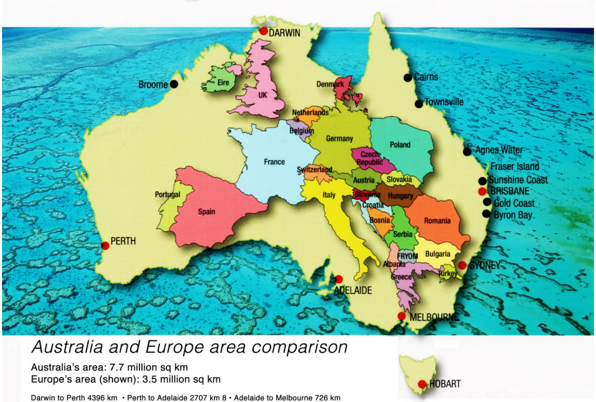
ENDOMORPHISMS AND SYNCHRONISATION

Gordon Royle
(+ Araújo, Bentz, Cameron & Schaefer)

Centre for the Mathematics of Symmetry & Computation
School of Mathematics & Statistics
University of Western Australia

July 2015

AUSTRALIA



Australia and Europe area comparison

Australia's area: 7.7 million sq km
Europe's area (shown): 3.5 million sq km

Darwin to Perth 4396 km • Perth to Adelaide 2707 km 8 • Adelaide to Melbourne 726 km
Melbourne to Sydney 887 km • Sydney to Brisbane 972 km • Brisbane to Cairns 1748 km

PERTH

UNIVERSITY OF WESTERN AUSTRALIA

DURHAM

PALATINATE PURPLE

The colour scheme of this presentation is based on the colour PALATINATE PURPLE which — as you probably know — is the cornerstone of Durham University's official corporate colour palette.

PALATINATE PURPLE

The colour scheme of this presentation is based on the colour PALATINATE PURPLE which — as you probably know — is the cornerstone of Durham University's official corporate colour palette.

Colour has a major part to play in the building of any brand. It is vital that we consider the use of colour carefully across all our communications. You will see that the colours we have chosen have an inviting warmth, distinctive quality and a timeless style whilst not feeling dated.

Primary corporate colours

The Durham University identity uses two colours - a single colour (255C) from the Pantone Matching System and black.

Secondary corporate colours

In addition to our corporate colour, there is a secondary colour palette to give you even greater flexibility and variety when you're producing communications for the University.

Colour	Pantone	Process	RGB	Hex
	Black C	C:0 M:0 Y:0 K:100	R:35 G:31 B:32	321F20
	255 C	C:51 M:91 Y:0 K:34	R:126 G:49 B:123	7E317B
	257 C	C:15 M:38 Y:00 K:00	R:216 G:172 B:244	D8ACE0
	634 C	C:100 M:0 Y:8.5 K:47	R:0 G:99 B:136	006388
	201C	C:0 M:100 Y:65 K:34	R:170 G:43 B:74	AA2B4A

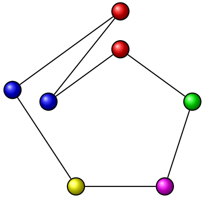
GRAPH HOMOMORPHISMS

A *(graph) homomorphism* from a graph X to a graph Y is a function

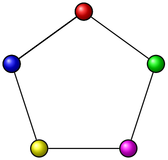
$$\varphi : V(X) \to V(Y)$$

such that

$$xy \in E(X) \Rightarrow \varphi(x)\varphi(y) \in E(Y).$$

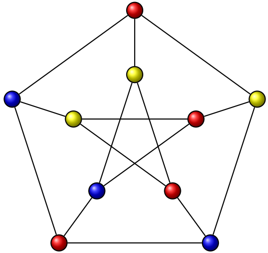


$C_7 \to C_5$

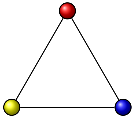


HOMOMORPHISMS AND COLOURINGS

A homomorphism from X to a *clique* K_q is a q-colouring of X.



$$P \to K_3$$



The *chromatic number* $\chi(X)$ of X is the *smallest* q such that

$$X \to K_q.$$

FRACTIONAL COLOURINGS

A *fractional colouring* of X is a real-valued function

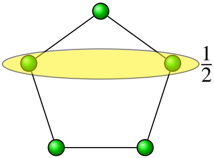
$$f : \mathcal{I}(X) \to \mathbb{R}$$

where $\mathcal{I}(X)$ is the set of independent (stable) sets of X, with the property that for every vertex v,

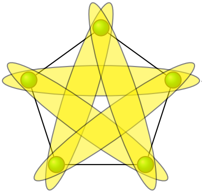
$$\sum_{I : v \in I} f(I) \geq 1.$$

The *fractional chromatic number* of X is $\chi^*(X) = \inf_f \sum f(I)$.

THE 5-CYCLE



THE 5-CYCLE



A fractional colouring of C_5 with weight $5/2$.

FRACTIONAL COLOURING AND HOMOMORPHISMS

The fractional chromatic number of a graph X is a *rational number* and equal to the minimum value of v/r such that

$$X \to K(v,r)$$

where $K(v,r)$ is the *Kneser graph* whose vertices are all the r-subsets of a v-set and where two vertices are adjacent if they are disjoint.

Many *other variants* of graph colouring can be expressed as the existence of homomorphisms to some family of graphs.

COMPLEXITY

Finding homomorphisms is *theoretically difficult*.

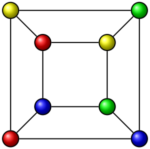
Y-COLOURING
INSTANCE: A graph X
QUESTION: Is there a homomorphism $X \to Y$.

Hell & Nešetřil showed that this problem is *NP*-complete for any non-bipartite graph Y.

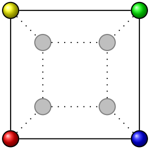
(This is a strong – but unsurprising – result.)

ENDOMORPHISMS

An *endomorphism* of X is a homomorphism from X to itself.



$Q_3 \to Q_3$



Under composition of mappings, the set of all endomorphisms of X forms the *endomorphism monoid* $\mathrm{End}(X)$.

As *automorphisms* are clearly endomorphisms, we have

$$\mathrm{Aut}(X) \subseteq \mathrm{End}(X).$$

COMPUTING

Given a graph X, how can we find $\mathrm{Aut}(X)$ and/or $\mathrm{End}(X)$?

- Finding $\mathrm{Aut}(X)$ is of unknown theoretical complexity, but in practice is *easy*.

  The software `nauty/Traces` *by McKay/Piperno is spectacularly good — recently I found the automorphism group of an arc-transitive* 10*-regular graph with* 76422528 *vertices in under an hour.*

- Finding $\mathrm{End}(X)$ is theoretically intractable, and in practice is *difficult*.

  There are graphs with as few as 45 *vertices that we cannot do in a reasonable time.*

"A week's programming can sometimes save an hour's thought!"

SYNCHRONISING GROUPS

From Wolfram's earlier talk —

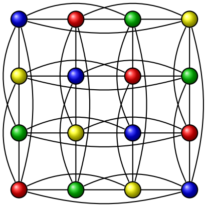
If $G \leq \mathrm{Sym}(\Omega)$ and $f \in \mathrm{T}(\Omega)$ and $S = \langle G, f \rangle$ then

- G *synchronises* f if S contains a constant function.
- G is a *synchronising group* if G synchronises *every* transformation $f \in \mathrm{T}(\Omega) \setminus \mathrm{Sym}(\Omega)$.

We know that a synchronising group must be *primitive*.

PRIMITIVE BUT NOT SYNCHRONISING

Not *all* primitive groups are synchronising:



The Cartesian product $X = K_4 \square K_4$ has primitive automorphism group $G = \mathrm{Sym}(4) \wr \mathrm{Sym}(2)$, a 4-clique, and a 4-colouring.

As the *colouring map* f is an endomorphism, so is every element of $\langle G, f \rangle$, and therefore G does not synchronise f.

THE GRAPH CONNECTION

The converse is also true.

A primitive group G is synchronising *if and only if*

$$\chi(X) > \omega(X)$$

for *every* non-trivial G-invariant graph X.

VERTEX-PRIMITIVE GRAPHS

If G is a primitive group acting on Ω, then the orbits of G on $\Omega \times \Omega$ are called the *orbitals* of G.

A *digraph* on Ω is G-invariant if and only if its *arc set* is a union of orbitals.

Undirected graphs arise provided every orbital containing, say (v, w), is "paired-up" with the orbital containing (w, v) (which is often itself).

So if G has k orbital pairs, then there are $2^k - 2$ graphs to examine.

Good news and bad news

Good news

For any *specific* primitive group G of reasonable size, this gives an implementable algorithm[1] for determining whether it is synchronising.

Bad news

For *families* of primitive groups, the existence of graphs with suitable cliques and colourings is equivalent to well-known difficult problems in, for example, finite geometry.

[1] with some more or less significant caveats

ALMOST SYNCHRONISING

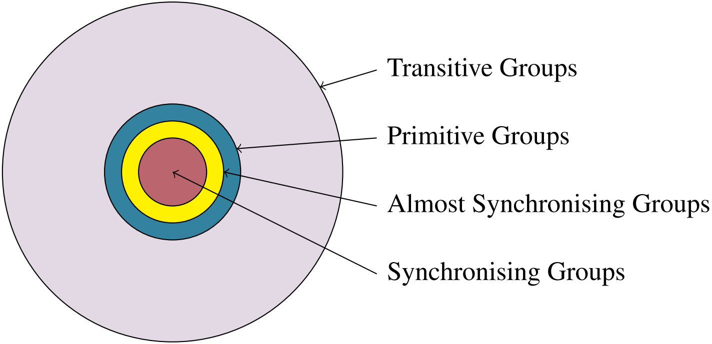
χ-colourings of vertex-transitive graphs with $\omega = \chi$ are *uniform transformations* — each colour class has the same size.

Perhaps uniform transformations are the *only reason* that some primitive groups are not synchronising?

DEFINITION A primitive group is *almost synchronising* if it synchronises every *non-uniform* transformation.

CONJECTURE Primitive groups are *almost synchronising*

THE LANDSCAPE



The transitive group landscape (not to scale) — the conjecture is that  is empty.

THE GRAPH CONNECTION, AGAIN

The arguments about groups, graphs and non-synchronised transformations carry over essentially unchanged:

PROPOSITION

The primitive group G fails to synchronise the transformation f if and only if there is a non-trivial G-invariant graph X such that $\chi(X) = \omega(X)$ and

$$f \in \mathrm{End}(X).$$

Thus the question involves determining (elements of) the endomorphism monoid of a graph.

Araújo & Cameron

Araújo & Cameron made progress at both ends of the "rank-spectrum" — primitive groups of degree n synchronise all non-uniform transformations of ranks 3, 4 and $n-2$.

Journal of Combinatorial Theory, Series B

Volume 106, May 2014, Pages 98–114

Primitive groups synchronize non-uniform maps of extreme ranks

João Araújo[a, b], Peter J. Cameron[c]

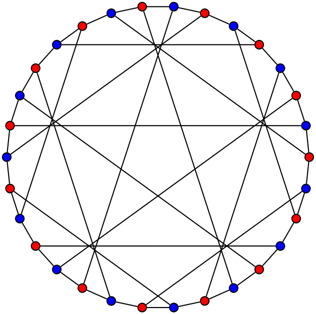
Show more

doi:10.1016/j.jctb.2014.01.006

Get rights and content

THE TUTTE-COXETER GRAPH

The *Tutte-Coxeter graph* — the incidence graph of the unique $GQ(2,2)$ — is a cubic 30-vertex graph of girth 8.



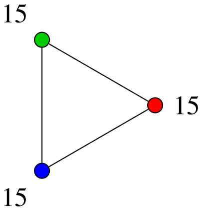
The chords of the non-ruled quadric in $PG(3,3)$, *Can. J. Math*, **10** (1958), 481–483. (Tutte)

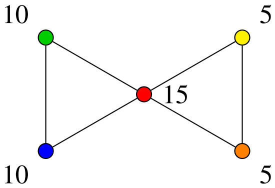
The chords of the non-ruled quadric in $PG(3,3)$, *Can. J. Math*, **10** (1958), 484–488. (Coxeter)

A COUNTEREXAMPLE

The *line graph* of Tutte-Coxeter is a 45-vertex quartic graph, which is 3-colourable, thus has 3 colour classes each of size 15.

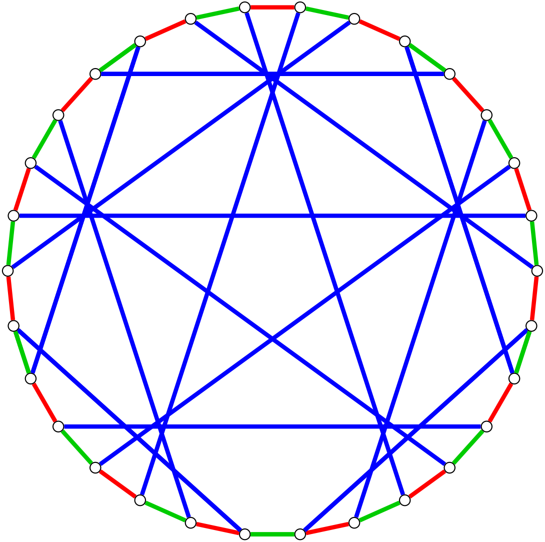
However, as well as this – necessarily uniform – endomorphism onto a triangle, it also has a *non-uniform* endomorphism onto a "*butterfly*" (aka *"bowtie"*).



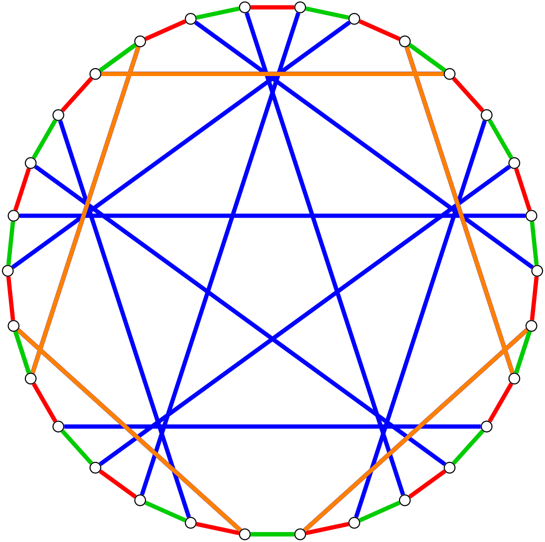


Thus, the conjecture takes on water at rank 5....

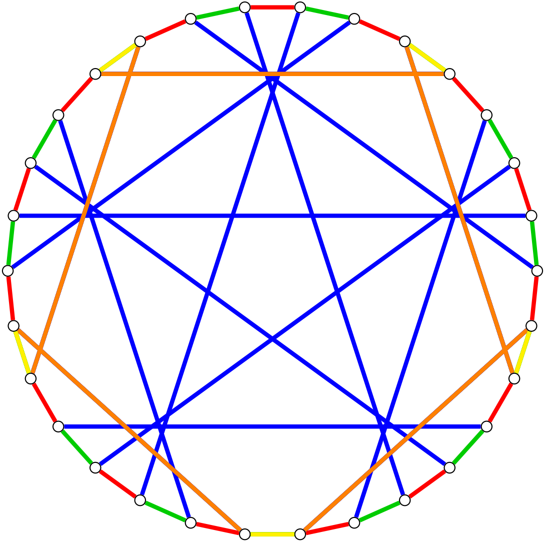
In a picture



IN A PICTURE



IN A PICTURE



MORE LIKE THIS?

The *Biggs-Smith* graph (BS) is a cubic 102-vertex distance-regular graph with automorphism group $PSL(2, 17)$ acting primitively on its edges.

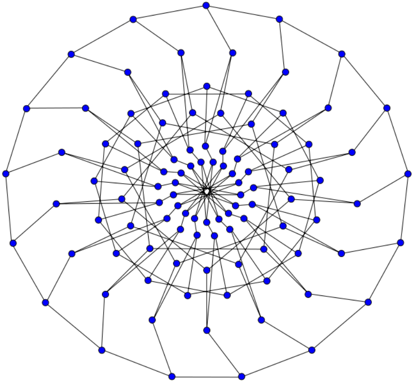


Image from Wikipedia

ANOTHER BUTTERFLY . . .

Letting L denote the line-graph of BS, we have

- L is *vertex-primitive*
- Every closed neighbourhood of L is a butterfly
- L has an endomorphism of kernel type $(6, 6, 45, 45, 51)$

. . . BUT NO MORE

Weiss[2] showed that the only *edge-primitive cubic graphs* are

- The complete bipartite graph $K_{3,3}$,
- The *Heawood* graph (incidence graph of $PG(2,2)$),
- The *Tutte-Coxeter* graph,
- The *Biggs-Smith* graph.

As a vertex-primitive graph of degree 4 with closed neighbourhood equal to a butterfly must be the line graph of one of these graphs, we are done.

[2] *Kantenprimitive Graphen vom Grad drei, JCT-B, 1973*

COMPUTER SEARCHING

Therefore to conduct a systematic search we need:

- Some *primitive groups*

  Thanks to *Colva Roney-Dougal*, these are available up to degree *4095* (way more than we need) in MAGMA.

- Some *vertex-primitive graphs* stabilised by these groups

  This is a few lines of code in either MAGMA or GAP.

- Some *endomorphisms* of these graphs

  This is a few lines of code in *James Mitchell's* packages in GAP.

Other people — many of whom are here — have also contributed to these software tools, packages and libraries

FINDING ENDOMORPHISMS — MINION

MINION is a freely available *constraint satisfaction problem* (CSP) solver developed at St Andrews.

If a problem can be expressed as a CSP, then MINION is often extremely effective: here is the code for a specific 45-vertex graph.

```
MINION 3
**VARIABLES**
DISCRETE v[45] {0..44}
```

The only variable is the 45-element array `v` to hold the endomorphism in image form.

THE TABLE

A list of 2-tuples, one for each arc of the graph (i.e. two per edge) is created and called `gr` (for graph) to be referred to later.

```
**TUPLELIST**
gr 180 2
1 0
0 1
2 0
...
...
41 44
43 42
42 43
```

THE CONSTRAINTS

```
**CONSTRAINTS**
eq(v[0],0)
lighttable([v[0],v[1]],gr)
lighttable([v[0],v[2]],gr)
...
lighttable([v[41],v[44]],gr)
lighttable([v[42],v[43]],gr)
**EOF**
```

The `eq` constraint sets the image of vertex 0 to be 0.

The important constraints — one for each edge have the following form:

```
lighttable([v[41],v[44]],gr)
```

says that the tuple `[v[44],v[45]]` must be in the tuple-list `gr`.

FINDING ENDOMORPHISMS — GAP

With recent developments MINION has mostly been eclipsed by GAP software from *St Andrews*.

```
gap> d := Digraph([
[59,64,77,148],
[63,71,112,136],
...
[13,46,68,78],
[14,62,70,131]]);
<digraph with 45 vertices, 180 edges>
gap>
gap> gens := GeneratorsOfEndomorphismMonoid(d);;
gap> Size(gens);
331
```

The generators are calculated in a fraction of a second.

THE SEMIGROUP

```
gap> s := SemigroupByGenerators(gens);
<transformation monoid on 45 pts with 330 generators>
gap> Size(s);
105120
gap> time;
376
gap>
gap> k := KernelOfTransformation(Random(s),45);;
gap> List(k,i->Size(i));
[ 10, 5, 15, 10, 5 ]
```

This graph has 25920 non-uniform endomorphisms of rank 7, and 51840 of rank 5.

SEARCH RESULTS

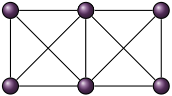
Systematic search confirms that the linegraph of the Tutte-Coxeter graph is indeed the smallest example.

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
16	28	6	336	1^{28} (12),7^4 (6144)
17	28	12	40320	1^{28} (1440), 4^7 (8985600)
18	28	15	336	1^{28} (12), 4^7 (305280)
19	28	18	336	1^{28} (12), 4^7 (11520)
20	28	18	336	1^{28} (12), 4^7 (23040)
21	28	21	336	1^{28} (12), 4^7 (51840)
22	35	18	40320	1^{35} (1152), 5^7 (1036800)
23	36	10	1036800	1^{36} (28800), 6^6 (270950400)
24	36	25	1036800	1^{36} (28800), 6^6 (28800)
25	45	4	1440	1^{45} (32), 5^2+10^2+15^1 (1152), 5^5+10^2 (576), 15^3 (576)
26	45	12	51840	1^{45} (1152), 9^5 (37440)

COLLECTING BUTTERFLIES

Partial searches — just on the *lower degree* graphs with more than 45 vertices — unearthed a number of other examples:

- A Cayley graph of $\mathbb{Z}_2^6$ with a rank 6 endomorphism whose image is a *"wide body"* butterfly

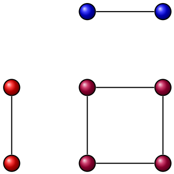
  This example generalises to an infinite family, all of rank 6
- Two 495 vertex graphs arising from two primitive representations of M_{12}.

NON-SYNCHRONISING RANKS

For a fixed degree n, say that a rank r is *non-synchronising* if there is some primitive group of degree n that fails to synchronise some transformation of degree n and rank r.

Peter Cameron asked whether (conjectured that?) the *number* of non-synchronising ranks is always small, say $O(\log n)$.

This can be shown to be *false* by a construction based on the *Cartesian product* of graphs.



$K_2 \square K_2$

Transitivity

Let X be a vertex-primitive graph with $\chi(X) = \omega(X) = k$, so

$$V(X) = C_1 \cup \ldots \cup C_k$$

where each C_i is an independent set.

Then $X \,\square\, X$ is *also* a vertex-primitive graph and there is a homomorphism

$$X \,\square\, X \to K_k \,\square\, K_k$$

with kernel classes $C_i \times C_j$.

If there were a homomorphism $\varphi : K_k \,\square\, K_k \to X$ then by transitivity

$$X \,\square\, X \to K_k \,\square\, K_k \xrightarrow{\varphi} X \to X \,\square\, X.$$

What's the point?

We hope to find *endomorphisms* of $X \,\square\, X$ (a big graph), by examining *homomorphisms* from $K_k \,\square\, K_k$ to X (small graphs).

So, what will we use for X first?

WHAT'S THE POINT?

We hope to find *endomorphisms* of $X \,\square\, X$ (a big graph), by examining *homomorphisms* from $K_k \,\square\, K_k$ to X (small graphs).

So, what will we use for X first?

For *maximum confusion*, we'll take $X = \overline{K_k \,\square\, K_k}$!

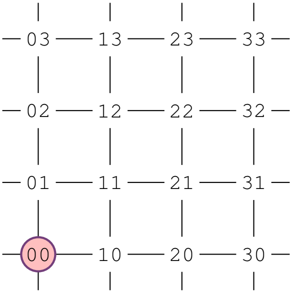
```
   |      |      |      |
 - 03 — 13 — 23 — 33 -
   |      |      |      |
 - 02 — 12 — 22 — 32 -
   |      |      |      |
 - 01 — 11 — 21 — 31 -
   |      |      |      |
 - 00 — 10 — 20 — 30 -
   |      |      |      |
```

HOMOMORPHISMS TO THE COMPLEMENT

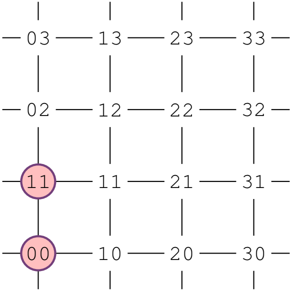
```
   |      |      |      |
 — 03 —— 13 —— 23 —— 33 —
   |      |      |      |
 — 02 —— 12 —— 22 —— 32 —
   |      |      |      |
 — 01 —— 11 —— 21 —— 31 —
   |      |      |      |
 — 00 —— 10 —— 20 —— 30 —
   |      |      |      |
```

.

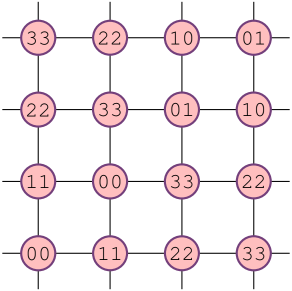
Homomorphisms to the complement



Homomorphisms to the complement

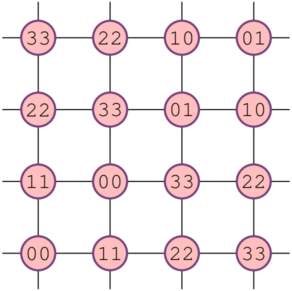


HOMOMORPHISMS TO THE COMPLEMENT



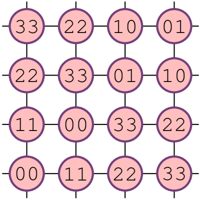
.

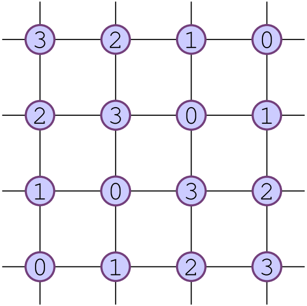
HOMOMORPHISMS TO THE COMPLEMENT

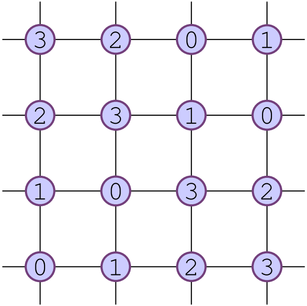


This gives a endomorphism of $\overline{(K_4 \square K_4)} \square \overline{(K_4 \square K_4)}$ of rank 6 with kernel type $(32, 32, 32, 32, 64, 64)$.

THE RULES


33 22 10 01
22 33 01 10
11 00 33 22
00 11 22 33


3 2 1 0
2 3 0 1
1 0 3 2
0 1 2 3


3 2 0 1
2 3 1 0
1 0 3 2
0 1 2 3

OVERLAPPING LATIN SQUARES

So *any two Latin squares* of order k will suffice — the *rank* of the resulting homomorphism is just the number of *distinct pairs*.

THEOREM[3] There are two r-orthogonal Latin squares of order k if and only if $r \in \{k, k^2\}$ or $k + 2 \leq r \leq k^2 - 2$, except for

- $k = 2$ and $r = 4$;
- $k = 3$ and $r \in \{5, 6, 7\}$;
- $k = 4$ and $r \in \{7, 10, 11, 13, 14\}$;
- $k = 5$ and $r \in \{8, 9, 20, 22, 23\}$;
- $k = 6$ and $r \in \{33, 36\}$.

So there are vertex-primitive graphs with k^4 vertices with endomorphisms of about k^2 different ranks.

[3] Belyavskaya, Colbourn & Zhu, Zhu & Zhang

OTHER APPROACHES

We could try fixing the *degree* d and increasing the size — this leads to characterising vertex-primitive graphs with an orbital of size d.

$\mathrm{Aut}(\Gamma)$	$\mathrm{Aut}(\Gamma)_v$	s	$\lvert V(\Gamma)\rvert$	Notes
$\mathbb{Z}_2^4 \rtimes \mathrm{Sym}(5)$	$\mathrm{Sym}(5)$	2	16	Clebsch
P`L(2, 9)	$\mathrm{AGL}(1,5) \times \mathbb{Z}_2$	2	36	Sylvester
$\mathrm{PGL}(2,11)$	D_{10}	1	66	
$\mathrm{Sym}(9)$	$\mathrm{Sym}(4) \times \mathrm{Sym}(5)$	3	126	Odd(5)
$\mathrm{Suz}(8)$	$\mathrm{AGL}(1,5)$	2	1 456	
$\mathrm{J}_3 \rtimes 2$	A`L(2, 4)	4	17 442	
Th	$\mathrm{Sym}(5)$	2	756 216 199 065 600	
$\mathrm{PSL}(2,p)$	$\mathrm{Alt}(5)$	2	$\frac{p^3-p}{120}$	$p \equiv \pm 1, \pm 9 \pmod{40}$
$\mathrm{PSL}(2,p^2)$	$\mathrm{Alt}(5)$	2	$\frac{p^6-p^2}{120}$	$p \equiv \pm 3 \pmod{10}$
P˚L$(2,p^2)$	$\mathrm{Sym}(5)$	2	$\frac{p^6-p^2}{120}$	$p \equiv \pm 3 \pmod{10}$
$\mathrm{PSp}(6,p)$	$\mathrm{Sym}(5)$	2	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{240}$	$p \equiv \pm 1 \pmod{8}$
$\mathrm{PCSp}(6,p)$	$\mathrm{Sym}(5)$	2	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{120}$	$p \equiv \pm 3 \pmod{8}, p \geq 11$

This table, for $d = 5$ is from uncompleted work by Fawcett, Giudici, Li, Praeger, Royle & Verret.

CONCLUSION

In summary,

- Primitive groups synchronise *low-* and *high-rank* transformations.
- The *almost-synchronising conjecture* is false ...
- ... and there are lots of *non-synchronising ranks*,
- but the full story is still not known.