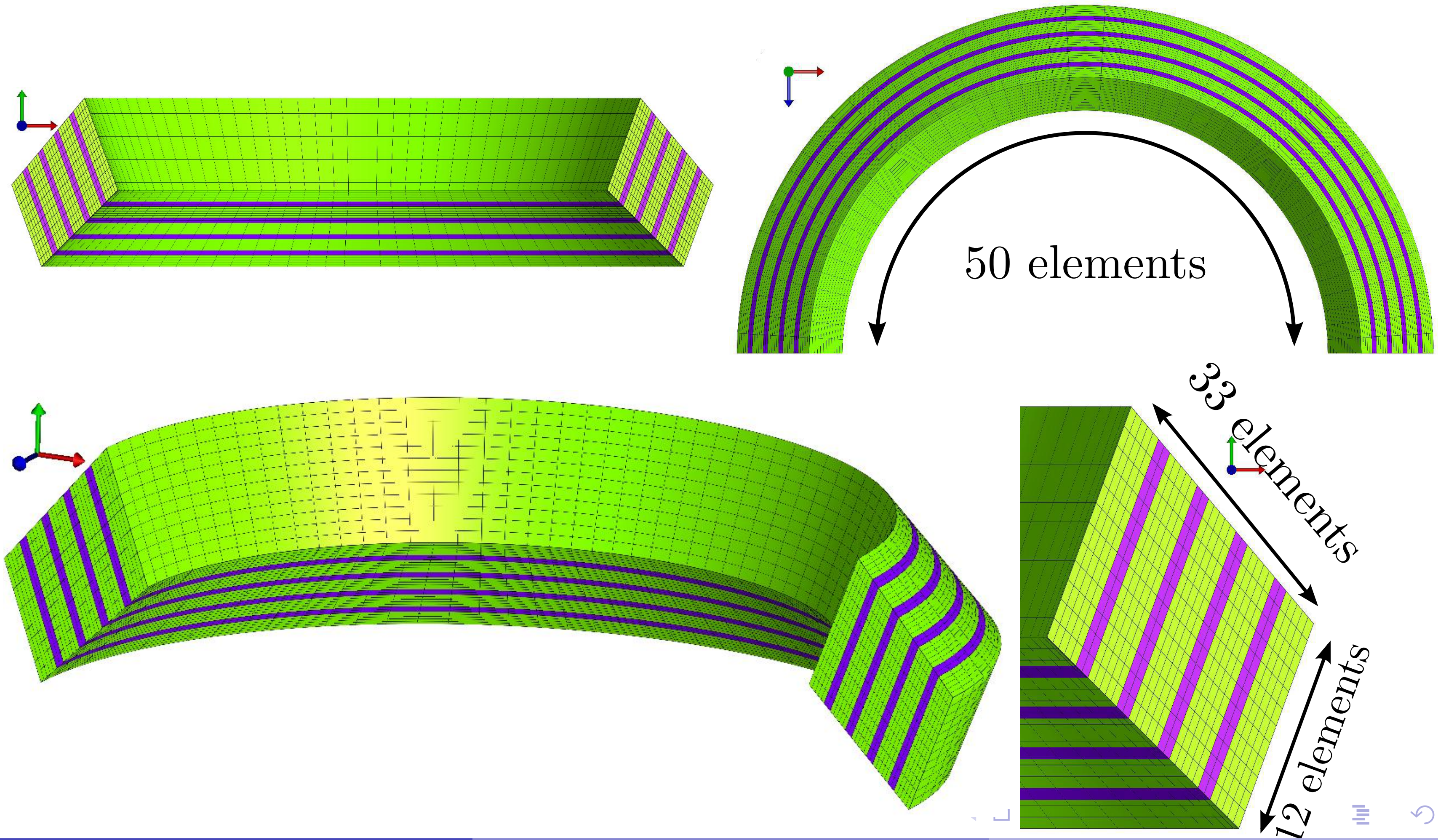
Rubber-metal conical bearing



Rubber-metal conical bearing

19800 elements, 22542 nodes $\rightarrow$ 67626 dofs

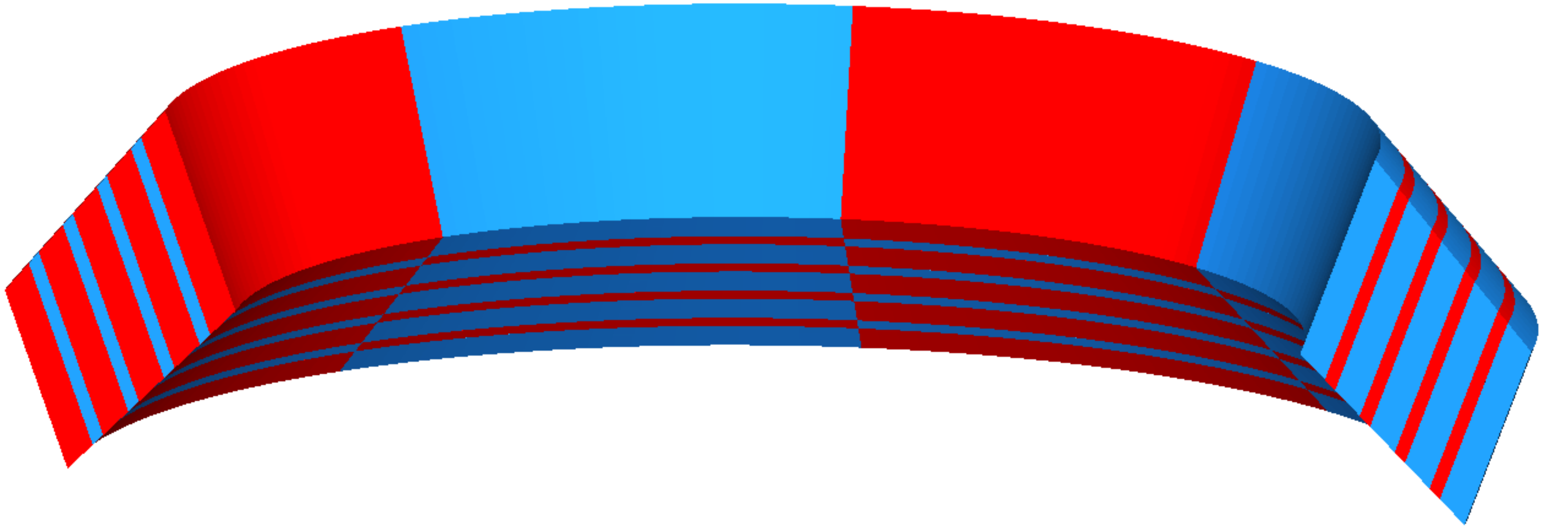
Symmetry conditions imposed



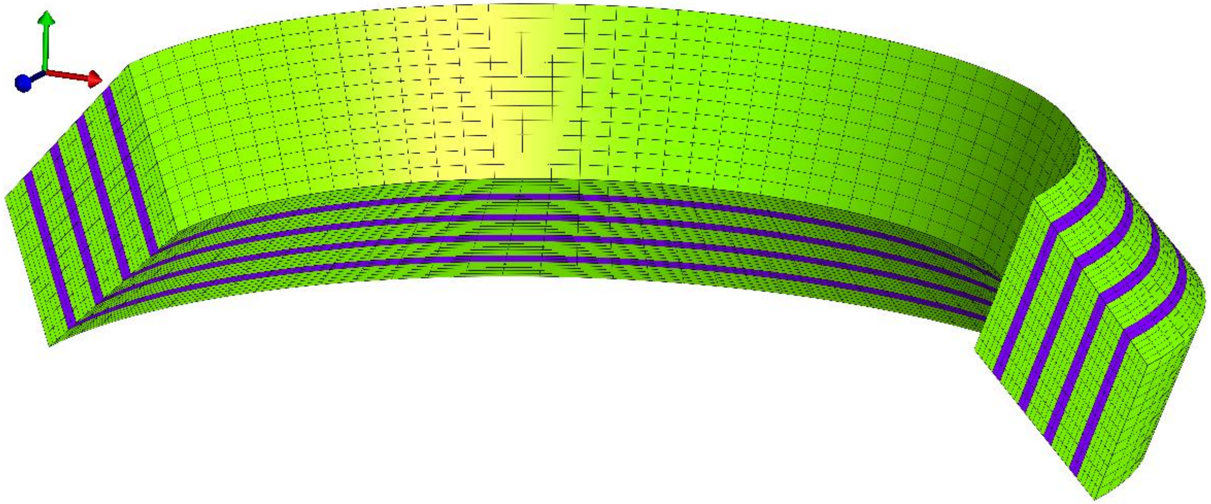
Rubber-metal conical bearing

36 elements $(4 \times 1 \times 9)$, $\begin{cases} p = 2 \rightarrow & 567\,\text{nodes}, \quad 1701\,\text{dofs} \\ p = 3 \rightarrow & 1296\,\text{nodes}, \quad 3888\,\text{dofs} \end{cases}$

Each layer is a different NURBS patch



Conical bearing: FEM vs IGA mesh



19800 elements
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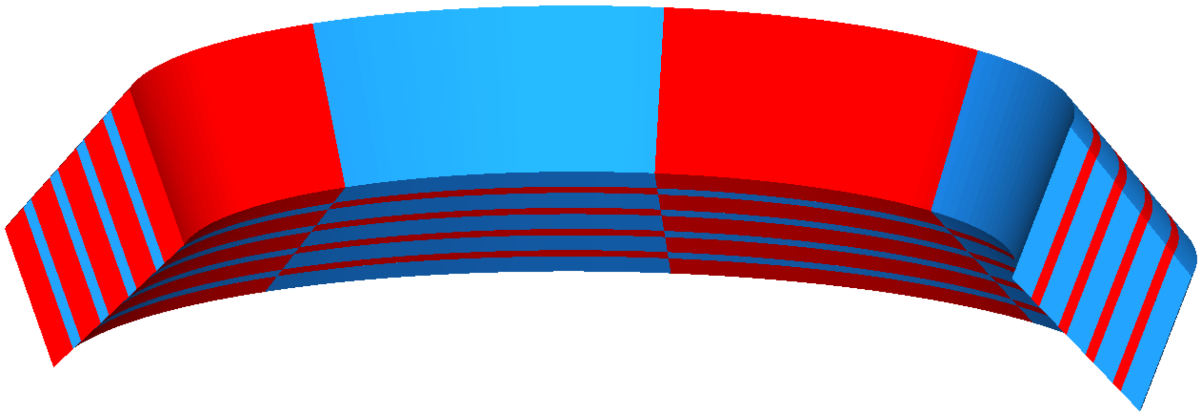


36 elements

$p = 2 \rightarrow$ 1701 dofs
$p = 3 \rightarrow$ 3888 dofs

Materials and numerical methods

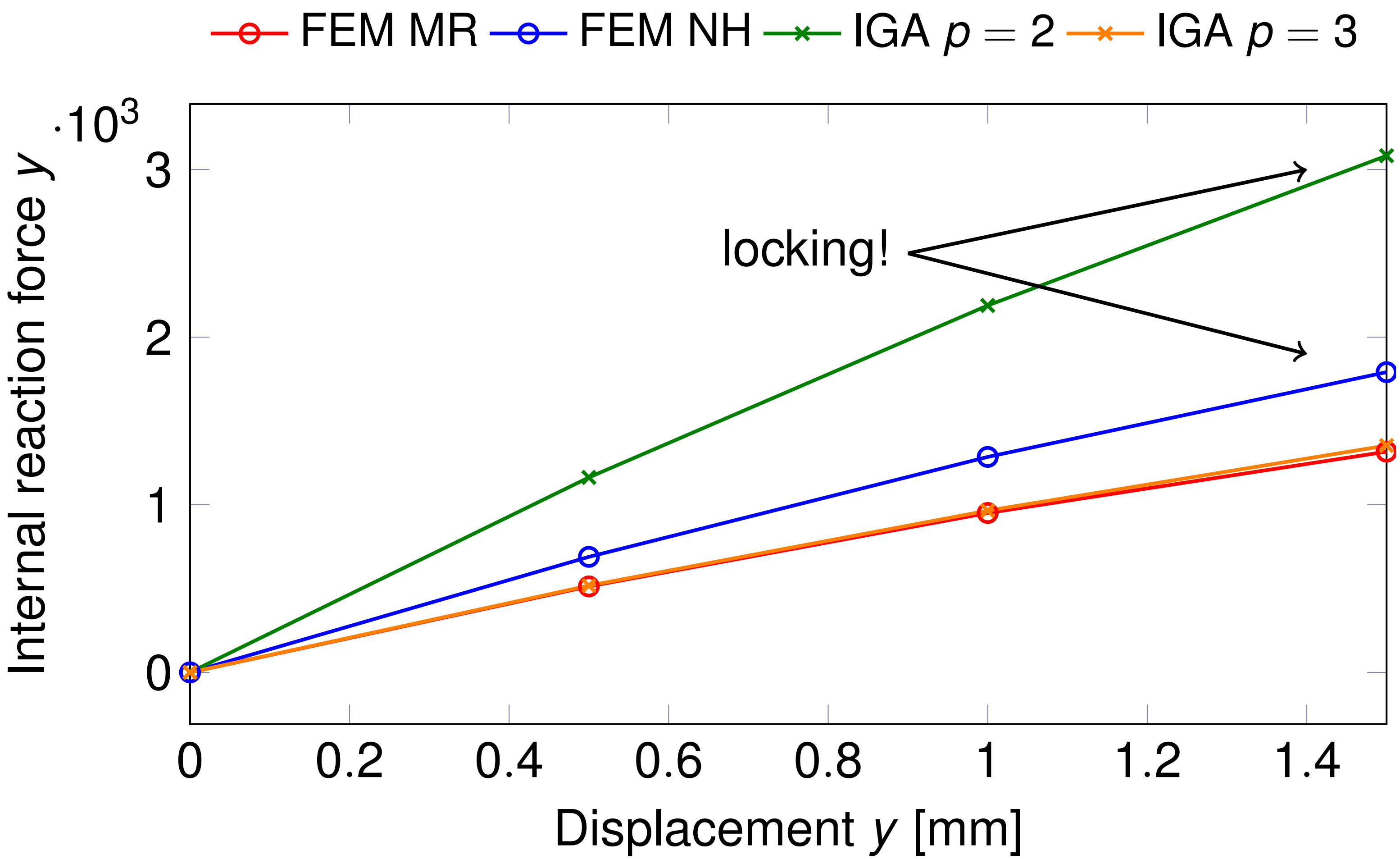
Steel:
$E = 210\,\text{GPa}$, $\nu = 0.3$

- Isotropic elasticity formulation used.

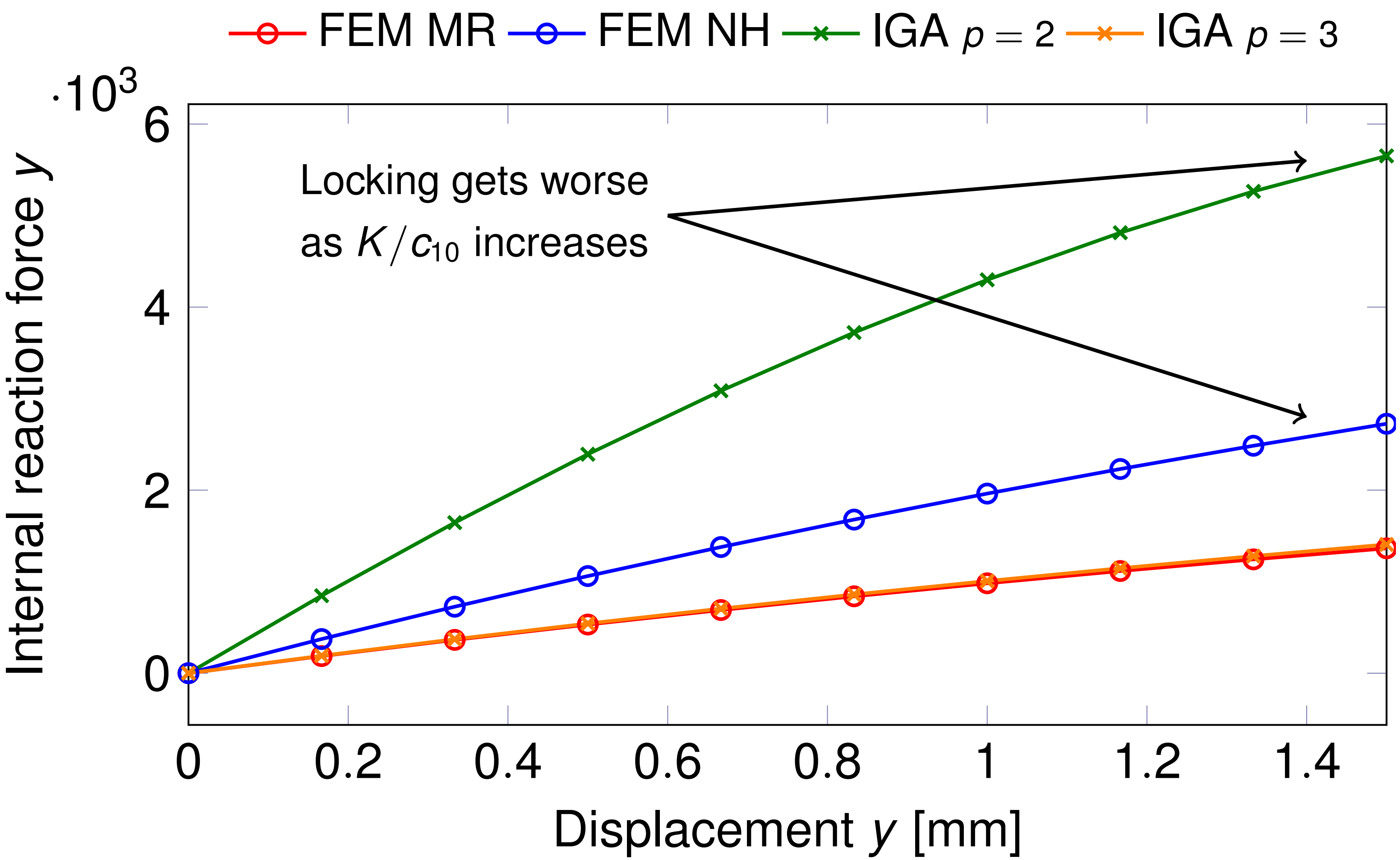
Rubber:
$C_{10} = 1\,\text{MPa}$, $K = 1000\,\text{MPa}$ ($E = 5.996\,\text{MPa}$, $\nu = 0.499$)

- For FEM:
  - Neo–Hokean with plain Galerkin formulation $\rightarrow$ solution locks
  - Mooney–Rivlin (three field: displacement+pressure+volume ratio) implemented with “selective-reduced integration” $\rightarrow$ suitable for incompressible materials.
- For IGA: Neo–Hokean with plain Galerkin formulation.

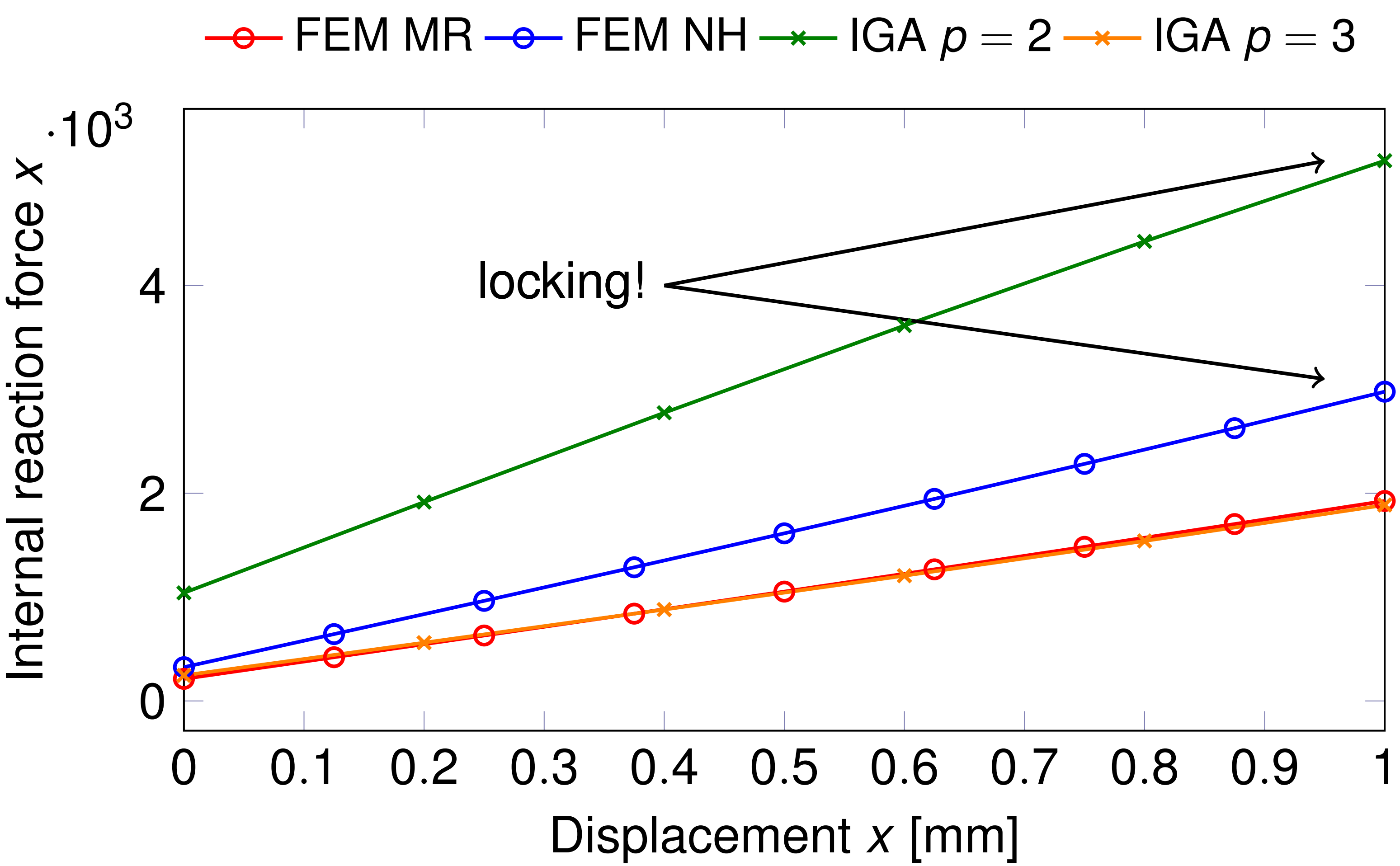
Conical bearing: phase 1



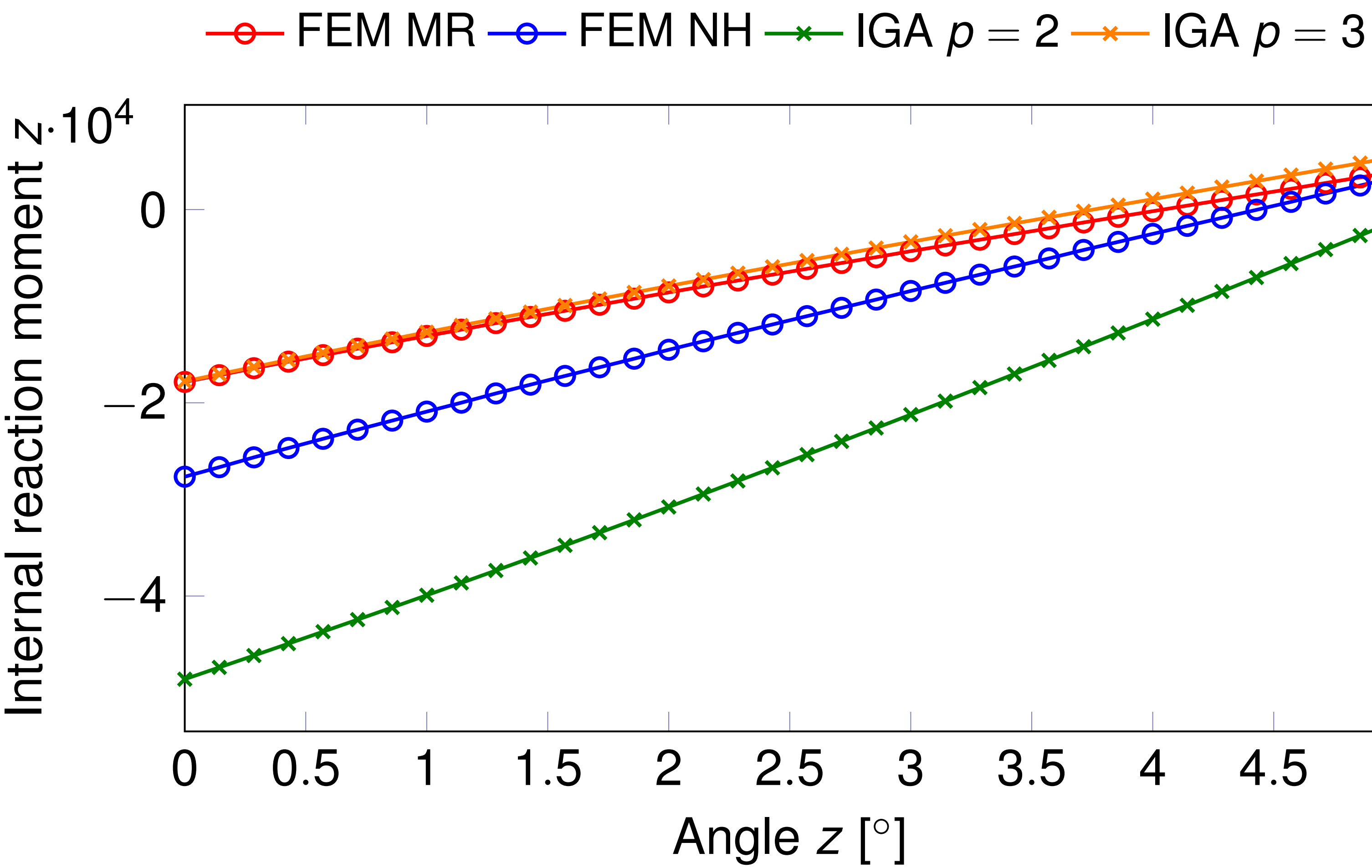
Conical bearing: $K/c_{10} = 3000$. Phase 1



Conical bearing: Phase 2

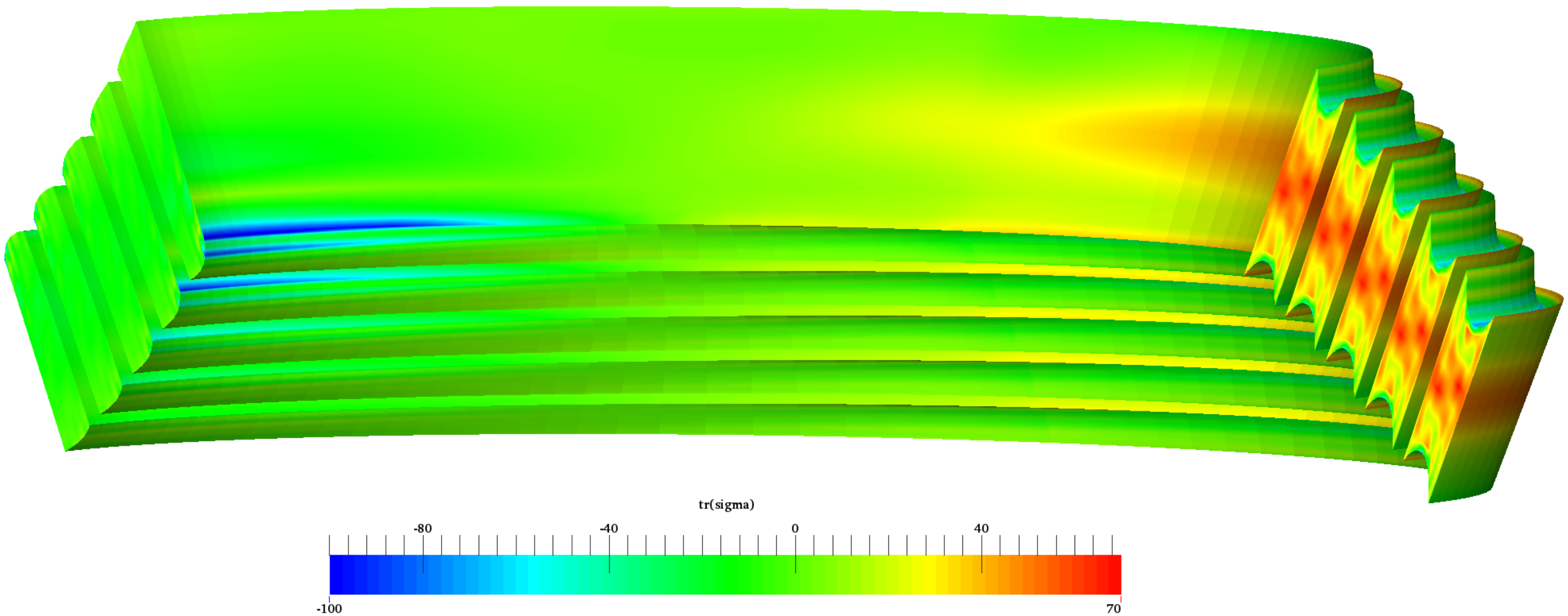


Conical bearing: Phase 3



Conical bearing: stress oscillations for IGA $p = 3$

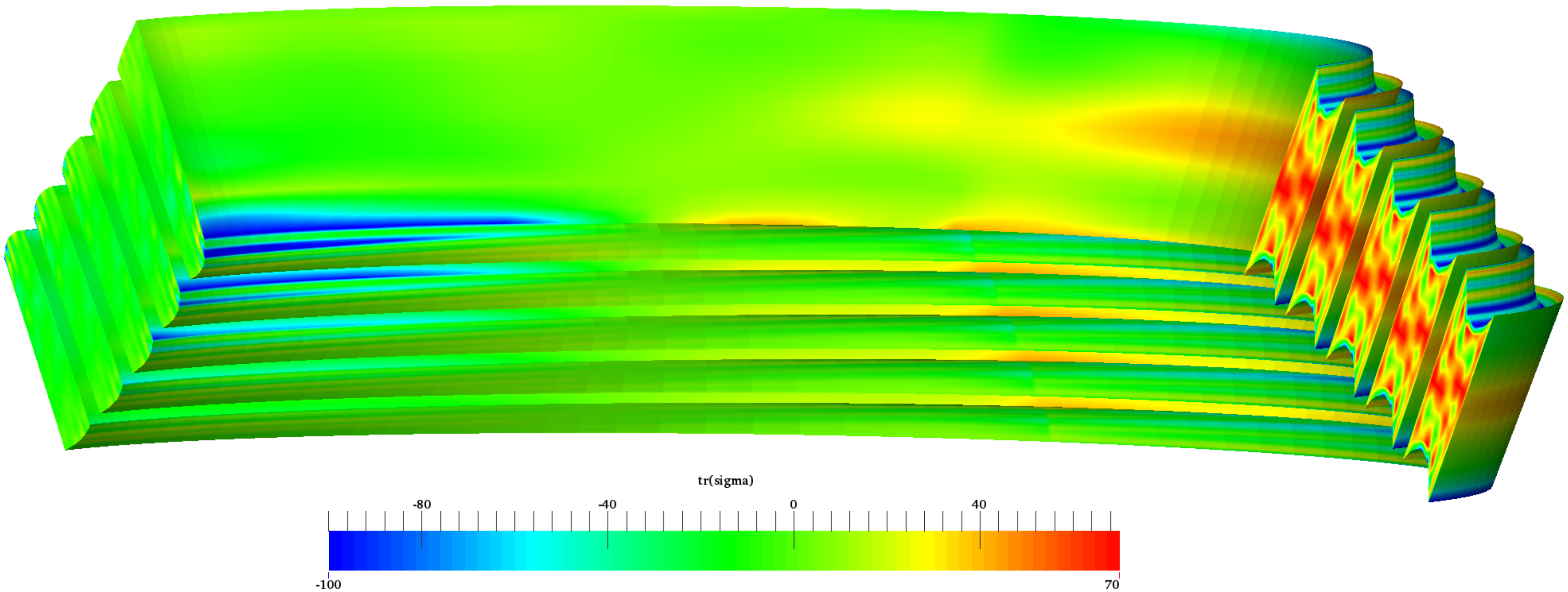
Oscillations appear in $\sigma^{\text{vol}} = \frac{1}{3}\,\text{tr}(\sigma)\,\mathbf{1}$



$\frac{1}{3}\,\text{tr}(\sigma)$ plotted

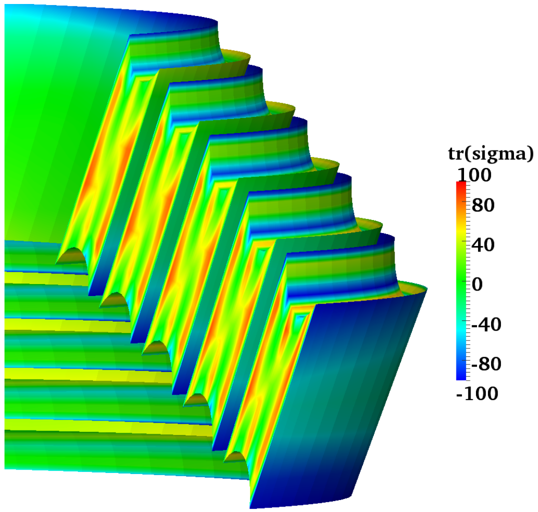
Conical bearing: stress oscillations for IGA $p = 3$

$$\mathbf{K/c_{10} = 3000}$$

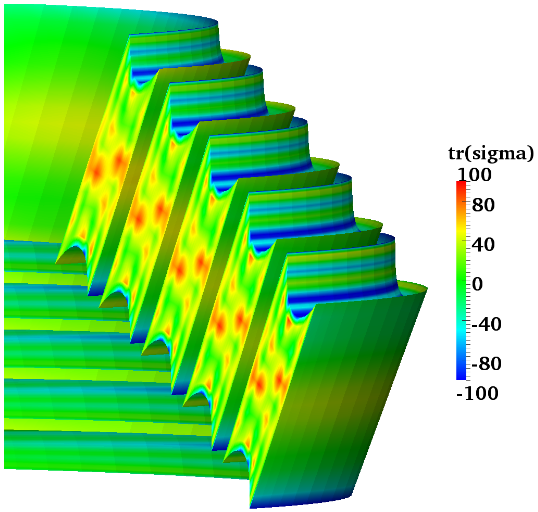


The scale is the same, the oscillations increase with K/c_{10}

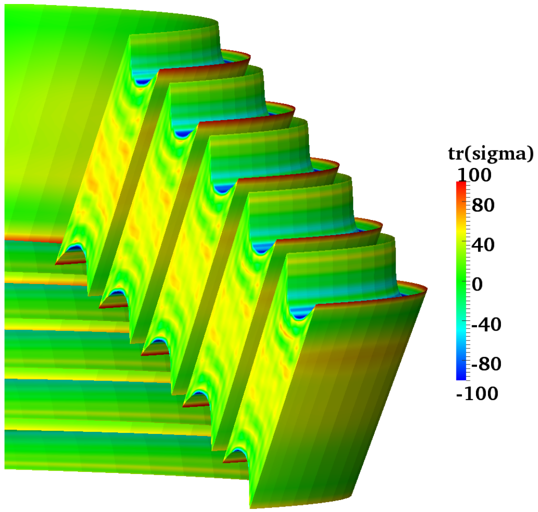
Conical bearing: stress oscillations for IGA $p = 3$



$4 \times 2 \times 2$
elements / layer



$4 \times 4 \times 4$
elements / layer



$4 \times 8 \times 8$
elements / layer

Linear elasticity model problem

Strong form problem

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} &= \boldsymbol{0} \quad \text{in}\, \Omega \\
\boldsymbol{u} &= \bar{\boldsymbol{u}} \quad \text{on}\, \Gamma_D \\
\boldsymbol{\sigma} \cdot \boldsymbol{n} &= \boldsymbol{t} \quad \text{on}\, \Gamma_N
\end{aligned}
$$

Linear elasticity model problem

Strong form problem

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \quad & \text{in } \Omega \\ \boldsymbol{u} = \bar{\boldsymbol{u}} \quad & \text{on } \Gamma_D \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} \quad & \text{on } \Gamma_N \end{aligned}$$

Isotropic linear elasticity

$$\begin{aligned} \boldsymbol{\sigma} &= 2\,\mu\,\varepsilon + \lambda\,\nabla \cdot \boldsymbol{u}\,\mathbf{1} \\ \varepsilon &= \nabla^s \boldsymbol{u} \\ \lambda &= \frac{\nu\,E}{(1+\nu)\,(1-2\,\nu)} \\ \mu &= \frac{E}{2\,(1+\nu)} \end{aligned}$$

$$\nu \to 1/2, \quad \lambda \to \infty$$

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\end{aligned}$$

Weak form: find $\boldsymbol{u} \in (H^1(\Omega))^3$ with $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_D and such that

$$\int_\Omega \nabla^s\boldsymbol{w} : \boldsymbol{\sigma}\,\mathrm{d}\Omega = \int_\Omega \boldsymbol{w}\cdot\boldsymbol{f}\,\mathrm{d}\Omega + \int_{\Gamma_N} \boldsymbol{w}\cdot\boldsymbol{t}\,\mathrm{d}\Gamma, \quad \forall\boldsymbol{w} \in (H^1_{\Gamma_D}(\Omega))^3$$

Linear elasticity model problem

Isotropic linear elasticity

$$\boldsymbol{\sigma} = 2\,\mu\,\boldsymbol{\varepsilon} + \lambda\,\nabla\cdot\boldsymbol{u}\,\mathbf{1}$$
$$\boldsymbol{\varepsilon} = \nabla^s \boldsymbol{u}$$
$$\lambda = \frac{\nu\,E}{(1+\nu)\,(1-2\,\nu)}$$
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$$\nu \to 1/2, \qquad \lambda \to \infty$$

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$$\begin{aligned} \nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} &\quad \text{in}\,\Omega \\ \boldsymbol{u} = \bar{\boldsymbol{u}} &\quad \text{on}\,\Gamma_D \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t} &\quad \text{on}\,\Gamma_N \end{aligned}$$

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$$\underbrace{\int_\Omega \nabla^s \boldsymbol{w} : \boldsymbol{\sigma}\,\mathrm{d}\Omega}_{a(\boldsymbol{w},\boldsymbol{u})} = \underbrace{\int_\Omega \boldsymbol{w}\cdot\boldsymbol{f}\,\mathrm{d}\Omega + \int_{\Gamma_N} \boldsymbol{w}\cdot\boldsymbol{t}\,\mathrm{d}\Gamma}_{L(\boldsymbol{w})}, \qquad \forall \boldsymbol{w} \in (H^1_{\Gamma_D}(\Omega))^3$$

Linear elasticity model problem

Strong form problem

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \quad & \text{in}\, \Omega \\ \boldsymbol{u} = \bar{\boldsymbol{u}} \quad & \text{on}\, \Gamma_D \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} \quad & \text{on}\, \Gamma_N \end{aligned}$$

Isotropic linear elasticity

$$\boldsymbol{\sigma} = 2\,\mu\,\varepsilon + \lambda\,\nabla \cdot \boldsymbol{u}\,\boldsymbol{1}$$

$$\varepsilon = \nabla^s \boldsymbol{u}$$

$$\lambda = \frac{\nu\,E}{(1+\nu)\,(1-2\,\nu)}$$

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Linear elasticity model problem

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$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \quad \text{in}\, \Omega$$
$$\boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{on}\, \Gamma_D$$
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Isotropic linear elasticity

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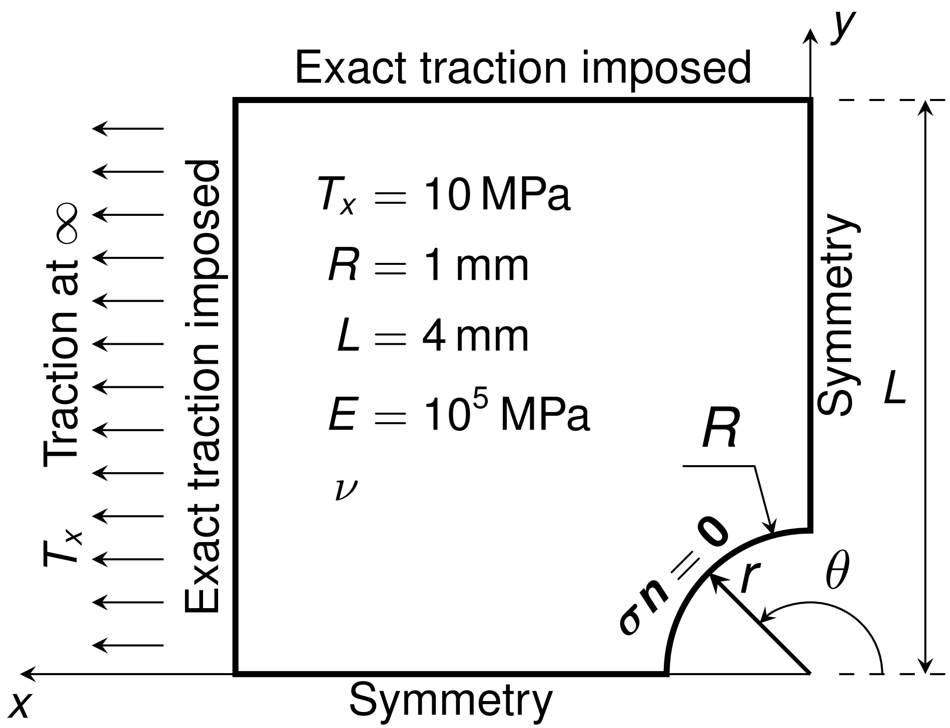
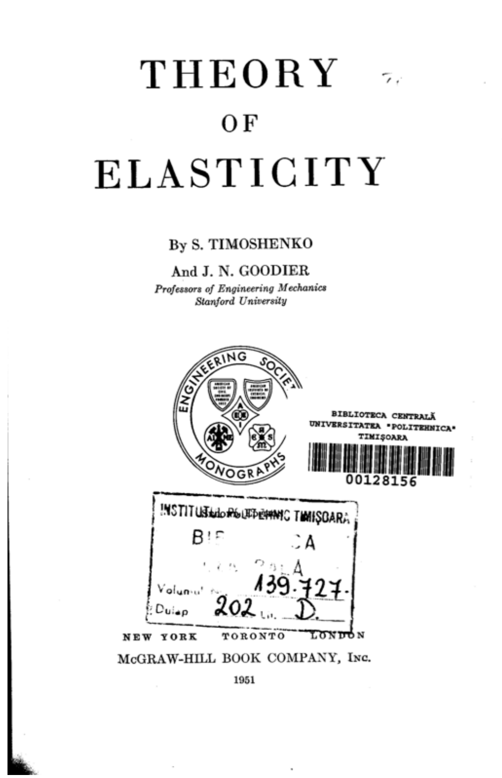
Target method

Rubber industry is interested in:

- A “locking-free” method: optimal order of convergence for the displacement and no spurious oscillations in the stress, in the range $\nu \in [0.3, 0.4999]$
- Stiffness matrix
  - Symmetric
  - Definite positive
- Efficient

Classical benchmark: plate with a hole





$$\sigma_{rr}(r,\ \theta) = \frac{T_x}{2}\left[1 - \frac{R^2}{r^2} + \left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)\cos 2\theta\right],$$

$$\sigma_{\theta\theta}(r,\ \theta) = \frac{T_x}{2}\left[1 + \frac{R^2}{r^2} - \left(1 + 3\frac{R^4}{r^4}\right)\cos 2\theta\right],$$

$$\sigma_{r\theta}(r,\ \theta) = -\frac{T_x}{2}\left(1 + 2\frac{R^2}{r^2} - 3\frac{R^4}{r^4}\right)\sin 2\theta,$$

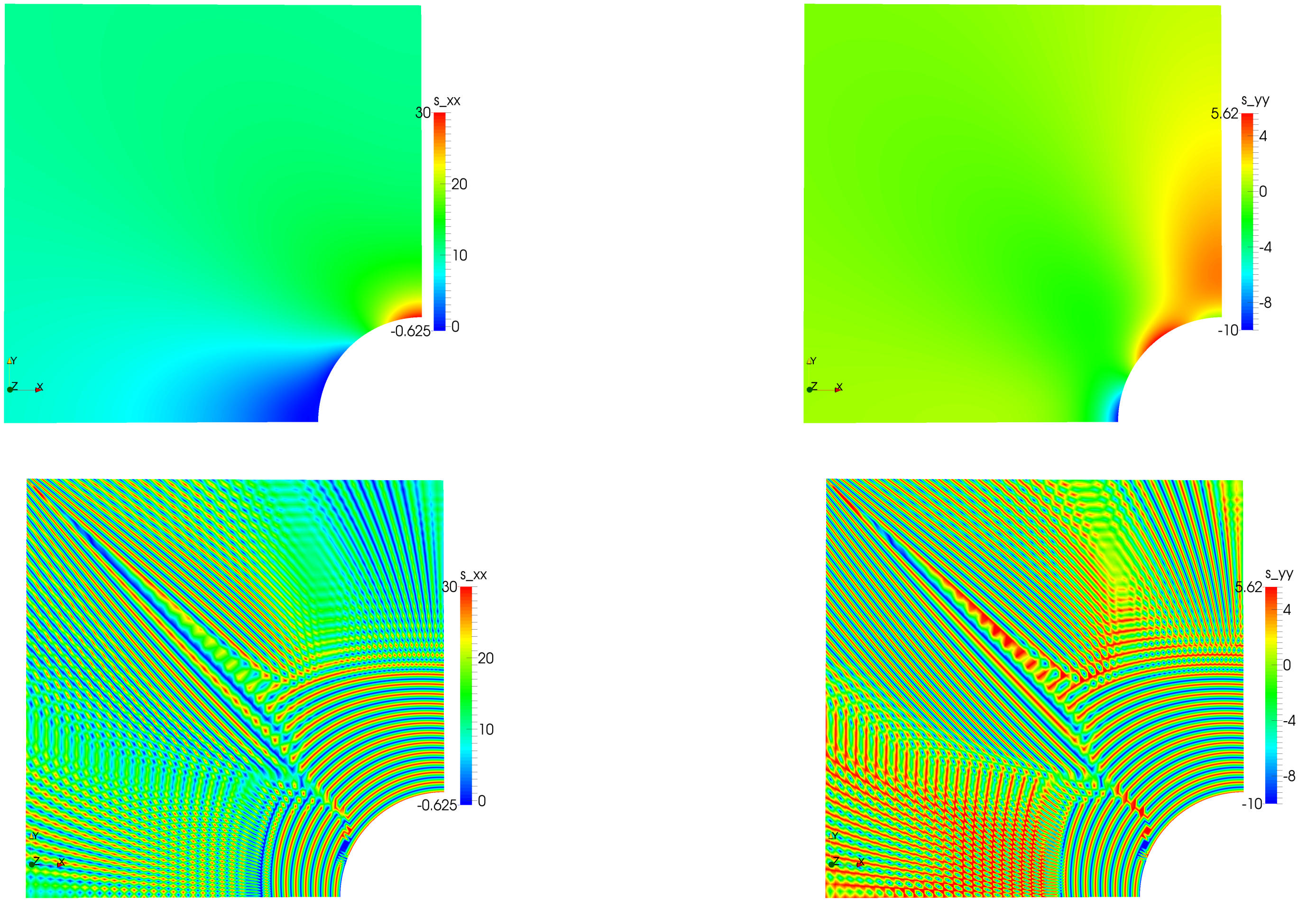
Standard formulation

Plain Galerkin (displacement) formulation

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} d\Omega + \int_\Omega \lambda \nabla \cdot \boldsymbol{w} \nabla \cdot \boldsymbol{u} \, d\Omega$$

- Stress oscillations and locking
- Symmetric ✓
- Sparse ✓
- Definite positive ✓

Exact vs plain formulation for $\nu = 0.49999$



Isogeometric $\bar{B}$ projection technique

$\bar{B}$ projection technique [Elguedj, Bazilevs, Calo, and Hughes, 2008]

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \pi (\nabla \cdot \boldsymbol{w}) \pi (\nabla \cdot \boldsymbol{u}) \mathrm{d}\Omega$$

$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{N}_i(\boldsymbol{x}) c_i(\phi) \quad L^2\text{-proj. on lower degree splines}$$

Isogeometric $\bar{B}$ projection technique

$\bar{B}$ **projection technique** [Elguedj, Bazilevs, Calo, and Hughes, 2008]

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \, d\Omega + \int_\Omega \lambda \, \pi(\nabla \cdot \boldsymbol{w}) \, \pi(\nabla \cdot \boldsymbol{u}) \, d\Omega$$

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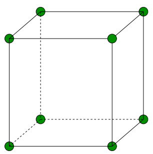
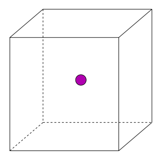
$\boldsymbol{w}, \boldsymbol{u} \in$  $\pi(\nabla \cdot \boldsymbol{w}), \pi(\nabla \cdot \boldsymbol{u}) \in$ 

... then perform degree-elevation and knot-insertion

Isogeometric $\bar{B}$ projection technique

$\bar{B}$ projection technique [Elguedj, Bazilevs, Calo, and Hughes, 2008]

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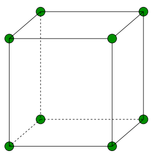
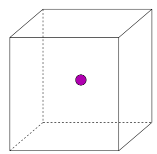
$\boldsymbol{w}, \boldsymbol{u} \in$  $\qquad \pi(\nabla \cdot \boldsymbol{w}), \pi(\nabla \cdot \boldsymbol{u}) \in$ 

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$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{N}_i(\boldsymbol{x}) \, c_i(\phi) \quad L^2\text{-proj. on lower degree splines}$$

isogeometric $\bar{B}$-method

$\boldsymbol{u} \in \mathcal{S}^p_{p-1} \times \mathcal{S}^p_{p-1} \times \mathcal{S}^p_{p-1}$ and $\pi(\cdot) \in \mathcal{S}^{p-1}_{p-2}$

Isogeometric $\bar{B}$ projection technique

$\bar{B}$ projection technique [Elguedj, Bazilevs, Calo, and Hughes, 2008]

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} d\Omega + \int_\Omega \lambda \pi (\nabla \cdot \boldsymbol{w}) \pi (\nabla \cdot \boldsymbol{u}) \, d\Omega$$

$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{N}_i(\boldsymbol{x}) \, c_i(\phi) \quad L^2\text{-proj. on lower degree splines}$$

$$c_i(\phi) = \tilde{\mathbf{M}}_{ij}^{-1} \int_\Omega \tilde{N}_j(\boldsymbol{x}) \, \phi(\boldsymbol{x}) d\Omega$$

- Unlocked solution ✓
- Symmetric ✓
- full matrix
- Definite positive ✓

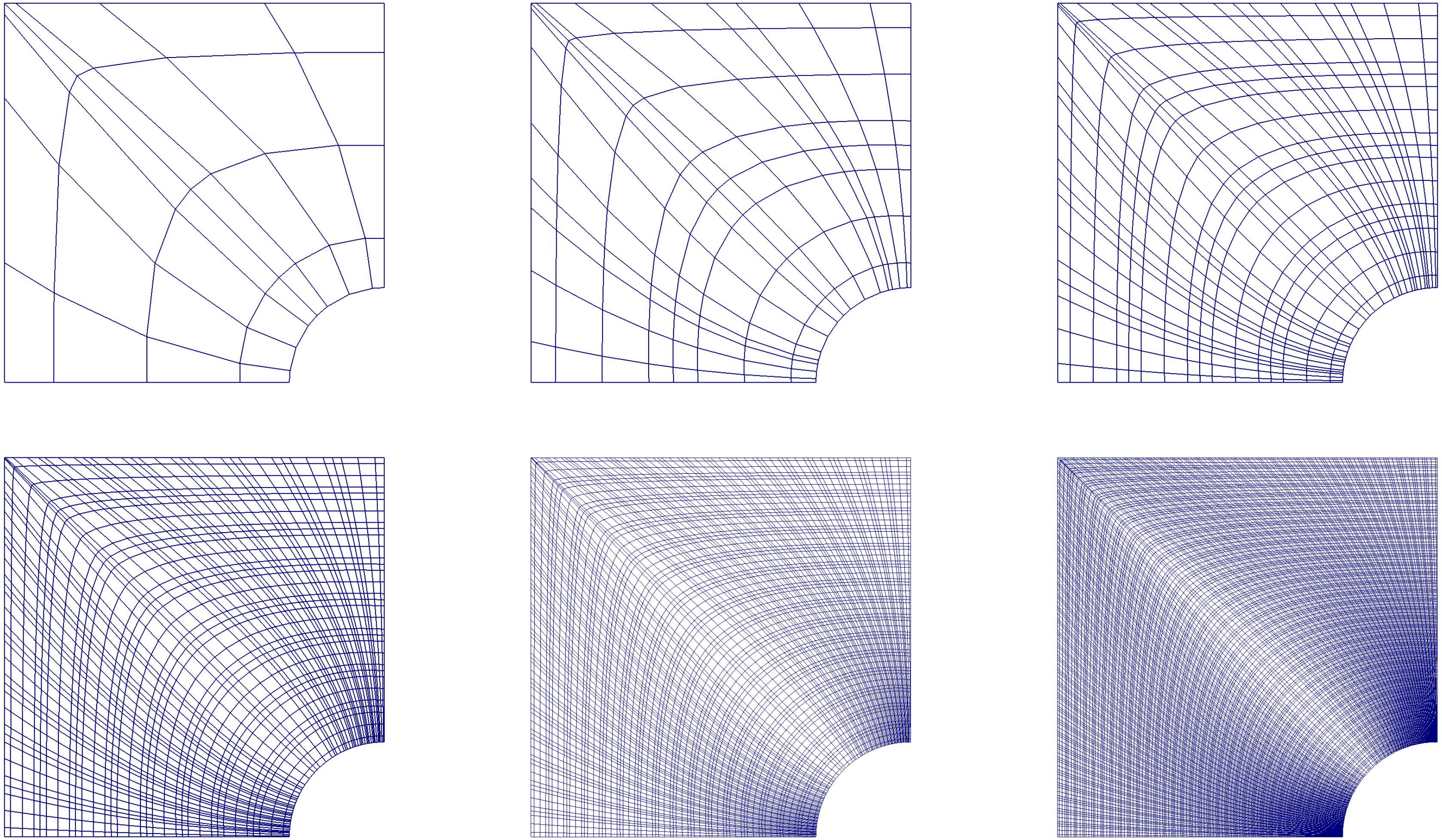
Isogeometric $\bar{B}$ projection technique

$\bar{B}$ projection technique [Elguedj, Bazilevs, Calo, and Hughes, 2008]

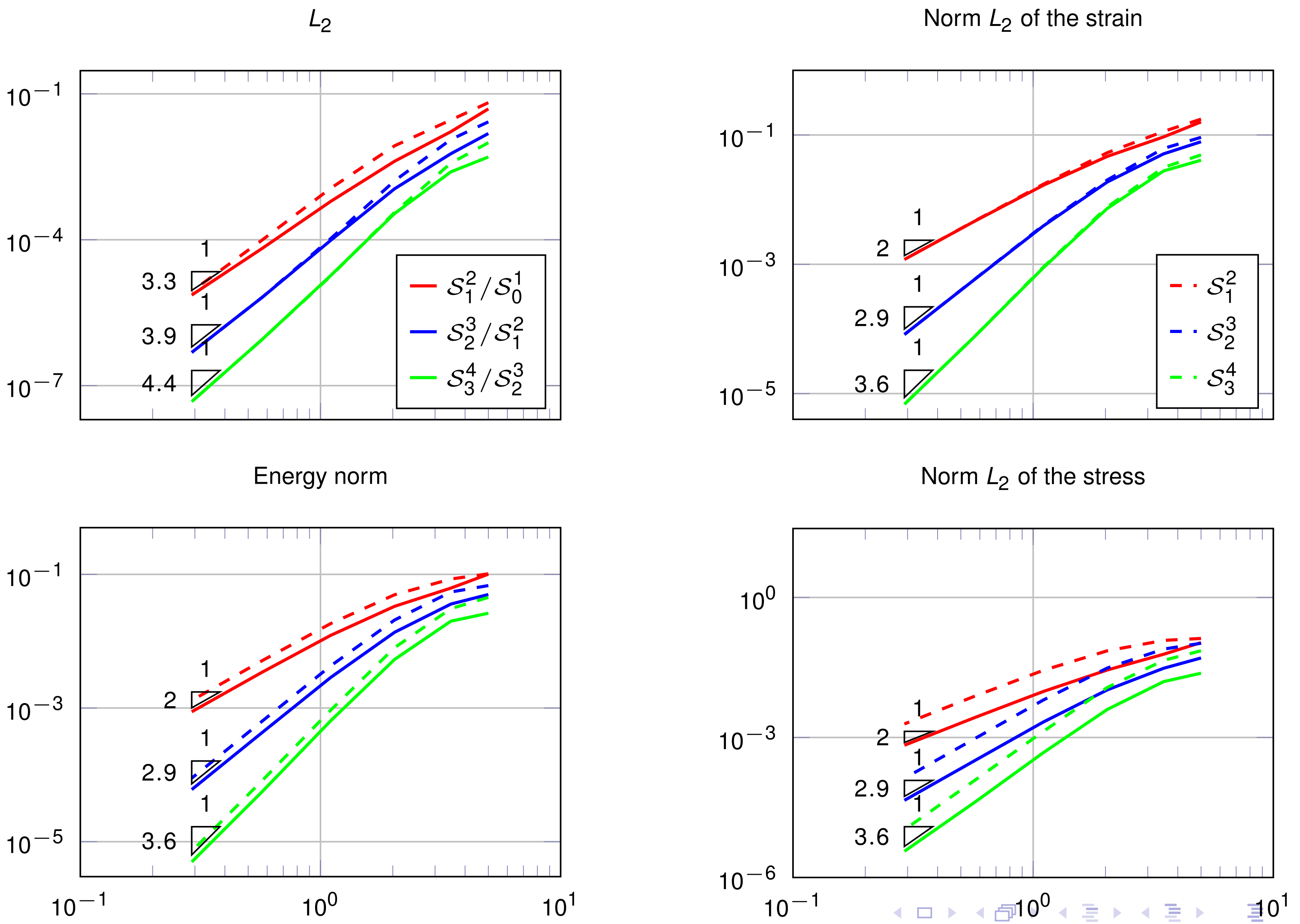
$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \pi (\nabla \cdot \boldsymbol{w}) \, \pi (\nabla \cdot \boldsymbol{u}) \, \mathrm{d}\Omega$$

$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{N}_i(\boldsymbol{x}) \, c_i(\phi) \quad L^2\text{-proj. on lower degree splines}$$

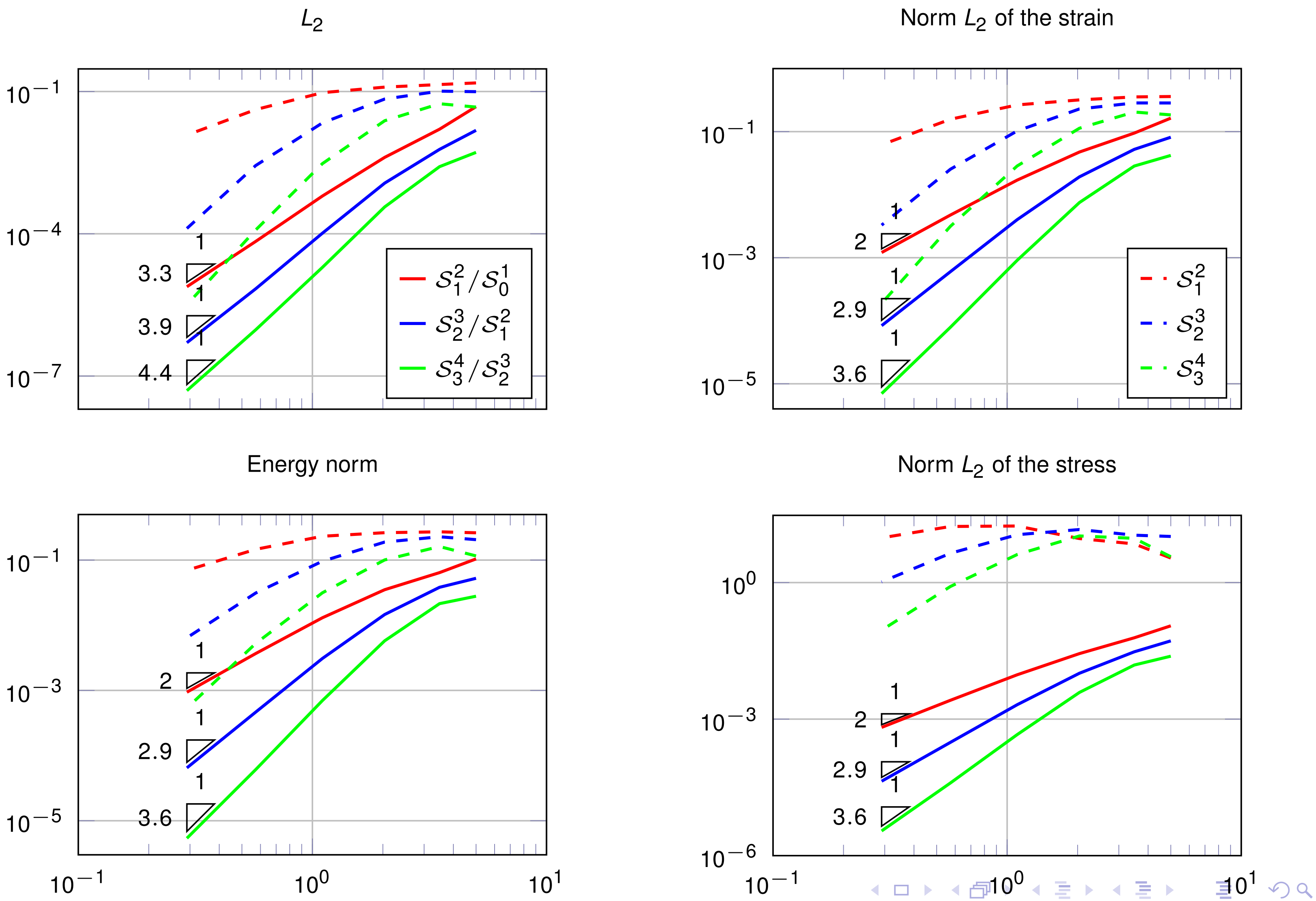
mesh refinement in our next tests



$\bar{B}$ for $\nu = 0.4$



$\bar{B}$ for $\nu = 0.49999$



$\bar{B}$ with lumped mass matrix

$\bar{B}$ with lumped mass matrix: projection technique

$$a^D(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \pi^D (\nabla \cdot \boldsymbol{w})\, \pi^D (\nabla \cdot \boldsymbol{u})\, \mathrm{d}\Omega a(\boldsymbol{w}, \boldsymbol{u})$$

$$\pi^D(\phi)(\boldsymbol{x}) = \sum_{i=1}^{N} \tilde{N}_i(\boldsymbol{x})\, c_i(\phi)$$

$$c_i(\phi) = [\tilde{\mathbf{M}}^D_{ij}]^{-1} \int_\Omega \tilde{N}_j(\boldsymbol{x})\, \phi(\boldsymbol{x}) \mathrm{d}\Omega$$

- Only first-order convergence
- Symmetric ✓
- Sparse ✓
- Definite positive ✓

$\bar{B}$ with lumped mass matrix

$\bar{B}$ with lumped mass matrix: projection technique

$$a^D(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} d\Omega + \int_\Omega \lambda \pi^D (\nabla \cdot \boldsymbol{w}) \pi^D (\nabla \cdot \boldsymbol{u}) \, d\Omega a(\boldsymbol{w}, \boldsymbol{u})$$

$$\pi^D(\phi)(\boldsymbol{x}) = \sum_{i=1}^{N} \tilde{N}_i(\boldsymbol{x}) \, c_i(\phi)$$

$$c_i(\phi) = [\tilde{\mathbf{M}}^D_{ij}]^{-1} \int_\Omega \tilde{N}_j(\boldsymbol{x}) \, \phi(\boldsymbol{x}) d\Omega$$

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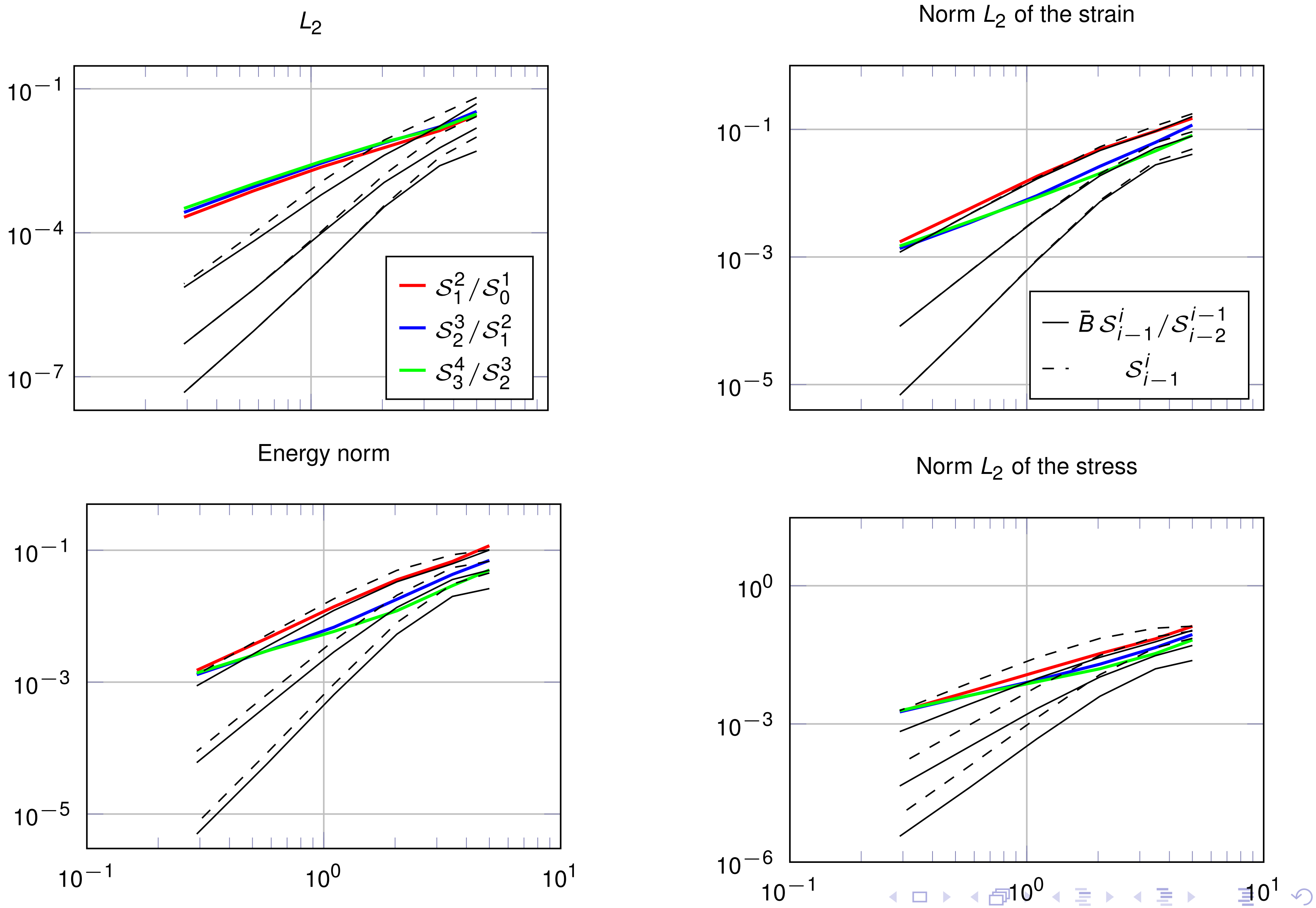
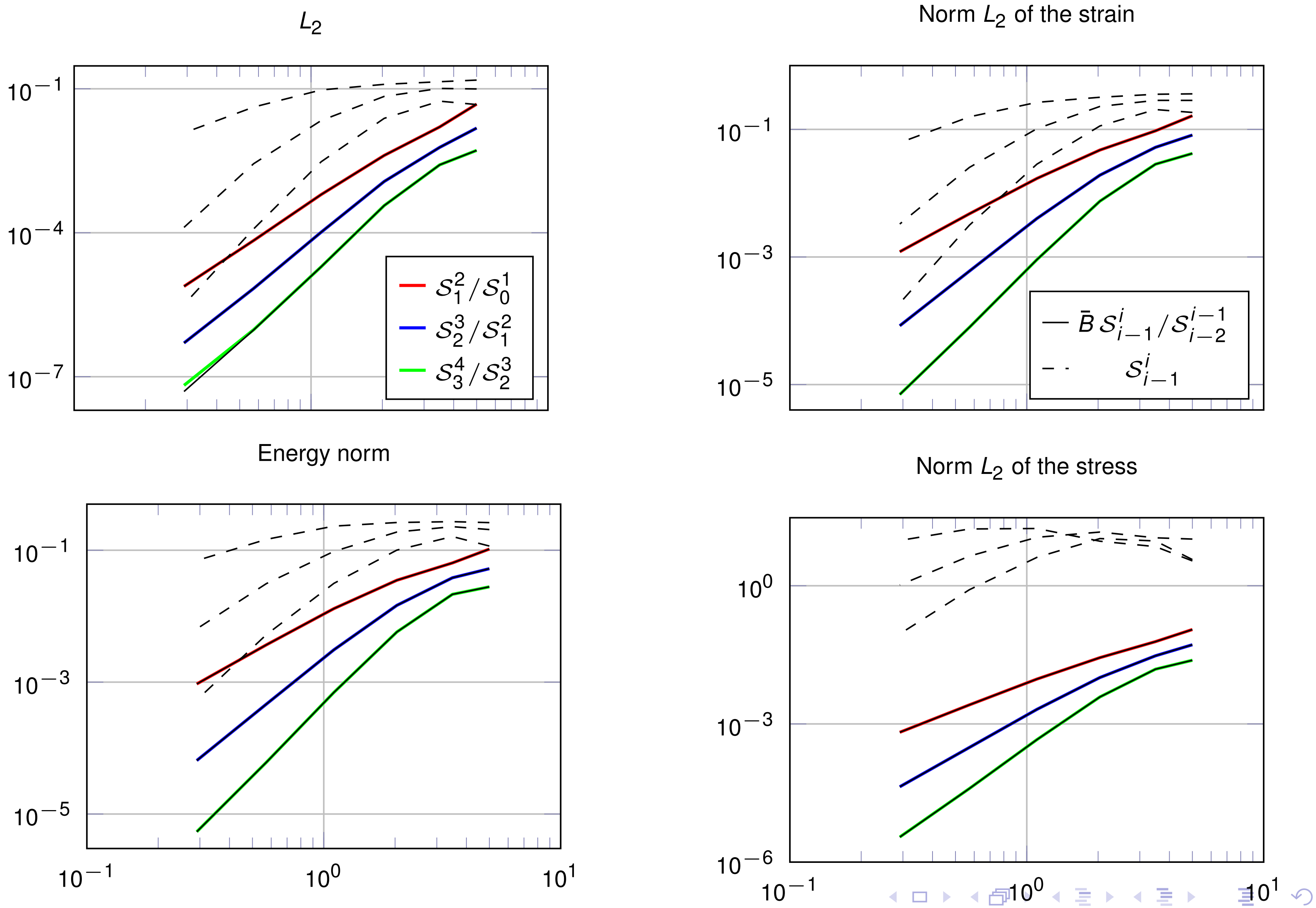
- Symmetric ✓
- Sparse ✓
- Definite positive ✓

$\bar{B}$ with lumped mass matrix for $\nu = 0.4$



$\bar{B}$ with lumped mass matrix for $\nu = 0.49999$



$\bar{B}$ with lumped mass matrix and iterative solver

$\bar{B}$, lumped mass + iterative solver [Elguedj, Bazilevs, Calo, and Hughes, 2008]

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \pi (\nabla \cdot \boldsymbol{w}) \pi (\nabla \cdot \boldsymbol{u}) \mathrm{d}\Omega$$

$$a^D(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \pi^D (\nabla \cdot \boldsymbol{w}) \pi^D (\nabla \cdot \boldsymbol{u}) \mathrm{d}\Omega$$

$$a^D\left(\boldsymbol{w}, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right) = L(\boldsymbol{w}) - a(\boldsymbol{w}, \boldsymbol{u}^n)$$

- optimally convergent ✓
- Symmetric ✓
- Sparse in $\boldsymbol{u}^{n+1}$ ✓
- Definite positive ✓

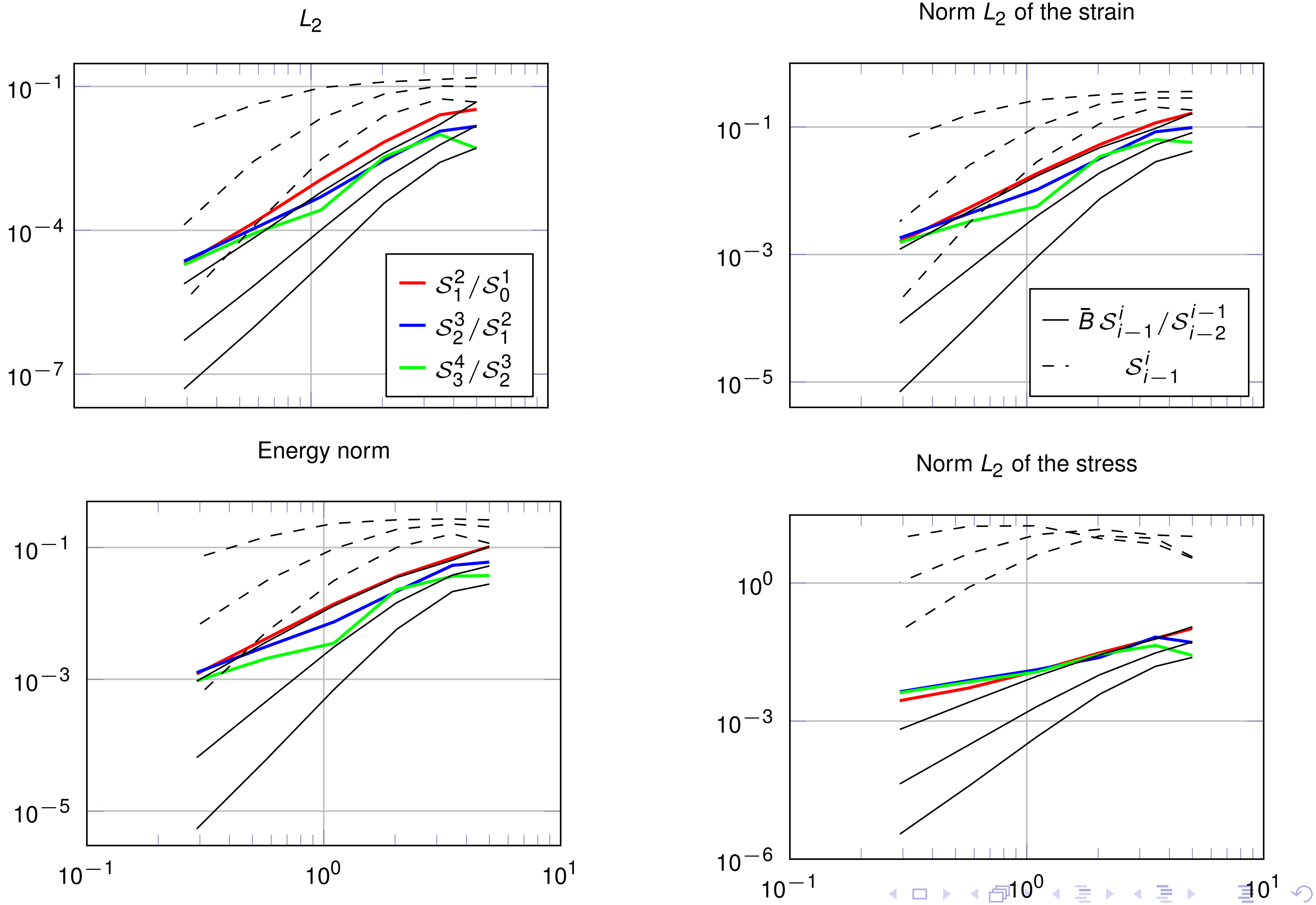
Symmetrized quasi-interpolant π^* [Lee, Lyche, and Mørken, 2001]

Symmetrized quasi-interpolant

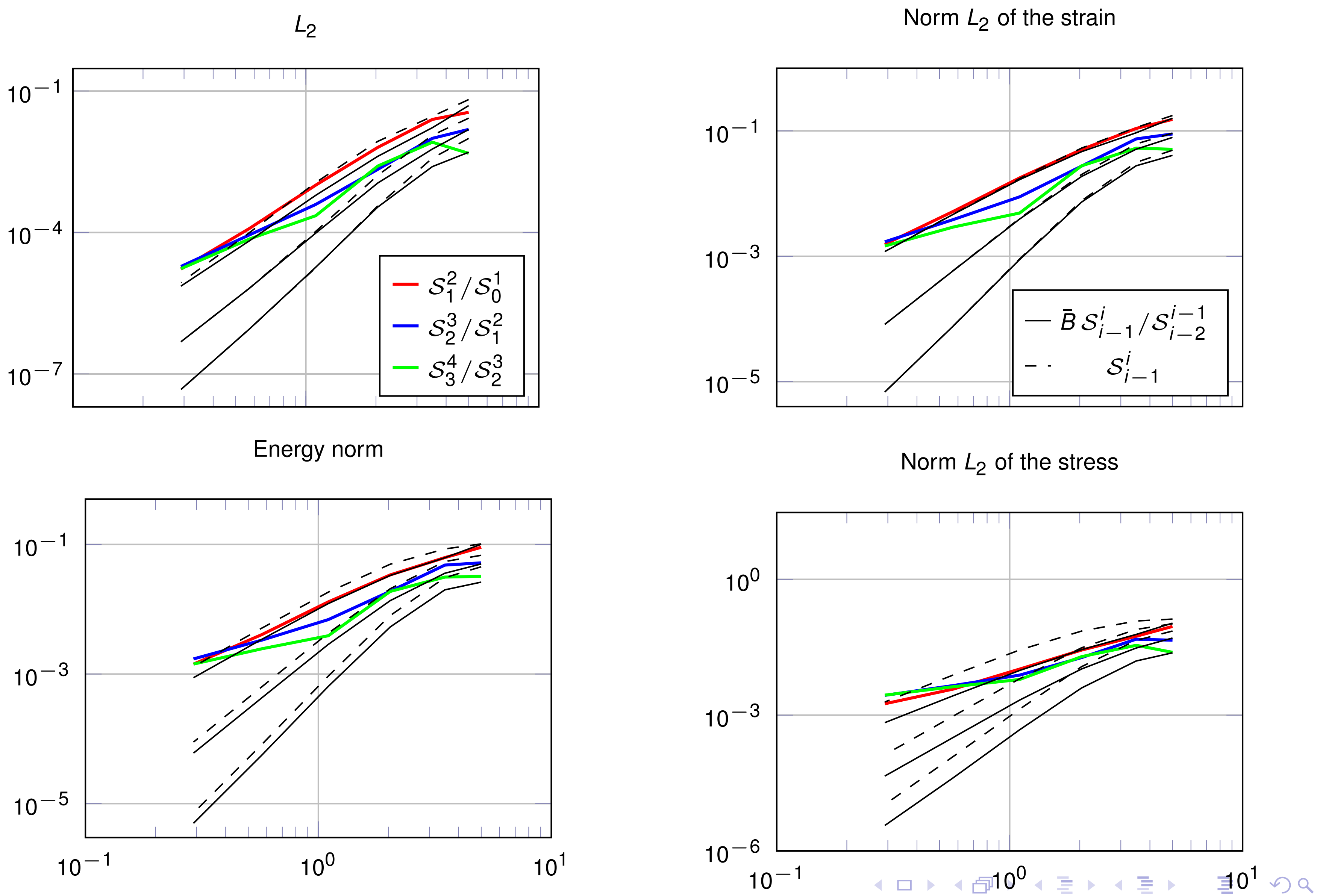
$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \pi^* (\nabla \cdot \boldsymbol{w}) \, \pi^* (\nabla \cdot \boldsymbol{u}) \, \mathrm{d}\Omega$$

- Poor approximation
- Symmetric ✓
- Sparse ✓
- Definite positive ✓

Symmetrized quasi-interpolant for $\nu = 0.49999$



Symmetrized quasi-interpolant for $\nu = 0.4$



Projection formulation as a mixed formulation

projection technique

$$\int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \lambda \pi (\nabla \cdot w)\, \pi (\nabla \cdot u) = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \quad \forall \boldsymbol{w}$$

Mixed formulation (for the unknowns *u* and *p*)

$$\int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \nabla \cdot \boldsymbol{w}\, \wp = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \qquad \forall \boldsymbol{w}$$

$$\int_\Omega (\nabla \cdot \boldsymbol{u})\, q - \lambda^{-1} \int_\Omega \wp\, q = 0, \quad \forall q$$

where the second equation states $\lambda\, \pi(\nabla \cdot \boldsymbol{u}) = \wp$

$\bar{B}$ method: $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{S}^p_{p-1} \times \mathcal{S}^p_{p-1} \times \mathcal{S}^p_{p-1}$ and $\wp, q \in \mathcal{S}^{p-1}_{p-2}$.

Projection formulation as a mixed formulation

projection technique

$$\int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \lambda \, \pi \left(\nabla \cdot w \right) \pi \left(\nabla \cdot u \right) = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \quad \forall \boldsymbol{w}$$

Mixed formulation (for the unknowns *u* and *p*)

$$\int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \nabla \cdot \boldsymbol{w} \, \wp = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \qquad \forall \boldsymbol{w}$$

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Well-posedness of mixed formulation

Babuška-Brezzi inf-sup condition

The discrete displacement space $\boldsymbol{V}_h \subset (H^1_{\Gamma_D}(\Omega))^2$ and the discrete pressure space $Q_h \subset L^2(\Omega)$ have to fulfill

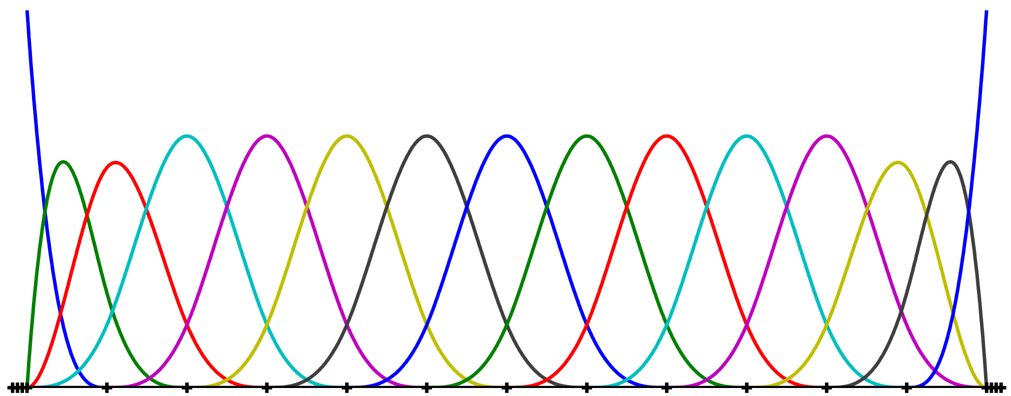
$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{\int_\Omega \nabla \cdot \boldsymbol{v}\, q \, \mathrm{d}\Omega}{\|q\|_{L^2} \|\boldsymbol{v}\|_{(H^1)^2}} \geq C_{is} > 0 \quad \text{(uniformly w.r.t. } h\text{)}.$$

The **inf-sup condition** above holds if "locally" and "on average" there are more knot lines of the displacement field than knot lines of the pressure field, roughly speaking... [Bressan and Sangalli, 2013] quick proof

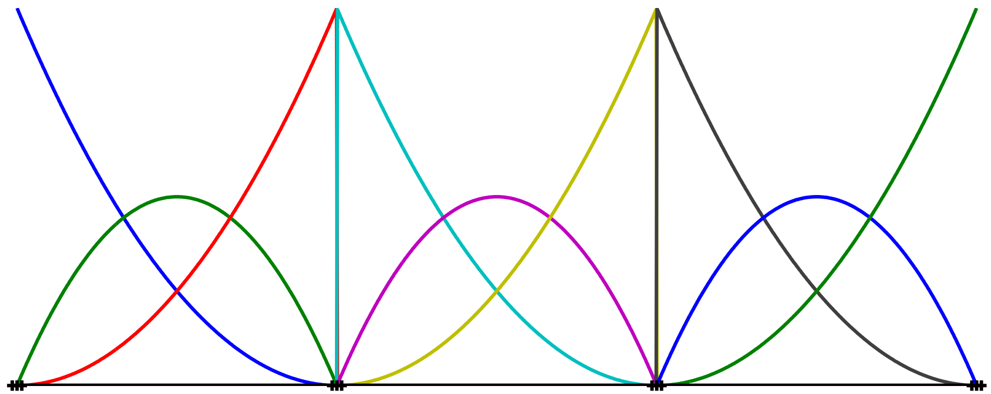
A special case for the mixed formulation

discontinuous pressure and subgrid displacement

cubic
$\boldsymbol{u}, \boldsymbol{v} \in$



quadratic
$\wp, q \in$



Mixed formulation with discontinuous pressures

[Antolín, Buffa, and Sangalli, 2014]

The mixed formulation with $\boldsymbol{u} \in (\mathcal{S}^{p}_{p-1})^3$ and $\wp \in \mathcal{S}^{p-1}_{-1,M}$

$$\int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \nabla \cdot \boldsymbol{w}\, \wp = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \qquad \forall \boldsymbol{w}$$

$$\int_\Omega (\nabla \cdot \boldsymbol{u})\, q - \lambda^{-1} \int_\Omega \wp\, q = 0, \quad \forall q$$

- Well-posed
- optimally convergent
- the 2nd equation is local to macroelements! $\lambda\, \pi_M(\nabla \cdot \boldsymbol{u}) = \wp$

Projection formulation with the local π_M

Mixed formulation with $\boldsymbol{u} \in (\mathcal{S}^{p}_{p-1})^3$ and $\wp \in \mathcal{S}^{p-1}_{-1,M}$

$$\int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \nabla \cdot \boldsymbol{w} \, \wp = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \qquad \forall \boldsymbol{w}$$

$$\int_\Omega (\nabla \cdot \boldsymbol{u}) \, q = \lambda^{-1} \int_\Omega \wp \, q, \quad \forall q$$

$$\Downarrow$$

projection technique with macroelement projection π_M

$$\int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \lambda \, \pi_M (\nabla \cdot w) \, \pi_M (\nabla \cdot u) = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \quad \forall \boldsymbol{w}$$

Macroelement projection π_M

Projection technique with macroelement projection π_M

$$a(\boldsymbol{w},\, \boldsymbol{u}) = \int_\Omega \mu \, \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \, \pi_M (\nabla \cdot \boldsymbol{w}) \; \pi_M (\nabla \cdot \boldsymbol{u}) \; \mathrm{d}\Omega$$

$$\tilde{N} \in \mathcal{S}^{p-1}_{-1,M} \quad \pi_M(\phi)(\boldsymbol{x}) = \sum_i \tilde{N}_i(\boldsymbol{x}) \left[\sum_j \overline{\mathbf{M}}_{ij}^{-1} \int_\Omega \tilde{N}_j(\boldsymbol{x}) \, \phi(\boldsymbol{x}) \mathrm{d}\Omega \right]$$

It works with minimal macroelements of p element per direction, but the theory is missing (unless $p = 1$: Pitkäranta and Stenberg [1984])

Macroelement projection π_M

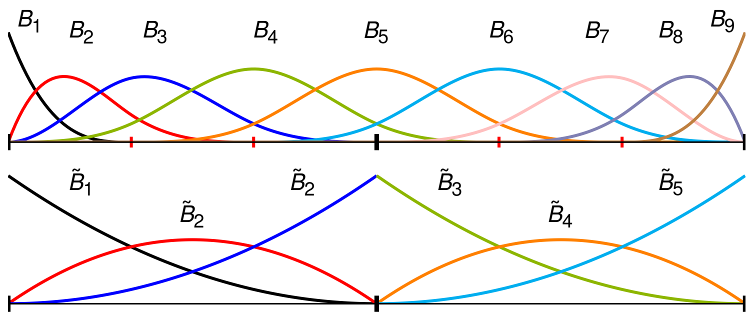
Projection technique with macroelement projection π_M

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega \mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} d\Omega + \int_\Omega \lambda \pi_M (\nabla \cdot \boldsymbol{w}) \, \pi_M (\nabla \cdot \boldsymbol{u}) \, d\Omega$$

$$\tilde{N} \in \mathcal{S}^{p-1}_{-1,M} \quad \pi_M(\phi)(\boldsymbol{x}) = \sum_i \tilde{N}_i(\boldsymbol{x}) \left[\sum_j \overline{\mathbf{M}}_{ij}^{-1} \int_\Omega \tilde{N}_j(\boldsymbol{x}) \, \phi(\boldsymbol{x}) d\Omega \right]$$

$\boldsymbol{u}, \boldsymbol{w} \in$

$\pi_M(\nabla \cdot \boldsymbol{u}), \pi_M(\nabla \cdot \boldsymbol{w}) \in$



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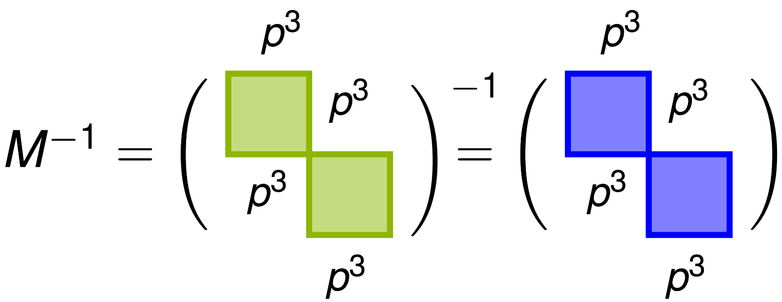

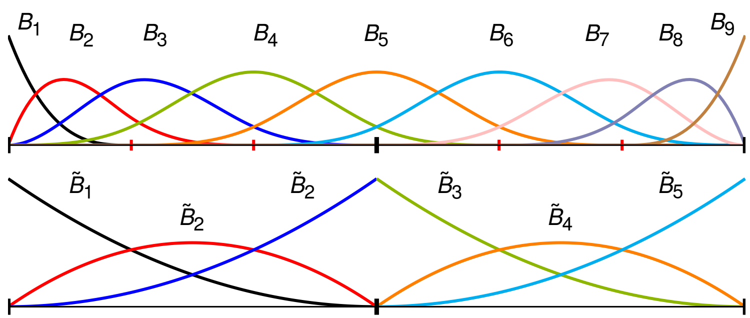

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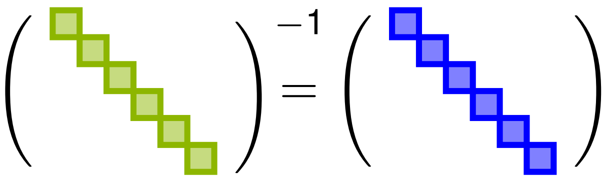
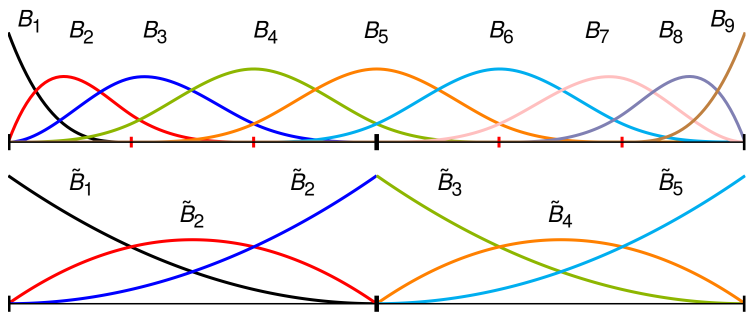
$M^{-1} =$ 



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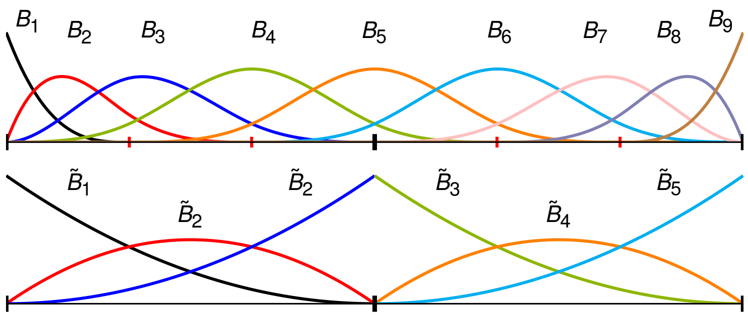

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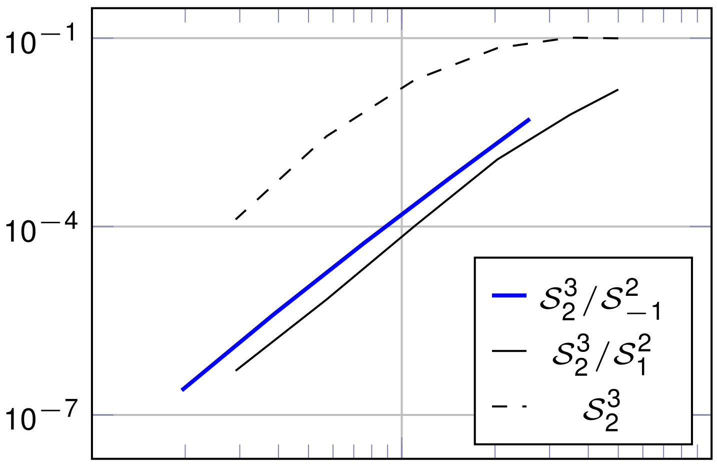
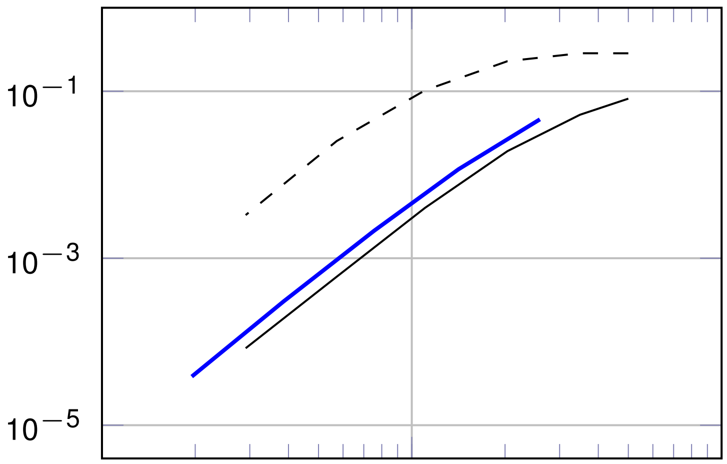
- Unlocked solution ✓
- Symmetric ✓
- Sparse ✓
- Definite positive ✓

Macroelement projection π_M for $\nu = 0.49999$

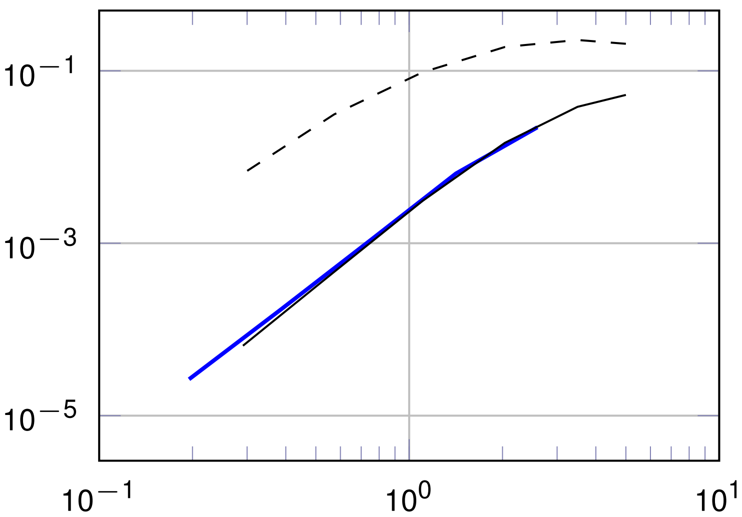
L_2



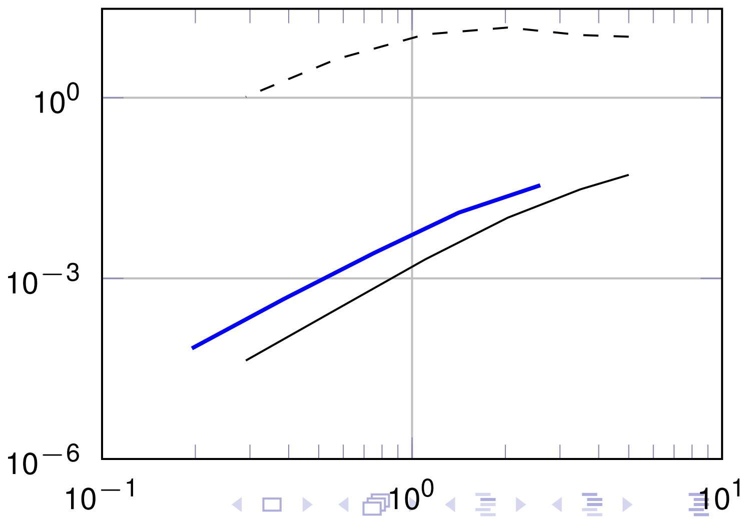
Norm L_2 of the strain



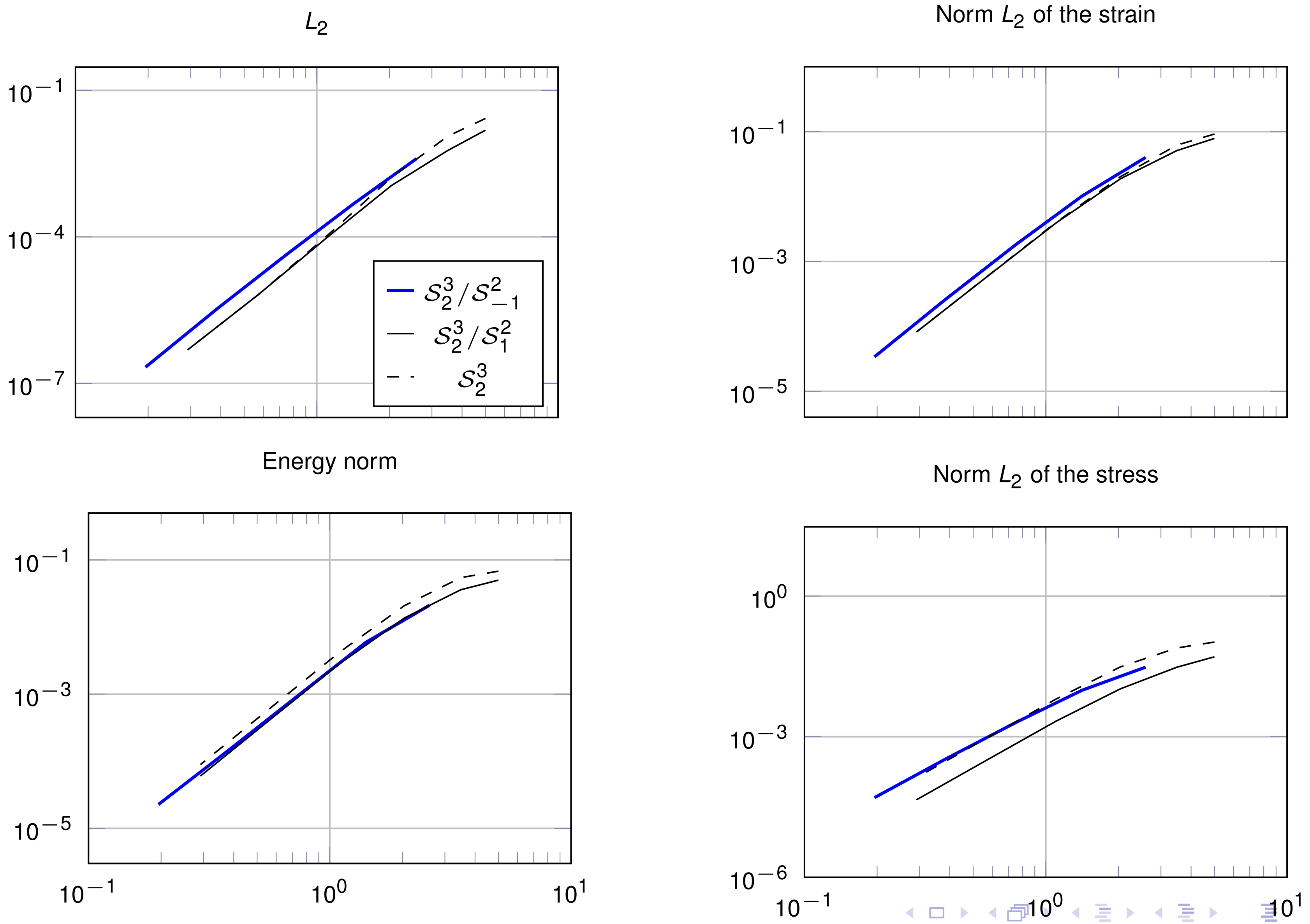
Energy norm



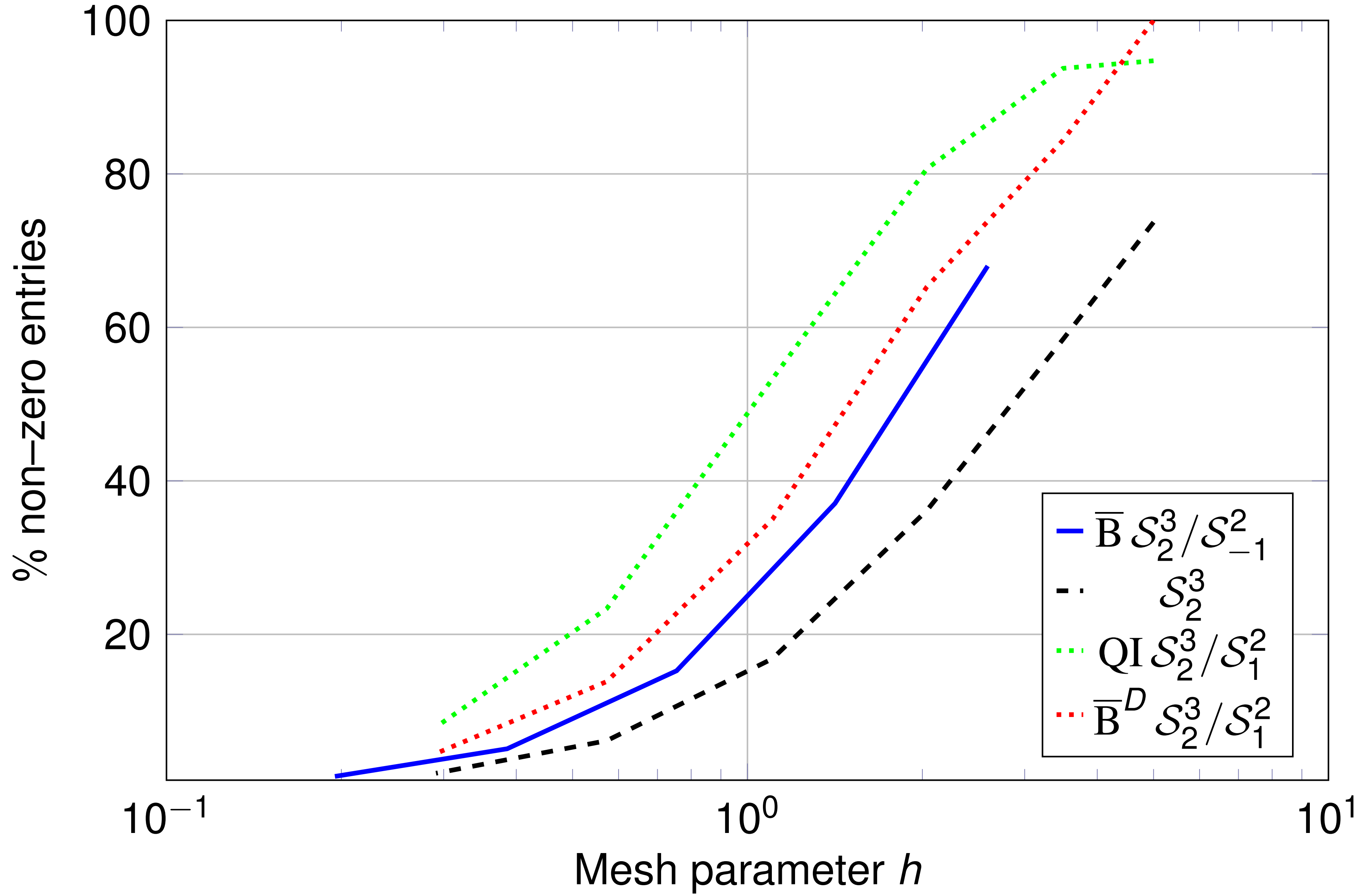
Norm L_2 of the stress



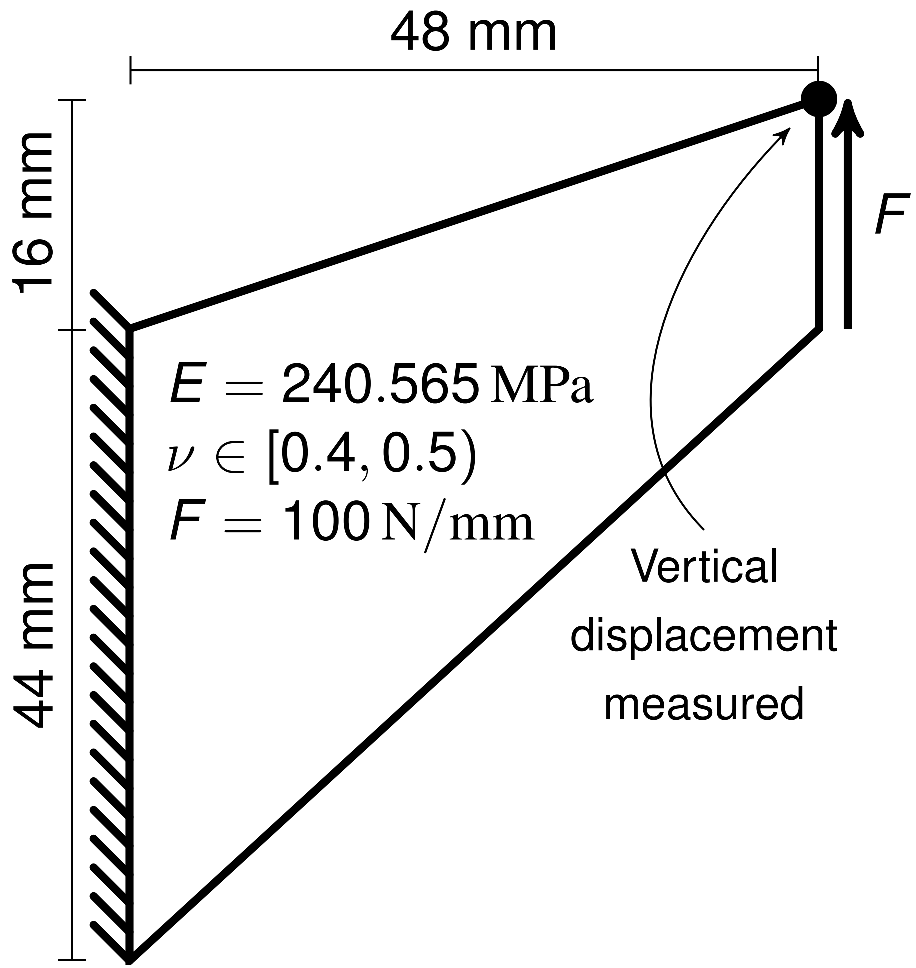
Macroelement projection π_M for $\nu = 0.4$



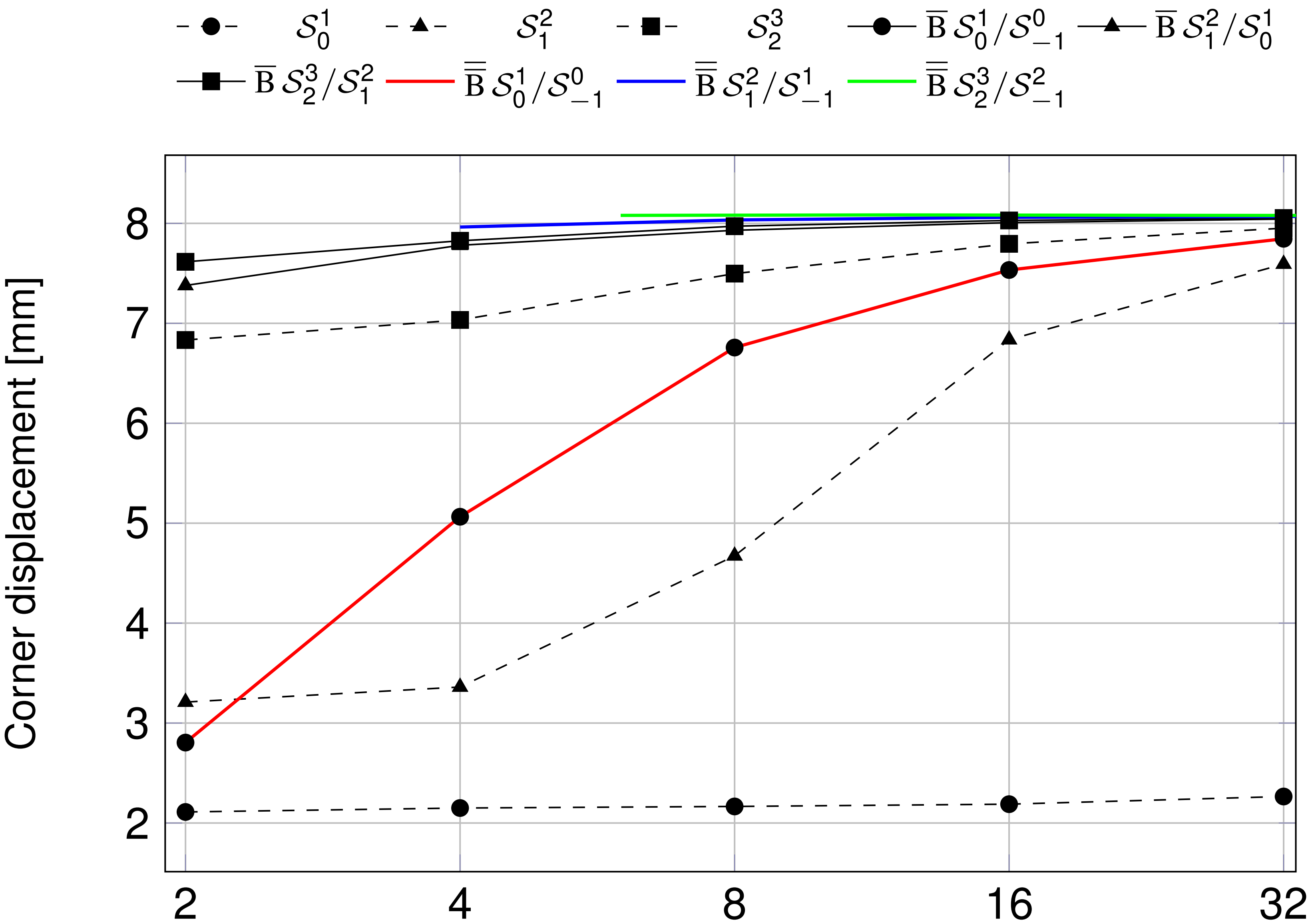
Macroelement projection π_M: sparsity



Cook Membrane



Cook Membrane $\nu = \mathbf{0.49999}$



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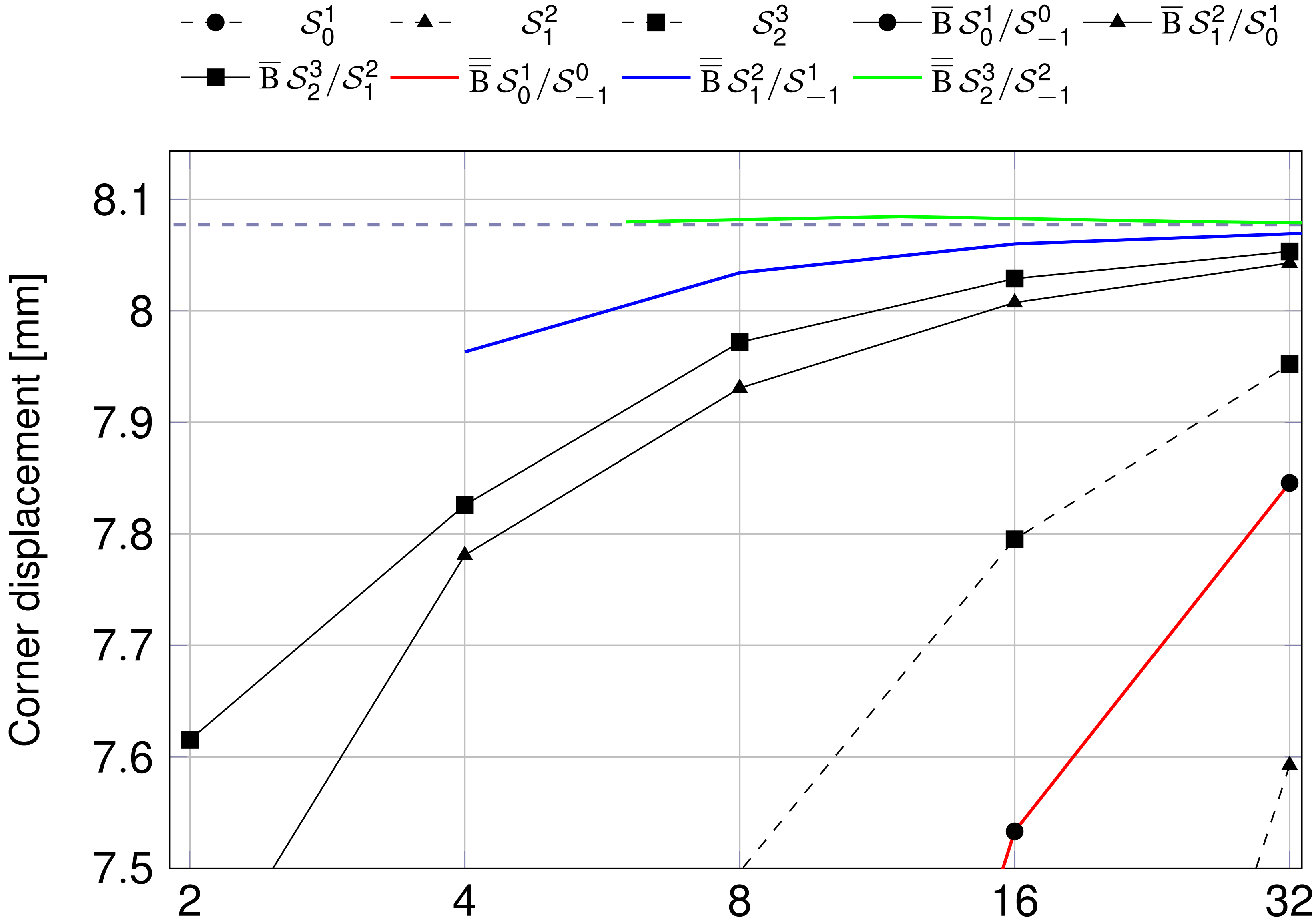

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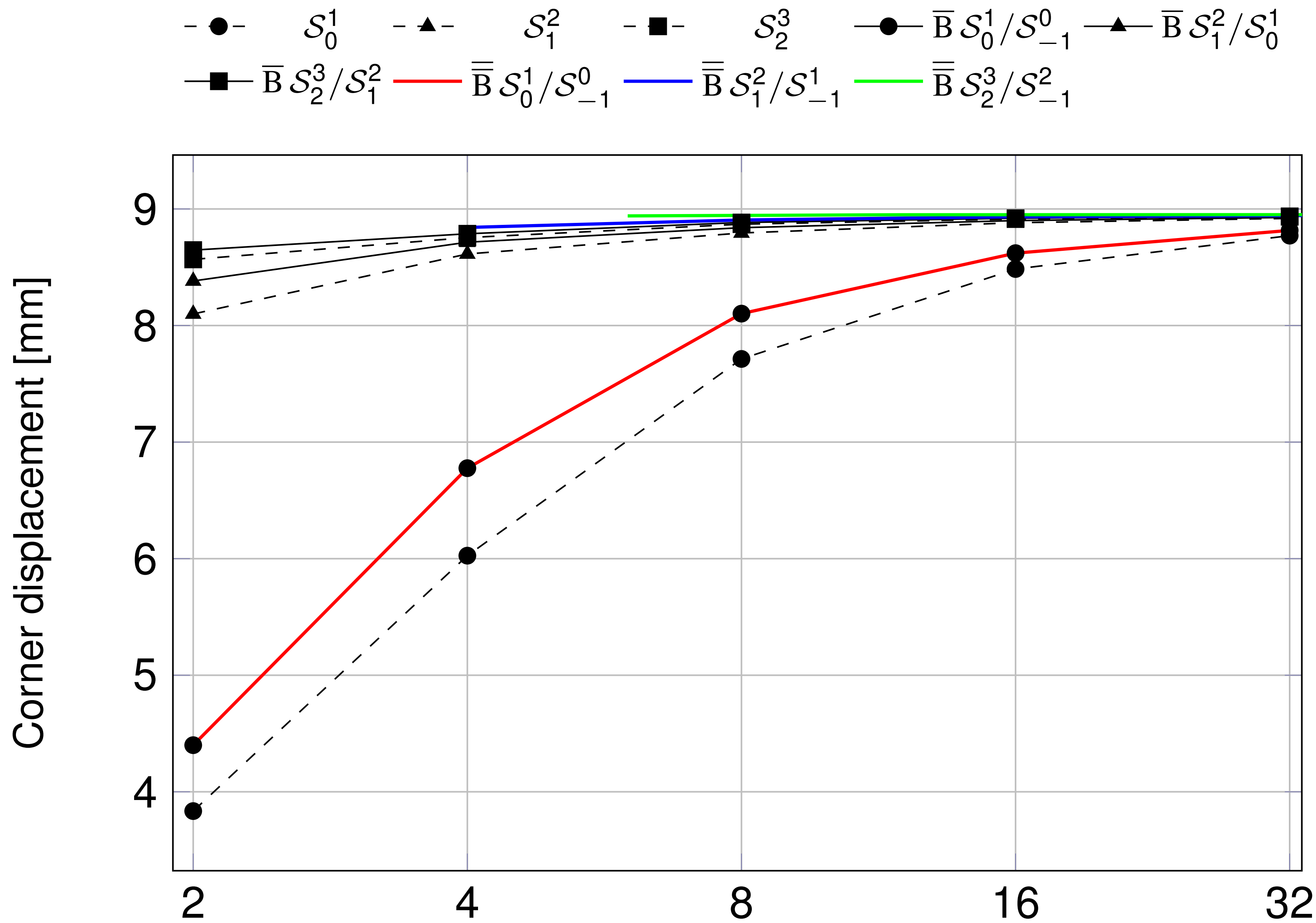

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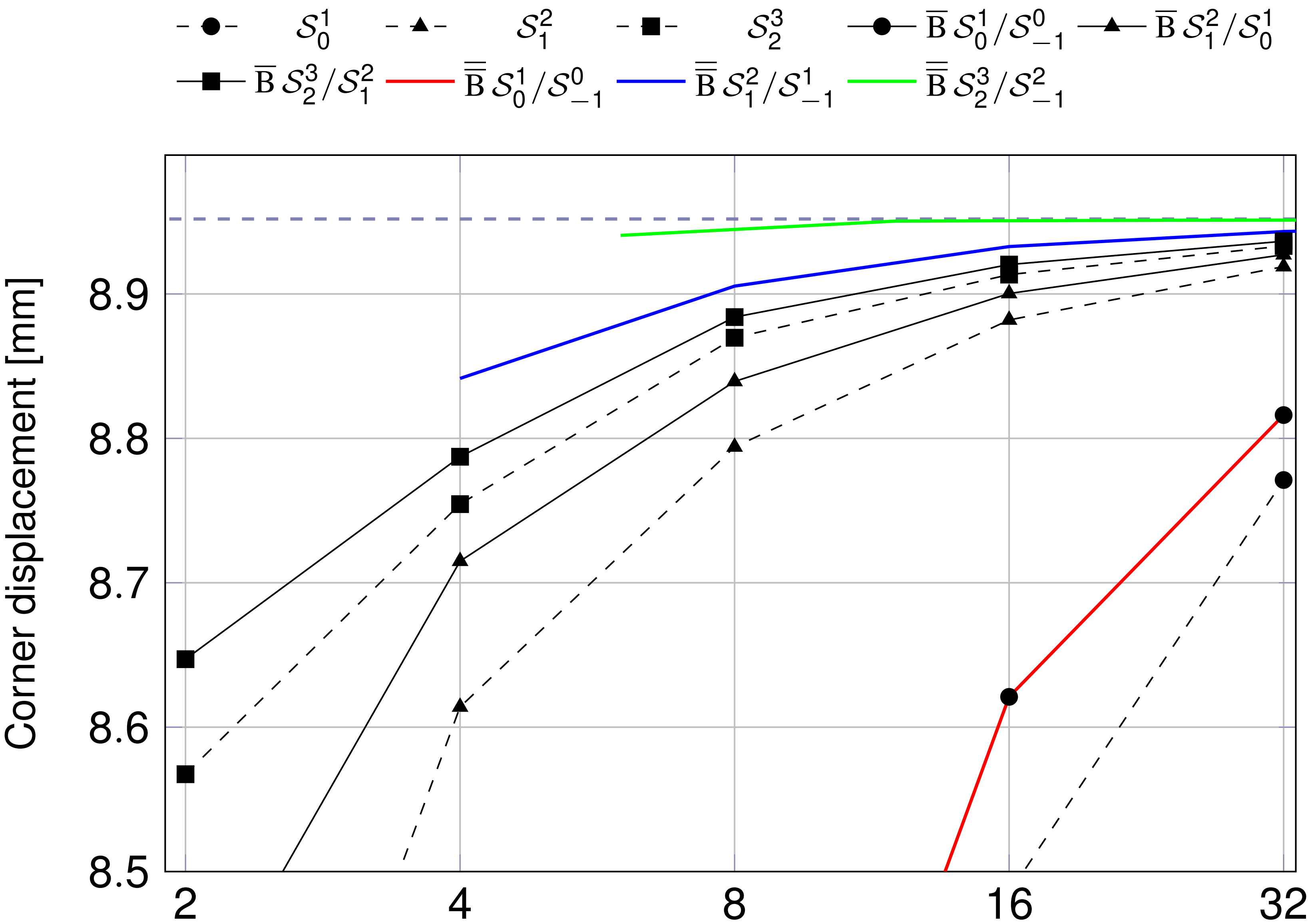
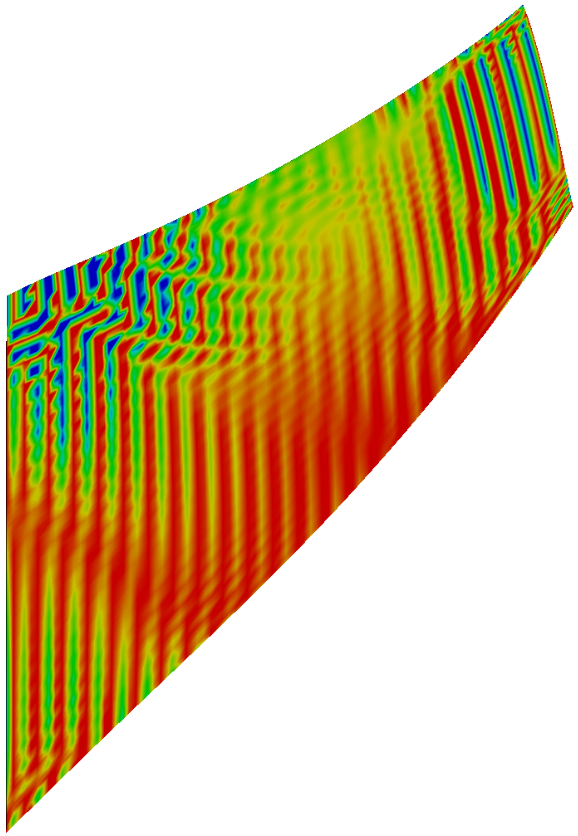
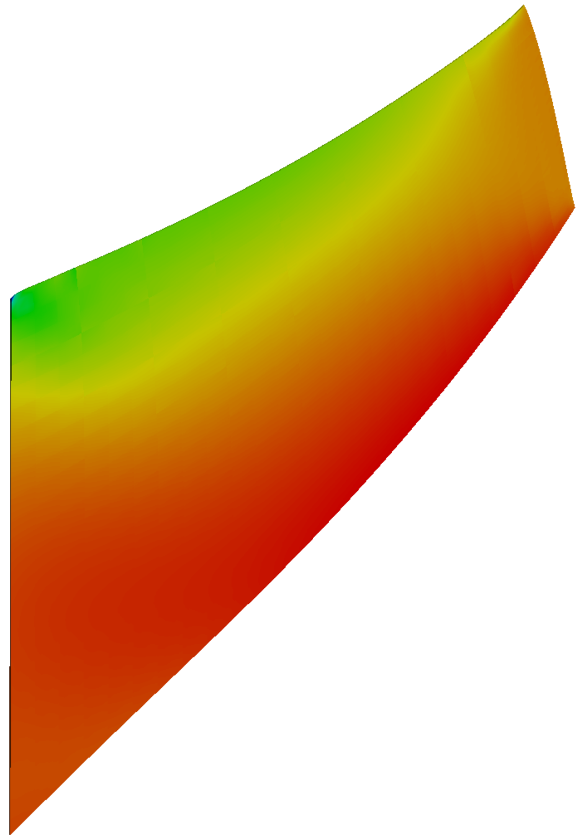


Cook Membrane σ_{xx} for $\nu = \mathbf{0.49999}$ $p = \mathbf{3}$

Standard formulation

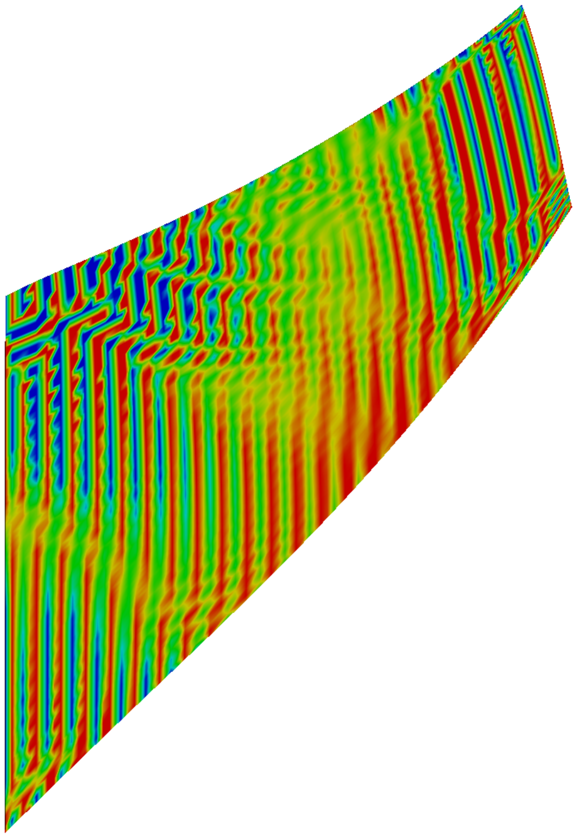


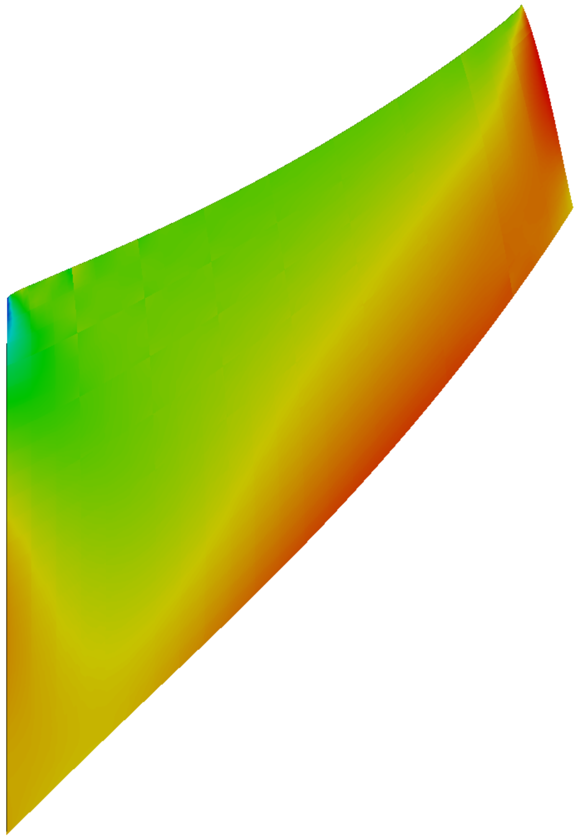
Discontinuous subgrid $\overline{\mathrm{B}}$



Cook Membrane σ_{yy} for $\nu = \mathbf{0.49999}$ $\boldsymbol{p} = \mathbf{3}$

Standard formulation **Discontinuous subgrid $\overline{\mathbf{B}}$**





Conclusion

Based on the $\bar{B}$-method, we have proposed a new formulation with:

- symmetric and positive definite stiffness matrix
- sparse, with little increase of non-zeros w.r.t. plain Galerkin
- locking-free and optimally accurate

It uses smooth & discontinuous functions together :-)

Open ERC-funded positions:

`http://www-dimat.unipv.it/sangalli/higeom`

*** Thank you for your attention ***

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Macroelement method [Bazilevs, Beirão da Veiga, Cottrell, Hughes, and Sangalli, 2006; Bressan and Sangalli, 2013]

Let $\mathcal{M} = \{M\}$ a covering of $\hat{\Omega}$ such that the (local) full-rank condition

$$\forall \hat{q}_h \in \hat{Q}_{h,0}, \hat{q}_h \text{ non-constant on } M, \exists \hat{\mathbf{v}}_h \in \hat{V}_{h,0} \text{ with } \mathrm{supp}(\hat{\mathbf{v}}_h) \subseteq M$$

$$\text{such that } \int_M \hat{q}_h \mathrm{div} \hat{\mathbf{v}}_h \neq 0$$

holds in each macroelement $M \in \mathcal{M}$ (+ some technicalities)

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holds in each macroelement $M \in \mathcal{M}$ (+ some technicalities)

then the global inf-sup condition hold:

$$\forall \hat{q}_h \in \hat{Q}_{h,0},\ \exists \hat{\mathbf{v}}_h \in \hat{V}_{h,0} \text{ such that } \frac{\int_{\hat{\Omega}} \hat{q}_h \,\mathrm{div} \hat{\mathbf{v}}_h}{\|\hat{q}_h\|_{L^2} \|\hat{\mathbf{v}}_h\|_{(H^1)^2}} \geq \hat{C}_{is} > 0$$

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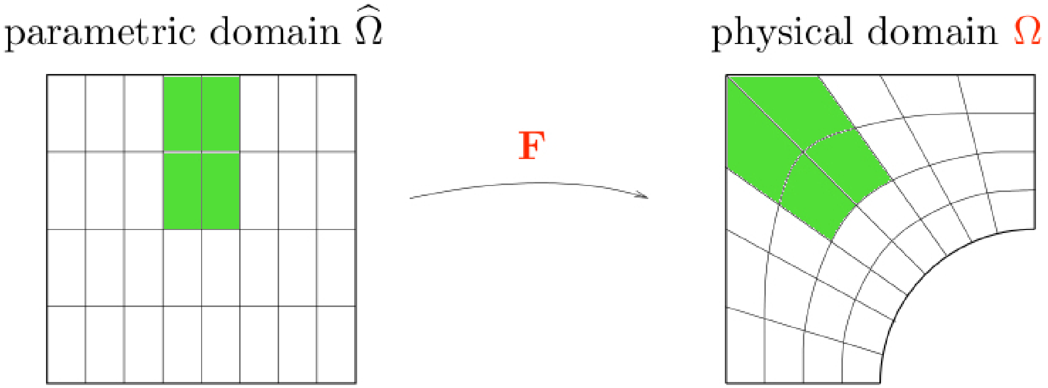

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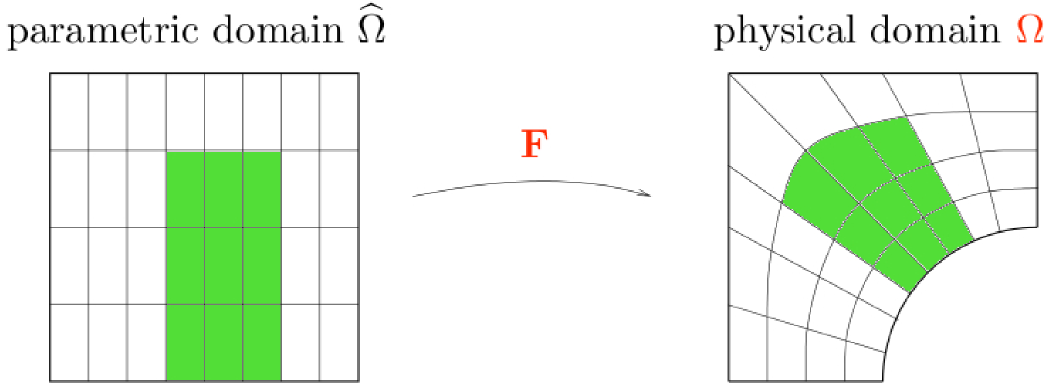

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$$\text{such that } \int_M \hat{q}_h \mathrm{div} \hat{\mathbf{v}}_h \neq 0$$

holds in each macroelement $M \in \mathcal{M}$ (+ some technicalities)

$$\int_M \hat{q}_h \mathrm{div} \hat{\mathbf{v}}_h = -\int_M \nabla \hat{q}_h \cdot \hat{\mathbf{v}}_h = -\int_M \frac{\partial \hat{q}_h}{\partial x_1} \hat{v}_{1,h} - \int_M \frac{\partial \hat{q}_h}{\partial x_2} \hat{v}_{2,h}$$

Macroelement method [Bazilevs, Beirão da Veiga, Cottrell, Hughes, and Sangalli, 2006; Bressan and Sangalli, 2013]

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$$\forall g \neq 0 \in S = S_1 \otimes S_2, \exists f \in T = T_1 \otimes T_2 : \int_M f(x_1, x_2) g(x_1, x_2) dx_1 dx_2 > 0$$

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Full-rank characterization of 1D scalar products

Let S a spline space on the partition $\zeta_0, \ldots, \zeta_m$, let $J = [a, b[$ be a subinterval of $[\zeta_0, \zeta_m[$ and define:

$$S_{\subset J} = \text{span}\{B_i \in S : \; B_i(x) = 0, \, \forall x \in \setminus J\}, \qquad (1)$$

$$S_{\cap J} = \text{span}\{B_i \in S : B_i|_J \neq 0\}, \qquad (2)$$

Theorem [Bressan and Sangalli, 2013]

Let T be a spline space on the partition $\eta_0, \ldots, \eta_n$ and S be a spline space on $\zeta_0, \ldots, \zeta_m$ with $[\zeta_0, \zeta_m] = [\eta_0, \eta_n]$; the following two properties are equivalent:

for all $g \in S$, $g \neq 0$, there exists $f \in T$ such that $\int_{\mathbb{R}} f(x)g(x)\,dx \neq 0$;

for all $0 \leq i < j \leq n$, $\dim T_{\cap[\eta_i,\eta_j[} \geq \dim S_{\subset[\eta_i,\eta_j[}$.

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