Categorical Constructions and the Ramsey Property

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Important notice

Thesis

Thesis. *Category theory is an appropriate context for understanding Ramsey property.*

Thesis

R. L. GRAHAM, K. LEEB, B. L. ROTHSCHILD: *Ramsey's theorem for a class of categories.* Adv. Math. 8 (1972) 417–443.

H. J. PRÖMEL, B. VOIGT: *Hereditary attributes of surjections and parameter sets.* European J. Combin. 7 (1986) 161–170.

J. NEŠETŘIL: *Ramsey classes and homogeneous structures.* Combinatorics, probability and computing, 14 (2005) 171–189.

L. NGUYEN VAN THÉ: *Universal flows of closed subgroups of S_∞ and relative extreme amenability.* Asymptotic Geometric Analysis, Fields Institute Communications Vol. 68, 2013, 229–245.

S. SOLECKI: *Dual Ramsey theorem for trees.* arXiv:1502.04442v1.

Outline

$$\boxed{\text{Ramsey property}}$$

$$\downarrow$$

$$\downarrow \ \textit{abstract interpretation}$$

$$\downarrow$$

$$\boxed{\begin{array}{c}\text{Ramsey property}\\ \text{in a category}\end{array}}$$

Outline

$$
\boxed{\text{Ramsey property}}
$$

$$
\Big\downarrow \ \textit{abstract interpretation}
$$

$$
\boxed{\begin{array}{c}\text{Ramsey property}\\ \text{in a category}\end{array}}
$$

$$
\Big\downarrow \ \textit{cat equivalence}
$$

$$
\boxed{\begin{array}{c}\text{Ramsey property}\\ \text{for other cat's}\end{array}}
$$

Outline

$$
\begin{array}{c}
\boxed{\text{Ramsey property}} \\
\downarrow \\
\downarrow \ \textit{abstract interpretation} \\
\downarrow \\
\boxed{\begin{array}{c}\text{Ramsey property}\\ \text{in a category}\end{array}} \\
\downarrow \\
\downarrow \ \textit{dual equivalence} \\
\downarrow \\
\boxed{\begin{array}{c}\text{Dual Ramsey property}\\ \text{for categories}\end{array}}
\end{array}
$$

Outline

Ramsey property

$\downarrow$

$\downarrow$ *abstract interpretation*

$\downarrow$

Ramsey property
in a category

$\downarrow$

$\downarrow$ *products of cat's*

$\downarrow$

Product Ramsey
theorems

Outline

A. S. KECHRIS, V. G. PESTOV, S. TODORČEVIĆ: *Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups.* GAFA Geometric and Functional Analysis, 15 (2005) 106–189.

Ingredients:

- Fraïssé theory,
- structural Ramsey theory,
- topological dynamics.

Outline

Homogeneity	•	Ramsey prop	•	Extreme amenability

↓
↓ *abstract interpretation*
↓

Homogeneity in a category	•	Ramsey prop in a category	•	Extreme amenability w.r.t. particular topology

Outline

Homogeneity •	Ramsey prop •	Extreme amenability

↓
↓ *abstract interpretation*
↓

Homogeneity •	Ramsey prop •	Extreme amenability
in a category	in a category	w.r.t. particular topology

↓
↓ *specialization*
↓

Homogeneity • Ramsey prop • Extreme amenability
for ultrahomog structs that are not Fraïssé limits
(e.g. uncountable ulrahomog structs)

Outline

Homogeneity •	Ramsey prop •	Extreme amenability

$\downarrow$
$\downarrow$ *abstract interpretation*
$\downarrow$

Homogeneity in a category •	Ramsey prop in a category •	Extreme amenability w.r.t. particular topology

$\downarrow$
$\downarrow$ *categorical duality*
$\downarrow$

Projective Homogeneity •	Dual Ramsey prop •	Extreme amenability

T. IRWIN, S. SOLECKI: *Projective Fraïssé limits and the pseudo-arc.* Trans. Amer. Math. Soc. 358, no. 7 (2006) 3077–3096.

Finite Ramsey Theorem.
For all $a, b \in \mathbb{N}$ and $k \geqslant 2$ there is a $c \in \mathbb{N}$ such that for every c-element set C and every coloring

$$\chi : \binom{C}{a} \to k$$

there is a b-element set $B \subseteq C$ such that $|\chi(\binom{B}{a})| = 1$.

Frank P. Ramsey
1903 – 1930
Image courtesy of Wikipedia

R. L. GRAHAM, B. L. ROTHSCHILD, J. H. SPENCER: *Ramsey Theory (2nd Ed).* John Wiley & Sons, 1990.

Finite Product Ramsey Theorem. *For all* $s, a_1, \ldots, a_s$, $b_1, \ldots, b_s \in \mathbb{N}$ *and* $k \geqslant 2$ *there exist* $c_1, \ldots, c_s \in \mathbb{N}$ *such that for all sets* $C_1, \ldots, C_s$ *of cardinalities* $c_1, \ldots, c_s$*, respectively, and every* k*-coloring of the set* $\binom{C_1}{a_1} \times \ldots \times \binom{C_s}{a_s}$ *there exist* $B_1 \subseteq C_1$ *of cardinality* b_1*, …,* $B_s \subseteq C_s$ *of cardinality* b_s *such that* $\binom{B_1}{a_1} \times \ldots \times \binom{B_s}{a_s}$ *is monochromatic.*

R. L. GRAHAM, B. L. ROTHSCHILD: *Ramsey's theorem for n-parameter sets.* Tran. Amer. Math. Soc. 159 (1971), 257–292.

Finite Dual Ramsey Theorem. *For all $a, b \in \mathbb{N}$ and $k \geqslant 2$ there is a $c \in \mathbb{N}$ such that for every c-element set C and every k-coloring of the set $\left[{C \atop a}\right]$ of all partitions of C with exactly a blocks there is a partition β of C with exaclty b blocks such that the set of all patitions from $\left[{C \atop a}\right]$ which are coarser than β is monochromatic.*

Structural Ramsey theory

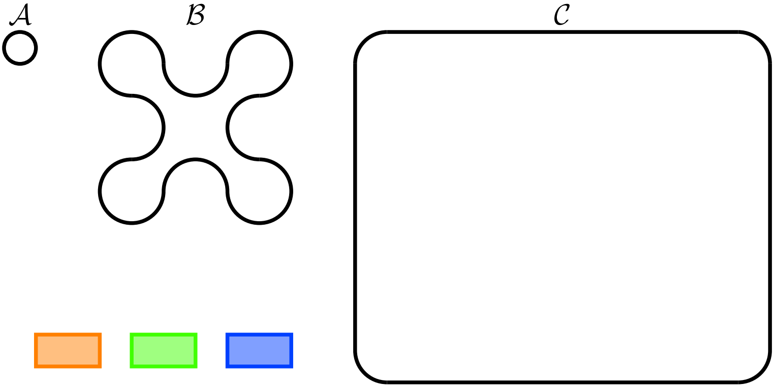
Deep structural property developed in the 1970's by Erdős, Graham, Leeb, Rothschild, Rödl, Nešetřil and many more.

Instead of sets, consider structures!

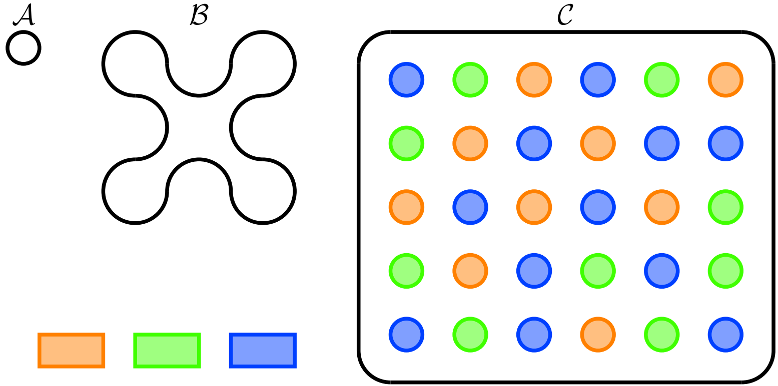
Definition. A class **K** of finite structures has the *Ramsey property* if:

for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{B}$ and all $k \geqslant 2$ there is a $\mathcal{C} \in \mathbf{K}$ such that $\longrightarrow$

Structural Ramsey theory

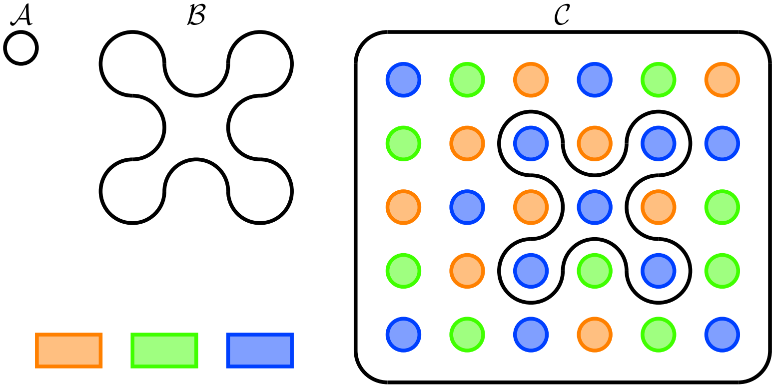


Structural Ramsey theory



for every coloring $\chi : \binom{\mathcal{C}}{\mathcal{A}} \to k$

Structural Ramsey theory



there is a $\tilde{\mathcal{B}} \in \binom{\mathcal{C}}{\mathcal{B}}$ such that $\left|\chi\left(\binom{\tilde{\mathcal{B}}}{\mathcal{A}}\right)\right| = 1$.

Categories

In order to specify a category $\mathbb{C}$ one has to specify:

1. a class of objects $\mathrm{Ob}(\mathbb{C})$,
2. a set of morphisms $\hom(A, B)$ for all $A, B \in \mathrm{Ob}(\mathbb{C})$,
3. an identity morphism id_A for all $A \in \mathrm{Ob}(\mathbb{C})$, and
4. the composition of morphisms $\cdot$ so that
   - $(f \cdot g) \cdot h = f \cdot (g \cdot h)$, and
   - $\mathrm{id}_B \cdot f = f \cdot \mathrm{id}_A$ for all $f \in \hom(A, B)$.

Ramsey properties in a category

J. NEŠETŘIL: *Ramsey classes and homogeneous structures.*
Combinatorics, probability and computing, 14 (2005) 171–189.

$\binom{B}{A} = \hom(A,B)/{\sim_A}$

- $f \sim_A g$ for $f, g \in \hom(A,B)$ iff $f = g \cdot \alpha$ for some $\alpha \in \mathrm{Aut}(A)$.

A category $\mathbb{C}$ has the *Ramsey property for objects* if:

for all $k \geqslant 2$ and all $A, B \in \mathrm{Ob}(\mathbb{C})$ such that $\hom(A,B) \neq \varnothing$ there is a $C \in \mathrm{Ob}(\mathbb{C})$ such that for every $\mathbb{SET}$-mapping $\chi : \binom{C}{A} \to k$ there is a $\mathbb{C}$-morphism $w : B \to C$ such that $|\chi(w \cdot \binom{B}{A})| = 1$.

Ramsey properties in a category

Example. $\mathbb{SET}_{\text{fin}}$

- objects are finite sets
- $\hom(A, B) =$ injective maps $A \to B$,
- identity is the identity map,
- composition: $f \cdot g = f \circ g$.

$\mathbb{SET}_{\text{fin}}$ has the Ramsey property for objects.

This is the finite Ramsey theorem.

Ramsey properties in a category

Example. $\mathbb{SET}_{\mathsf{fin}}^{(\twoheadleftarrow)}$

- objects are finite sets
- $\hom(A, B) =$ surjective maps $A \twoheadleftarrow B$,
- identity is the identity map,
- composition: $f \cdot g = g \circ f$.

$\mathbb{SET}_{\mathsf{fin}}^{(\twoheadleftarrow)}$ has the Ramsey property for objects.

This is the finite dual Ramsey theorem.

Ramsey properties in a category

Example. $\mathbb{BA}_{\text{fin}}$

- objects are finite boolean algebras
- $\text{hom}(A, B) =$ embeddings $A \to B$,
- identity is the identity map,
- composition: $f \cdot g = f \circ g$.

$\mathbb{BA}_{\text{fin}}$ has the Ramsey property for objects.

This is the finite Ramsey theorem for boolean algebras.

Ramsey properties in a category

A category $\mathbb{C}$ has the *Ramsey property for morphisms* if:

for all $k \geqslant 2$ and all $A, B \in \mathrm{Ob}(\mathbb{C})$ such that $\hom(A, B) \neq \varnothing$ there is a $C \in \mathrm{Ob}(\mathbb{C})$ such that for every $\mathbb{SET}$-mapping $\chi : \hom(A, C) \to k$ there is a $\mathbb{C}$-morphism $w : B \to C$ such that $|\chi(w \cdot \hom(A, B))| = 1$.

Ramsey properties in a category

Example. $\mathbb{CH}_{\text{fin}}$

- objects are finite chains $(\{1, \dots, n\}, \leqslant)$, $n \geqslant 1$
- $\hom(A, B) =$ embeddings $A \to B$,
- identity is the identity map,
- composition: $f \cdot g = f \circ g$.

Ramsey property for $\mathbb{CH}_{\text{fin}} \iff$ Ramsey property for finite chains.

Ramsey properties in a category

Example. $\mathbb{CH}_{\text{fin}}^{(\twoheadleftarrow)}$

- objects are finite chains $(\{0, 1, \ldots, n\}, \leqslant)$, $n \geqslant 1$
- $\hom(A, B) =$ surjective monotonous maps $A \twoheadleftarrow B$,
- identity is the identity map,
- composition: $f \cdot g = g \circ f$.

Ramsey property for $\mathbb{CH}_{\text{fin}}^{(\twoheadleftarrow)} \iff$ *dual* Ramsey property for partitions of finite chains into intervals.

Ramsey property and categorical equivalence

Proposition. *Assume that* $\mathbb{C}$ *and* $\mathbb{D}$ *are equivalent categories. Then* $\mathbb{C}$ *has the Ramsey property for morphisms (objects) iff* $\mathbb{D}$ *has the Ramsey property for morphisms (objects).*

- Categories $\mathbb{C}$ and $\mathbb{D}$ are *equivalent* if there exist functors $E : \mathbb{C} \to \mathbb{D}$ and $H : \mathbb{D} \to \mathbb{C}$, and natural isomorphisms $\eta : \mathrm{ID}_{\mathbb{C}} \to HE$ and $\varepsilon : \mathrm{ID}_{\mathbb{D}} \to EH$.

Ramsey property and categorical equivalence

Example. Finite Stone duality:

$$\boxed{\text{Finite Ramsey Theorem}} \iff \boxed{\text{Dual Ramsey Theorem for Finite BA's}}$$

Dual Ramsey Theorem for Finite BA's. *Let* $\mathrm{Con}(\mathcal{B})$ *denote the set of congruences of* $\mathcal{B}$*, and let*

$$\mathrm{Con}(\mathcal{B}, \mathcal{A}) = \{\Phi \in \mathrm{Con}(\mathcal{B}) : \mathcal{B}/\Phi \cong \mathcal{A}\}.$$

For every finite bolean algebra $\mathcal{B}$*, every* $\Phi \in \mathrm{Con}(\mathcal{B})$ *and every* $k \geqslant 2$ *there is a finite boolean algebra* $\mathcal{C}$ *such that for every* k*-coloring of* $\mathrm{Con}(\mathcal{C}, \mathcal{B}/\Phi)$ *there is a congruence* $\Psi \in \mathrm{Con}(\mathcal{C}, \mathcal{B})$ *such that the set of all congruences from* $\mathrm{Con}(\mathcal{C}, \mathcal{B}/\Phi)$ *which are coarser than* Ψ *is monochromatic.*

Ramsey property and categorical equivalence

Example. Finite Stone duality:

$$\boxed{\begin{matrix}\text{Finite Dual}\\ \text{Ramsey Theorem}\end{matrix}} \iff \boxed{\begin{matrix}\text{Ramsey Theorem}\\ \text{for Finite BA'a}\end{matrix}}$$

Example. Hu's equivalence:

$$\boxed{\begin{matrix}\text{Ramsey Theorem}\\ \text{for Finite BA'a}\end{matrix}} \iff \boxed{\begin{matrix}\text{Ramsey Theorem for}\\ \text{Finite Powers of a Primal Algebra}\end{matrix}}$$

Ramsey property and categorical equivalence

Example. By the standard duality of fin dim vector spaces:

$$\boxed{\begin{array}{c}\text{Ramsey Theorem}\\ \text{for Finite Vector Spaces}\end{array}} \iff \boxed{\begin{array}{c}\text{Dual Ramsey Theorem}\\ \text{for Finite Vector Spaces}\end{array}}$$

Dual Ramsey Theorem for Finite Vector Spaces. *Let F be a finite field and for a vector space V over F let*

$$\left[{V \atop d}\right]_{\text{lin}} = \{V/W : W \leqslant V, \dim(V/W) = d\}.$$

For all $a, b \in \mathbb{N}$ and $k \geqslant 2$ there is a $c \in \mathbb{N}$ such that for every c-dimensional vector space V over F and every k-coloring of $\left[{V \atop a}\right]_{\text{lin}}$ there is a partition $\beta \in \left[{V \atop b}\right]_{\text{lin}}$ such that the set of all patitions from $\left[{V \atop a}\right]_{\text{lin}}$ which are coarser than β is monochromatic.

Ramsey property and products of categories

Theorem. *Assume that both* $\mathbb{C}$ *and* $\mathbb{D}$ *satisfy the following finiteness condition:*

- $\hom(A, B)$ *is finite for all objects* A *and* B *in the category.*

If both $\mathbb{C}$ *and* $\mathbb{D}$ *have the Ramsey property for morphisms (objects) then* $\mathbb{C} \times \mathbb{D}$ *has the Ramsey property for morphisms (objects).*

- Objects of $\mathbb{C} \times \mathbb{D}$ are pairs (A, B) where $A \in \mathrm{Ob}(\mathbb{C})$ and $B \in \mathrm{Ob}(\mathbb{D})$.
- Morphisms of $\mathbb{C} \times \mathbb{D}$ are pairs $(f, g) : (A_1, B_1) \to (A_2, B_2)$ where $f : A_1 \to A_2$ in $\mathbb{C}$ and $g : B_1 \to B_2$ in $\mathbb{D}$.
- The composition is componentwise.

Ramsey property and products of categories

Corollary. *Assume that* $\hom(A, B)$ *is finite for all* $A, B \in \mathrm{Ob}(\mathbb{C})$*. If* $\mathbb{C}$ *has the Ramsey property for morphisms (objects) then* $\mathbb{C}^n$ *has the Ramsey property for morphisms (objects).*

Ramsey property and products of categories

Corollary. *Assume that* $\hom(A, B)$ *is finite for all* $A, B \in \mathrm{Ob}(\mathbb{C})$. *If* $\mathbb{C}$ *has the Ramsey property for morphisms (objects) then* $\mathbb{C}^n$ *has the Ramsey property for morphisms (objects).*

Metatheorem. *Every Ramsey property for classes of finite structures, be it "direct" or dual, has the finite product version.*

Ramsey property and products of categories

Corollary. *Assume that* $\hom(A, B)$ *is finite for all* $A, B \in \mathrm{Ob}(\mathbb{C})$*. If* $\mathbb{C}$ *has the Ramsey property for morphisms (objects) then* $\mathbb{C}^n$ *has the Ramsey property for morphisms (objects).*

Metatheorem. *Every Ramsey property for classes of finite structures, be it "direct" or dual, has the finite product version.*

Example.

$$\boxed{\begin{array}{c}\text{Finite}\\ \text{Ramsey Theorem}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Finite Product}\\ \text{Ramsey Theorem}\end{array}}$$

R. L. GRAHAM, B. L. ROTHSCHILD, J. H. SPENCER: *Ramsey Theory (2nd Ed).* John Wiley & Sons, 1990.

Ramsey property and products of categories

Example.

$$\boxed{\begin{array}{c}\text{Finite}\\ \text{Dual Ramsey Theorem}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Finite Product}\\ \text{Dual Ramsey Theorem}\end{array}}$$

Finite Product Dual Ramsey Theorem. *For all s, $a_1, \ldots, a_s$, $b_1, \ldots, b_s \in \mathbb{N}$ and $k \geqslant 2$ there exist $c_1, \ldots, c_s \in \mathbb{N}$ such that for all sets $C_1, \ldots, C_s$ of cardinalities $c_1, \ldots, c_s$, respectively, and every k-coloring of the set $\begin{bmatrix} C_1 \\ a_1 \end{bmatrix} \times \ldots \times \begin{bmatrix} C_s \\ a_s \end{bmatrix}$, there exist a partition β_1 of C_1 with b_1 blocks, …, a partition β_s of C_s with b_s blocks such that the following set is monochromatic:*

$$\Big\{(\gamma_1, \ldots, \gamma_s) \in \begin{bmatrix} C_1 \\ a_1 \end{bmatrix} \times \ldots \times \begin{bmatrix} C_s \\ a_s \end{bmatrix} : \gamma_i \text{ is coarser than } \beta_i, 1 \leqslant i \leqslant s\Big\}.$$

Example.

$$\boxed{\begin{array}{c}\text{Ramsey Theorem}\\ \text{for Finite BA'a}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Finite Product Ramsey}\\ \text{Theorem for Finite BA'a}\end{array}}$$

Finite Product Ramsey Theorem for Finite BA'a. *For all positive integers s, k and all finite boolean algebras* $\mathcal{A}_1, \ldots, \mathcal{A}_s, \mathcal{B}_1, \ldots, \mathcal{B}_s$ *there exist finite boolean algebras* $\mathcal{C}_1, \ldots, \mathcal{C}_s$ *such that for every k-coloring of the set* $\binom{\mathcal{C}_1}{\mathcal{A}_1} \times \ldots \times \binom{\mathcal{C}_s}{\mathcal{A}_s}$*, where* $\binom{\mathcal{C}}{\mathcal{A}}$ *is the set of all subalgebras of* $\mathcal{C}$ *that are isomorphic to* $\mathcal{A}$*, there exist* $\tilde{\mathcal{B}}_1 \in \binom{\mathcal{C}_1}{\mathcal{B}_1}, \ldots, \tilde{\mathcal{B}}_s \in \binom{\mathcal{C}_s}{\mathcal{B}_s}$ *such that the set* $\binom{\tilde{\mathcal{B}}_1}{\mathcal{A}_1} \times \ldots \times \binom{\tilde{\mathcal{B}}_s}{\mathcal{A}_s}$ *is monochromatic.*

Ramsey property and products of categories

Example. Finite product Ramsey theorem for

- finite linearly ordered graphs,
- finite linearly ordered posets,
- finite linearly ordered metric spaces with rational distances,
- (and so on)

Self-dual Ramsey results

Example. $\mathbb{SET}_{\text{fin}} \times \mathbb{SET}_{\text{fin}}^{(\twoheadleftarrow)}$ has the Ramsey property for objects since both the factors do, so:

For all $a, b \in \mathbb{N}$ and $k \geqslant 2$ there exists a $c \in \mathbb{N}$ such that for every set C with $|C| = c$ and for every coloring

$$\chi : \binom{C}{a} \times \left[\begin{matrix} C \\ a \end{matrix}\right] \to k$$

there is a set $B \subseteq C$ with $|B| = b$ and a partition β of C with b blocks such that the following set is monochromatic:

$$\binom{B}{a} \times \{\gamma \in \left[\begin{matrix} C \\ a \end{matrix}\right] : \gamma \text{ is coarser than } \beta\}.$$

(Cf. S. SOLECKI: *Abstract approach to finite Ramsey theory and a self-dual Ramsey theorem.* Adv. Math. 248 (2013), 1156–1198.)

Silly self-dual Ramsey results

Example. $\mathbb{BA}_{\mathsf{fin}} \times \mathbb{V}_{\mathsf{fin}}^{(\twoheadleftarrow)}$ has the Ramsey property for obj's, so:

Let F be a finite field. For all $a, b \in \mathbb{N}$ and $k \geqslant 2$ there exists a $c \in \mathbb{N}$ such that for every finite boolean algebra $\mathcal{C}$ with c atoms, every vector space V over F of dimension c and for every coloring

$$\chi : \binom{\mathcal{C}}{\mathcal{P}(a)} \times \begin{bmatrix} V \\ a \end{bmatrix}_{\mathsf{lin}} \to k$$

there is a subalgebra $\mathcal{B}$ of $\mathcal{C}$ with b atoms and a $\beta \in \begin{bmatrix} V \\ b \end{bmatrix}_{\mathsf{lin}}$ such that the following set is monochromatic:

$$\binom{\mathcal{B}}{\mathcal{P}(a)} \times \{\gamma \in \begin{bmatrix} V \\ a \end{bmatrix}_{\mathsf{lin}} : \gamma \textit{ is coarser than } \beta\}.$$

- $\mathbb{V}_{\mathsf{fin}}^{(\twoheadleftarrow)}$ are finite vector spaces over a finite field F with surjective linear maps $V \twoheadleftarrow W$.

Ramsey property and extremely amenable groups

A. S. KECHRIS, V. G. PESTOV, S. TODORČEVIĆ: *Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups.* GAFA Geometric and Functional Analysis, 15 (2005) 106–189.

Theorem. *TFAE for a countable locally finite ultrahomogeneous first-order structure F:*

1. $\mathrm{Aut}(F)$ *is extremely amenable*
2. $\mathrm{Age}(F)$ *has the Ramsey property and consists of rigid elements.*

- A group G is *extremely amenable* if every continuous action of G on a compact Hausdorff space X has a common fixed point.

KPT theory in a category – the setup

Let $\mathbb{C}$ be a category and $\mathbb{C}_0$ a full subcategory of $\mathbb{C}$ such that:

(C1) all morphisms in $\mathbb{C}$ are monic ($=$ left cancellable);

(C2) $\mathrm{Ob}(\mathbb{C}_0)$ is a set;

(C3) for all $A, B \in \mathrm{Ob}(\mathbb{C}_0)$ the set $\hom(A, B)$ is finite;

(C4) for every $F \in \mathrm{Ob}(\mathbb{C})$ there is an $A \in \mathrm{Ob}(\mathbb{C}_0)$ such that $A \to F$;

(C5) for every $B \in \mathrm{Ob}(\mathbb{C}_0)$ the set $\{A \in \mathrm{Ob}(\mathbb{C}_0) : A \to B\}$ is finite.

$\mathbb{C}_0$ are (templates of) *finite objects* in $\mathbb{C}$.

$\mathrm{Age}(F) = \{A \in \mathrm{Ob}(\mathbb{C}_0) : A \to F\}$.

KPT theory in a category – the setup

Example. $\mathbb{CH}$

- objects are all chains,
- $\mathrm{hom}(A, B) =$ embeddings $A \to B$,
- composition: $f \cdot g = f \circ g$,
- $\mathbb{CH}_0$ objects are finite chains $(\{1, \dots, n\}, \leqslant)$, $n \geqslant 1$.

KPT theory in a category – the setup

Example. $\mathbb{HAUS}^{(\twoheadleftarrow)}$

- objects are Hausdorff spaces,
- $\hom(A, B) =$ continuous surjective maps $A \twoheadleftarrow B$,
- composition: $f \cdot g = g \circ f$,
- $\mathbb{HAUS}_0^{(\twoheadleftarrow)}$ objects are finite discrete spaces $\{1, \ldots, n\}$, $n \geqslant 1$.

An age of a structure in an op-category will be referred to as the *projective age* and denoted by $\partial\mathrm{Age}(A)$.

Example. $\mathcal{K} =$ Cantor set 2^ω.
$\partial\mathrm{Age}(\mathcal{K}) =$ all finite discrete spaces.

KPT theory in a category – the setup

Example. $\mathbb{OHAUS}^{(\twoheadleftarrow)}$

- objects are all lin ordered Hausdorff spaces,
- $\hom(A, B) =$ continuous monotonous surjective maps $A \twoheadleftarrow B$,
- composition: $f \cdot g = g \circ f$,
- $\mathbb{OHAUS}_0^{(\twoheadleftarrow)}$ objects are finite chains $(\{1, \ldots, n\}, \leqslant)$, $n \geqslant 1$.

Example. $\mathcal{K}_{\leqslant} = \mathcal{K}$ with the lexicographic order.
$\partial\mathrm{Age}(\mathcal{K}_{\leqslant}) =$ all finite chains.

Homogeneous objects

$F \in \mathrm{Ob}(\mathbb{C})$ is *homogeneous* if for every $A \in \mathrm{Age}(F)$ and every pair of morphisms $e_1, e_2 : A \to F$ there is a $g \in \mathrm{Aut}(F)$ such that $g \cdot e_1 = e_2$.

$$\begin{tikzcd} F \arrow[rr, "g"] & & F \\ & A \arrow[ul, "e_1"] \arrow[ur, "e_2"'] & \end{tikzcd}$$

Example. Ultrahomogeneous structures in “direct” categories.

Following Irwin and Solecki, homogeneous structures in an op-category will be referred to as *projectively homogeneous*.

$$\begin{tikzcd} F \arrow[dr, two heads, "q_1"'] & & F \arrow[ll, "g"'] \arrow[dl, two heads, "q_2"] \\ & A & \end{tikzcd}$$

Example. Both $\mathcal{K}$ and $\mathcal{K}_{\leqslant}$ are projectively homogeneous (each in its category).

Locally finite objects

$F \in \mathrm{Ob}(\mathbb{C})$ is *locally finite* if

1. for every $A, B \in \mathrm{Age}(F)$ and every $e : A \to F$, $f : B \to F$ there are a $D \in \mathrm{Age}(F)$, $r : D \to F$, $p : A \to D$ and $q : B \to D$ such that $r \cdot p = e$ and $r \cdot q = f$, and
2. for every $H \in \mathrm{Ob}(\mathbb{C})$, $r' : H \to F$, $p' : A \to H$ and $q' : B \to H$ such that $r' \cdot p' = e$ and $r' \cdot q' = f$ there is an $s : D \to H$ such that the diagram below commutes.

$$\begin{tikzcd}
 & F & \\
A \arrow[ur, "e"] & & B \arrow[ul, "f"']
\end{tikzcd}$$

Locally finite objects

$F \in \mathrm{Ob}(\mathbb{C})$ is *locally finite* if

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2. for every $H \in \mathrm{Ob}(\mathbb{C})$, $r' : H \to F$, $p' : A \to H$ and $q' : B \to H$ such that $r' \cdot p' = e$ and $r' \cdot q' = f$ there is an $s : D \to H$ such that the diagram below commutes.

$$\begin{tikzcd}
D \arrow[rr, "r"] & & F & \\
& A \arrow[ul, "p"] \arrow[ur, "e"'] & & B \arrow[ul, "f"'] \arrow[ulll, "q"]
\end{tikzcd}$$

Locally finite objects

$F \in \mathrm{Ob}(\mathbb{C})$ is *locally finite* if

1. for every $A, B \in \mathrm{Age}(F)$ and every $e : A \to F$, $f : B \to F$ there are a $D \in \mathrm{Age}(F)$, $r : D \to F$, $p : A \to D$ and $q : B \to D$ such that $r \cdot p = e$ and $r \cdot q = f$, and
2. for every $H \in \mathrm{Ob}(\mathbb{C})$, $r' : H \to F$, $p' : A \to H$ and $q' : B \to H$ such that $r' \cdot p' = e$ and $r' \cdot q' = f$ there is an $s : D \to H$ such that the diagram below commutes.

$$\begin{tikzcd}
D \arrow[rr, "r"] & & F & & H \arrow[ll, "r'"'] \\
& A \arrow[ul, "p"] \arrow[ur, "e"] \arrow[urrr, "p'"] & & B \arrow[ulll, "q"] \arrow[ul, "f"] \arrow[ur, "q'"'] &
\end{tikzcd}$$

Locally finite objects

$F \in \mathrm{Ob}(\mathbb{C})$ is *locally finite* if

1. for every $A, B \in \mathrm{Age}(F)$ and every $e : A \to F$, $f : B \to F$ there are a $D \in \mathrm{Age}(F)$, $r : D \to F$, $p : A \to D$ and $q : B \to D$ such that $r \cdot p = e$ and $r \cdot q = f$, and
2. for every $H \in \mathrm{Ob}(\mathbb{C})$, $r' : H \to F$, $p' : A \to H$ and $q' : B \to H$ such that $r' \cdot p' = e$ and $r' \cdot q' = f$ there is an $s : D \to H$ such that the diagram below commutes.

$$
\begin{tikzcd}
D \arrow[rr, "r"] \arrow[rrrr, dotted, bend left, "\exists s"] & & F & & H \arrow[ll, "r'"'] \\
& A \arrow[ul, "p"] \arrow[ur, "e"'] \arrow[urrr, "p'"] & & B \arrow[ulll, "q"'] \arrow[ul, "f"] \arrow[ur, "q'"'] &
\end{tikzcd}
$$

Locally finite objects

Example. Every object in $\mathbb{CH}$ is locally finite.

Locally finite structures in an op-category will be referred to as *projectively locally finite*.

Example. Both $\mathcal{K}$ and $\mathcal{K}_{\leqslant}$ are projectively locally finite (each in its category).

Finitely separated automorphisms

The automorphisms of $F \in \mathrm{Ob}(\mathbb{C})$ are *finitely separated* if the following holds for all $f, g \in \mathrm{Aut}(F)$:

> if $f \neq g$ then there is an $A \in \mathrm{Age}(F)$ and an $e : A \to F$ such that $f \cdot e \neq g \cdot e$.

Example. Automorphisms of every relational structure are finitely separated.

Example. The automorphisms of both $\mathcal{K}$ and $\mathcal{K}_{\leqslant}$ are finitely separated (each in its category).

The topology generated by the age of an object

$F \in \mathrm{Ob}(\mathbb{C})$

For $A \in \mathrm{Age}(F)$ and $e_1, e_2 \in \hom(A, F)$ let

$$N_F(e_1, e_2) = \{f \in \mathrm{Aut}(F) : f \cdot e_1 = e_2\}.$$

Lemma. *Let F be a locally finite object in $\mathbb{C}$. Then*

$$\mathcal{M}_F = \{N_F(e_1, e_2) : A \in \mathrm{Age}(F); e_1, e_2 \in \hom(A, F)\}$$

is a base of a topology α_F on $\mathrm{Aut}(F)$. If, in addition, the automorphisms of F are fintely separated, $\mathrm{Aut}(F)$ endowed with the topology α_F is a Hausdorff topological group.

Example. In the category $\mathbb{REL}(\Delta)$ of relational structures of a fixed relational type Δ and embeddings, $\alpha_{\mathcal{F}}$ is the pointwise convergence topology for every Δ-structure $\mathcal{F}$.

Example. In the category of Hausdorff topological spaces and topological embeddings $\alpha_{\mathbb{R}}$ is nontrivial, but it is not the pointwise convergence topology.

The topology generated by the age of an object

Example. In $\mathbb{HAUS}^{(\twoheadleftarrow)}$: $\alpha_{\mathcal{K}} =$ compact-open topology on $\mathcal{K}$.

Example. In $\mathbb{HAUS}^{(\twoheadleftarrow)}$: $\alpha_{\mathcal{K}_{\leqslant}} =$ "compact interval-open interval" topology on $\mathcal{K}_{\leqslant}$.

Example. In the op-category of metric spaces and nonexpansive maps $\alpha_{\mathbb{R}}$ is antidiscrete.

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in $\mathbb{C}$ whose automorphisms are finitely separated. TFAE:*

1. $\mathrm{Aut}(F)$ *endowed with* α_F *is extr amenable,*
2. $\mathrm{Age}(F)$ *has the Ramsey property for morphisms.*

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in $\mathbb{C}$ whose automorphisms are finitely separated. TFAE:*

1. $\mathrm{Aut}(F)$ *endowed with* α_F *is extr amenable,*
2. $\mathrm{Age}(F)$ *has the Ramsey property for morphisms.*

Corollary 1. *Let F be an ultrahomogeneous relational structure. Then* $\mathrm{Aut}(F)$ *with with the pointwise convergence topology is extremely amenable if and only if* $\mathrm{Age}(F)$ *has the Ramsey property.*

D. BARTOŠOVÁ: *Universal minimal flows of groups of automorphisms of uncountable structures.* Canadian Mathematical Bulletin, 2012.

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in $\mathbb{C}$ whose automorphisms are finitely separated. TFAE:*

1. $\mathrm{Aut}(F)$ *endowed with* α_F *is extr amenable,*
2. $\mathrm{Age}(F)$ *has the Ramsey property for morphisms.*

Corollary 2. *Let F be a projectively locally finite projectively homogeneous structure. Then* $\mathrm{Aut}(F)$ *endowed with the topology* α_F *is extremely amenable if and only if* $\partial\mathrm{Age}(F)$ *has the dual Ramsey property.*

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in* $\mathbb{C}$ *whose automorphisms are finitely separated. TFAE:*

1. $\mathrm{Aut}(F)$ *endowed with* α_F *is extr amenable,*
2. $\mathrm{Age}(F)$ *has the Ramsey property for morphisms.*

Corollary 3. *Let F be a projectively homogeneous 0-dimensional Hausdorff space. Then* $\mathrm{Homeo}(F)$ *endowed with the compact-open topology is extremely amenable if and only if* $\partial\mathrm{Age}(F)$ *has the dual Ramsey property.*

(Cf. D. BARTOŠOVÁ: *Universal minimal flows of groups of automorphisms of uncountable structures.* Canadian Mathematical Bulletin, 2012.)

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in $\mathbb{C}$ whose automorphisms are finitely separated. TFAE:*

1. $\mathrm{Aut}(F)$ *endowed with α_F is extr amenable,*
2. $\mathrm{Age}(F)$ *has the Ramsey property for morphisms.*

Example. In $\mathbb{HAUS}^{(\twoheadleftarrow)}$: $\mathrm{Homeo}(\mathcal{K})$ endowed with the compact-open topology is not extremely amenable.

Example. In $\mathbb{OHAUS}^{(\twoheadleftarrow)}$: Let G be the homeomorphism group of $\mathcal{K}_{\leqslant}$ endowed with $\alpha_{\mathcal{K}_{\leqslant}} =$ "*compact interval – open interval*" topology. Then G is extremely amenable.

Minimal flows and the expansion property

A. S. KECHRIS, V. G. PESTOV, S. TODORČEVIĆ: *Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups.* GAFA Geometric and Functional Analysis, 15 (2005) 106–189.

Theorem. *Let $\mathcal{F}$ be a locally finite Fraïssé structure, $\mathcal{F}^*$ a Fraïssé order expansion of $\mathcal{F}$ and X^* the set of admissible linear orders on F. TFAE:*

1. *X^* is a minimal* $\mathrm{Aut}(\mathcal{F})$*-flow*
2. $\mathrm{Age}(\mathcal{F}^*)$ *has the ordering property w.r.t.* $\mathrm{Age}(\mathcal{F})$.

Minimal flows and the expansion property

L. NGUYEN VAN THÉ: *More on the Kechris-Pestov-Todorcevic correspondence: precompact expansions.* Fund. Math. 222 (2013), 19–47.

Theorem. *Let $\mathcal{F}$ be a locally finite Fraïssé structure, $\mathcal{F}^*$ a Fraïssé precompact expansion of $\mathcal{F}$ and X^* the set of admissible expansions on F. TFAE:*

1. *X^* is a minimal* $\mathrm{Aut}(\mathcal{F})$*-flow*
2. $\mathrm{Age}(\mathcal{F}^*)$ *has the expansion property w.r.t.* $\mathrm{Age}(\mathcal{F})$*.*

Minimal flows and the expansion property

$\Theta = (\theta_i)_{i<n}$ – a finite relational language

$\Omega_F = \bigcup\{\mathrm{hom}(A, F) : A \in \mathrm{Ob}(\mathbb{C}_0)\}$

For $F \in \mathrm{Ob}(\mathbb{C})$, a Θ-*expansion* of F is a tuple $(F, (\rho_i)_{i<n})$ where ρ_i is a finitary relation on Ω_F.

Minimal flows and the expansion property

$\mathbb{C}(\Theta)$ – a category of Θ expansions of objects from $\mathbb{C}$:

- objects are Θ-expansions of objects from $\mathbb{C}$;
- $f : (F, (\rho_i)_{i<n}) \to (H, (\sigma_i)_{i<n})$ is a $\mathbb{C}(\Theta)$-morphism if
  - $f \in \hom_{\mathbb{C}}(F, H)$, and
  - $(e_0, \ldots, e_{m-1}) \in \rho_i \Rightarrow (f \cdot e_0, \ldots, f \cdot e_{m-1}) \in \sigma_i$, for all $i < n$.

$\mathrm{Age}(F, (\theta_i)_{i<n})$ has the *expansion property* w.r.t. $\mathrm{Age}(F)$ if

for every $A \in \mathrm{Age}(F)$ there is a $B \in \mathrm{Age}(F)$ such that for all $(A, (\rho_i)_{i<n}), (B, (\sigma_i)_{i<n}) \in \mathrm{Age}(F, (\theta_i)_{i<n})$ we have a morphism $(A, (\rho_i)_{i<n}) \to (B, (\sigma_i)_{i<n})$ in $\mathbb{C}(\Theta)$.

Minimal flows and the expansion property

$F \in \mathrm{Ob}(\mathbb{C})$, $G = \mathrm{Aut}(F)$

$E_F = \{$all the tuples $(\rho_i)_{i<n}$ where $\rho_i \subseteq \Omega_{\mathcal{F}}^{m_i}\}$

G acts on E_F *logically*, that is

$$(\rho_i)_{i<n}^{g} = (\rho_i^{g})_{i<n} \quad \text{and}$$
$$(e_0, \ldots, e_{m-1}) \in \rho_i^{g} \Rightarrow (g^{-1} \cdot e_0, \ldots, g^{-1} \cdot e_{m-1}) \in \rho_i$$

Minimal flows and the expansion property

Theorem. *Let F be a locally finite homogeneous object in $\mathbb{C}$ and let $G = \mathrm{Aut}(F)$. Let $(F, (\rho_i)_{i<n})$ be a Θ-expansion of F which is locally finite in $\mathbb{C}(\Theta)$. Let $X^\Theta = \overline{(\rho_i)^G_{i<n}}$ be a G-flow where the action of G is logical. TFAE:*

1. *X^Θ is a minimal G-flow.*
2. $\mathrm{Age}(F, (\rho_i)_{i<n})$ *has the expansion property w.r.t.* $\mathrm{Age}(F)$.

Example. Let S be an infinite set, let $G = \mathrm{Sym}(S)$ and let $(S, \leqslant)$ be an ultrahomogeneous chain. Then

$$X^{\Theta} = \overline{\leqslant^{G}} = \text{all lin orders on } S$$

is a minimal G-flow.

Example. Let $G = \mathrm{Aut}(\mathcal{K})$ and recall that $\mathcal{K}_{\leqslant}$ is the Cantor set with the lexicographic order. Then $X^{\Theta} = \overline{\leqslant^{G}}$ is a minimal G-flow.

Universal minimal flows

A. S. KECHRIS, V. G. PESTOV, S. TODORČEVIĆ: *Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups.* GAFA Geometric and Functional Analysis, 15 (2005) 106–189.

Theorem. *Let $\mathcal{F}$ be a locally finite Fraïssé structure, $\mathcal{F}^*$ a Fraïssé order expansion of $\mathcal{F}$ and X^* the set of admissible linear orders on F. TFAE:*

1. *X^* is the universal minimal* $\mathrm{Aut}(\mathcal{F})$*-flow*
2. $\mathrm{Age}(\mathcal{F}^*)$ *has the Ramsey property and the ordering property w.r.t.* $\mathrm{Age}(\mathcal{F})$*.*

Universal minimal flows

Work in progress . . .