Homogeneity of the pseudoarc and permutation groups

Sławomir Solecki

University of Illinois at Urbana–Champaign

Research supported by NSF grant DMS-1266189

July 2015

Most of this work is joint with Todor Tsankov.

Outline of Topics

1. The pseudoarc and projective Fraïssé limits
2. Projective “types”
3. Homogeneity for points with minimal types
4. The transfer theorem
5. Questions (and comments on Menger compacta)

Two objects:

Two objects:

the pseudoarc $P =$

Two objects:

the pseudoarc $P =$ a certain compact, connected, second countable space

Two objects:

the pseudoarc P = a certain compact, connected, second countable space

the pre-pseudoarc $\mathbb{P}$ =

Two objects:

the pseudoarc P = a certain compact, connected, second countable space

the pre-pseudoarc $\mathbb{P}$ = the Cantor set and a certain compact equivalence relation R on it with $\mathbb{P}/R = P$ and with a certain relationship to a family of finite structures

Aim: Develop a model theoretic/combinatorial point of view (projective Fraïssé limit) that can be used to:

Aim: Develop a model theoretic/combinatorial point of view (projective Fraïssé limit) that can be used to:

1. find canonical "models" for interesting topological spaces, for example, the pseudoarc, Menger compacta, etc;

Aim: Develop a model theoretic/combinatorial point of view (projective Fraïssé limit) that can be used to:

1. find canonical "models" for interesting topological spaces, for example, the pseudoarc, Menger compacta, etc;

2. find a unified approach to topological homogeneity results and put these results on firm footing;

Aim: Develop a model theoretic/combinatorial point of view (projective Fraïssé limit) that can be used to:

1. find canonical "models" for interesting topological spaces, for example, the pseudoarc, Menger compacta, etc;

2. find a unified approach to topological homogeneity results and put these results on firm footing;

3. resolve topological questions about homeomorphism groups.

The pseudoarc and projective Fraïssé limits

The pseudoarc

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{C}$ is compact

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

This P is called the **pseudoarc**.

$\mathcal{K}([0,1]^n)$ = compact subsets of $[0,1]^n$ with the Vietoris topology, $n \geq 2$

$\mathcal{K}([0,1]^n)$ is compact, $n \geq 2$

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^n)$, $n \geq 2$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

This P is called the **pseudoarc**.

$\mathcal{K}([0,1]^\omega)$ = compact subsets of $[0,1]^\omega$ with the Vietoris topology

$\mathcal{K}([0,1]^\omega)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^\omega)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

This P is called the **pseudoarc**.

Continuum = compact and connected

Continuum = compact and connected

The pseudoarc is a **hereditarily indecomposable** continuum

Continuum = compact and connected

The pseudoarc is a **hereditarily indecomposable** continuum, that is, if $C_1, C_2 \subseteq P$ are continua with $C_1 \cap C_2 \neq \emptyset$, then $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.

Continuum = compact and connected

The pseudoarc is a **hereditarily indecomposable** continuum, that is, if $C_1, C_2 \subseteq P$ are continua with $C_1 \cap C_2 \neq \emptyset$, then $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.

It was discovered by **Knaster** in 1922.

Projective Fraïssé limits

$\mathcal{F}$ a family of finite structures

$\mathcal{F}$ a family of finite structures

Consider $\mathcal{F}$ with **epimorphisms** between structures.

$\mathcal{F}$ a family of finite structures

Consider $\mathcal{F}$ with **epimorphisms** between structures.

$\mathcal{F}$ is called a **projective Fraïssé family** if it has
Joint Epimorphism Property and **Projective Amalgamation Property**.

$\mathcal{F}$ a family of finite structures

Consider $\mathcal{F}$ with **epimorphisms** between structures.

$\mathcal{F}$ is called a **projective Fraïssé family** if it has
Joint Epimorphism Property and **Projective Amalgamation Property**.

Irwin–S.: If $\mathcal{F}$ is a projective Fraïssé family, then there exists a unique projective limit

$$\mathbb{F} = \varprojlim \mathcal{F}$$

that is projectively universal and projectively homogeneous.

$\mathcal{F}$ a family of finite structures

Consider $\mathcal{F}$ with **epimorphisms** between structures.

$\mathcal{F}$ is called a **projective Fraïssé family** if it has
Joint Epimorphism Property and **Projective Amalgamation Property**.

Irwin–S.: If $\mathcal{F}$ is a projective Fraïssé family, then there exists a unique projective limit

$$\mathbb{F} = \varprojlim \mathcal{F}$$

that is projectively universal and projectively homogeneous.

$\mathbb{F}$ also has **Projective Extension Property**.

Connection with the pseudoarc

$\mathcal{P}$ = all finite, linear, reflexive graphs with graph relation R

$\mathcal{P}$ = all finite, linear, reflexive graphs with graph relation R

Irwin–S.: $\mathcal{P}$ is a projective Fraïssé family.

Let $\mathbb{P} = \varprojlim \mathcal{P}$ be the projective Fraïssé limit of $\mathcal{P}$ with relation $R^{\mathbb{P}}$.

Let $\mathbb{P} = \varprojlim \mathcal{P}$ be the projective Fraïssé limit of $\mathcal{P}$ with relation $R^{\mathbb{P}}$.

$R^{\mathbb{P}}$ is a compact equivalence relation on $\mathbb{P}$, whose equivalence classes have at most 2 elements each.

Let $\mathbb{P} = \varprojlim \mathcal{P}$ be the projective Fraïssé limit of $\mathcal{P}$ with relation $R^{\mathbb{P}}$.

$R^{\mathbb{P}}$ is a compact equivalence relation on $\mathbb{P}$, whose equivalence classes have at most 2 elements each.

Irwin–S.: $\mathbb{P}/R^{\mathbb{P}}$ is the pseudoarc.

Let $\mathbb{P} = \varprojlim \mathcal{P}$ be the projective Fraïssé limit of $\mathcal{P}$ with relation $R^{\mathbb{P}}$.

$R^{\mathbb{P}}$ is a compact equivalence relation on $\mathbb{P}$, whose equivalence classes have at most 2 elements each.

Irwin–S.: $\mathbb{P}/R^{\mathbb{P}}$ is the pseudoarc.

There is a natural continuous homomorphism

$$\mathrm{Aut}(\mathbb{P}) \to \mathrm{Homeo}(\mathbb{P}/R^{\mathbb{P}})$$

with dense range.

Bing: The pseudoarc is homogeneous

Bing: The pseudoarc is homogeneous, that is, for any $x, y \in P$, there exists $f \in \mathrm{Homeo}(P)$ such that $f(x) = y$.

Projective "types"

M is a **structure** if

M is a **structure** if

- M is a compact, 0-dimensional, second countable space,
- R^M is a closed binary relation on M,
- each continuous function $M \to X$, with X finite, factors through an epimorphism $M \to A$ for some $A \in \mathcal{P}$.

Let $f\colon M \to X$ be continuous, with X finite.

Let $f\colon M \to X$ be continuous, with X finite. So f is a "projective tuple."

Let $f\colon M \to X$ be continuous, with X finite. So f is a "projective tuple."

Let $p \in M$.

Let $f\colon M \to X$ be continuous, with X finite. So f is a "projective tuple." Let $p \in M$.

Define

$$t_{(M,p)}(f) = \{f(K)\colon p \in K \subseteq M,\ K \text{ a } \textbf{structure}\}.$$

Let $f\colon M \to X$ be continuous, with X finite. So f is a "projective tuple."
Let $p \in M$.

Define

$$t_{(M,p)}(f) = \{f(K)\colon p \in K \subseteq M,\ K \text{ a } \textbf{structure}\}.$$

$t_{(M,p)}(f)$ is a family of subsets of the finite set X.

X finite set, $x \in X$

c is a **chain at** x if c is a maximal family of subsets of X linearly ordered by inclusion and with $\{x\} \in c$.

X finite set, $x \in X$

c is a **chain at** x if c is a maximal family of subsets of X linearly ordered by inclusion and with $\{x\} \in c$.

$t_{(M,p)}(f)$ is called

X finite set, $x \in X$

c is a **chain at** x if c is a maximal family of subsets of X linearly ordered by inclusion and with $\{x\} \in c$.

$t_{(M,p)}(f)$ is called

minimal if $t_{(M,p)}(f)$ is a chain at $f(p)$;

X finite set, $x \in X$

c is a **chain at** x if c is a maximal family of subsets of X linearly ordered by inclusion and with $\{x\} \in c$.

$t_{(M,p)}(f)$ is called

minimal if $t_{(M,p)}(f)$ is a chain at $f(p)$;

almost minimal if $t_{(M,p)}(f) = c_1 \cup c_2$, for some chains c_1 and c_2 at $f(p)$.

Homogeneity for points with minimal types

Lemma

Let $p \in \mathbb{P}$, $f : \mathbb{P} \to X$ *continuous,* X *finite.*

Lemma

Let $p \in \mathbb{P}$, $f : \mathbb{P} \to X$ continuous, X finite.
Then $t_{(\mathbb{P},p)}(f)$ is almost minimal.

$p \in \mathbb{P}$ **has minimal types** if $t_{(\mathbb{P},p)}(f)$ is minimal for each continuous $f : \mathbb{P} \to X$ with X finite.

$p \in \mathbb{P}$ **has minimal types** if $t_{(\mathbb{P},p)}(f)$ is minimal for each continuous $f\colon \mathbb{P} \to X$ with X finite.

Theorem (S.–Tsankov, 2015)

Let $p, q \in \mathbb{P}$. Assume that $R^{\mathbb{P}}(p) = \{p\}$ and $R^{\mathbb{P}}(q) = \{q\}$ and p and q have minimal types.

$p \in \mathbb{P}$ **has minimal types** if $t_{(\mathbb{P},p)}(f)$ is minimal for each continuous $f : \mathbb{P} \to X$ with X finite.

Theorem (S.–Tsankov, 2015)

Let $p, q \in \mathbb{P}$. Assume that $R^{\mathbb{P}}(p) = \{p\}$ and $R^{\mathbb{P}}(q) = \{q\}$ and p and q have minimal types. Then there exists $f \in \mathrm{Aut}(\mathbb{P})$ such that $f(p) = q$.

Proof uses the following **strong Projective Extension Property**.

Proof uses the following **strong Projective Extension Property**.

Lemma

Proof uses the following **strong Projective Extension Property**.

Lemma

Given: $p \in \mathbb{P}$ *with minimal types and* $R^{\mathbb{P}}(p) = \{p\}$,
$A, B \in \mathcal{P}$, $a \in A$, $b \in B$;
$f\colon \mathbb{P} \to A$, $g\colon B \to A$ *epimorphisms with* $f(p) = a$, $g(b) = a$.

Proof uses the following **strong Projective Extension Property**.

Lemma

Given: *$p \in \mathbb{P}$ with minimal types and $R^{\mathbb{P}}(p) = \{p\}$,*
$A, B \in \mathcal{P}$, $a \in A$, $b \in B$;
$f\colon \mathbb{P} \to A$, $g\colon B \to A$ epimorphisms with $f(p) = a$, $g(b) = a$.
Conclusion: *there exists an epimorphism $h\colon \mathbb{P} \to B$ such that $h(p) = b$.*

The transfer theorem

Aim: transfer partial homogeneity from $\mathbb{P}$ to full homogeneity of $\mathbb{P}/R^{\mathbb{P}}$.

Theorem (S.–Tsankov, 2015)

For each $y \in \mathbb{P}/R^{\mathbb{P}}$*, there exists* $x \in \mathbb{P}/R^{\mathbb{P}}$ *and a homeomorphism* $\phi\colon \mathbb{P}/R^{\mathbb{P}} \to \mathbb{P}/R^{\mathbb{P}}$ *such that*

Theorem (S.–Tsankov, 2015)

For each $y \in \mathbb{P}/R^{\mathbb{P}}$, there exists $x \in \mathbb{P}/R^{\mathbb{P}}$ and a homeomorphism $\phi\colon \mathbb{P}/R^{\mathbb{P}} \to \mathbb{P}/R^{\mathbb{P}}$ such that

(i) $x = p/R^{\mathbb{P}}$ *for some* $p \in \mathbb{P}$ *having minimal types and with* $R^{\mathbb{P}}(p) = \{p\}$;

Theorem (S.–Tsankov, 2015)

For each $y \in \mathbb{P}/R^{\mathbb{P}}$, *there exists* $x \in \mathbb{P}/R^{\mathbb{P}}$ *and a homeomorphism* $\phi\colon \mathbb{P}/R^{\mathbb{P}} \to \mathbb{P}/R^{\mathbb{P}}$ *such that*

(i) $x = p/R^{\mathbb{P}}$ *for some* $p \in \mathbb{P}$ *having minimal types and with* $R^{\mathbb{P}}(p) = \{p\}$;

(ii) $\phi(x) = y$.

Theorem (S.–Tsankov, 2015)

For each $y \in \mathbb{P}/R^{\mathbb{P}}$, there exists $x \in \mathbb{P}/R^{\mathbb{P}}$ and a homeomorphism $\phi \colon \mathbb{P}/R^{\mathbb{P}} \to \mathbb{P}/R^{\mathbb{P}}$ such that

(i) $x = p/R^{\mathbb{P}}$ *for some* $p \in \mathbb{P}$ *having minimal types and with* $R^{\mathbb{P}}(p) = \{p\}$;

(ii) $\phi(x) = y$.

An important ingredient of the proof is a notion of **weak commutation of diagrams**.

Weak commutation of diagrams

Weak commutation of diagrams

Given $A \in \mathcal{P}$, define the pre-dual $\widehat{A} \in \mathcal{P}$ of A with a bijection

$$A \ni a \to \widehat{a} \text{ an edge in } \widehat{A}.$$

Weak commutation of diagrams

Given $A \in \mathcal{P}$, define the pre-dual $\widehat{A} \in \mathcal{P}$ of A with a bijection

$$A \ni a \to \widehat{a} \text{ an edge in } \widehat{A}.$$

Weak commutation for epimorphisms

$$f\colon \mathbb{P} \to A,\ g\colon \mathbb{P} \to B \text{ and } h\colon \widehat{A} \to \widehat{B}:$$

Weak commutation of diagrams

Given $A \in \mathcal{P}$, define the pre-dual $\widehat{A} \in \mathcal{P}$ of A with a bijection

$$A \ni a \to \widehat{a} \text{ an edge in } \widehat{A}.$$

Weak commutation for epimorphisms

$$f\colon \mathbb{P} \to A,\ g\colon \mathbb{P} \to B \text{ and } h\colon \widehat{A} \to \widehat{B}:$$

$$h[\widehat{f(p)}] \subseteq \widehat{g(p)} \ \text{ for each } p \in \mathbb{P}.$$

From partial homogeneity of $\mathbb{P}$ and the above transfer theorem we get the following corollary.

From partial homogeneity of $\mathbb{P}$ and the above transfer theorem we get the following corollary.

Corollary (Bing)

The pseudoarc is homogeneous.

Questions
(and comments on Menger compacta)

Define relations S and T on $\mathbb{P}$ as follows

Define relations S and T on $\mathbb{P}$ as follows

$S(x, y)$ if and only if $x, y \in K$ for some substructure $K \subsetneq \mathbb{P}$;

Define relations S and T on $\mathbb{P}$ as follows

$S(x, y)$ if and only if $x, y \in K$ for some substructure $K \subsetneq \mathbb{P}$;

$T(x, y, z)$ if and only if $x, y \in K$ and $z \notin K$ for some substructure $K \subseteq \mathbb{P}$ with $R(K) = K$.

Define relations S and T on $\mathbb{P}$ as follows

$S(x, y)$ if and only if $x, y \in K$ for some substructure $K \subsetneq \mathbb{P}$;

$T(x, y, z)$ if and only if $x, y \in K$ and $z \notin K$ for some substructure $K \subseteq \mathbb{P}$ with $R(K) = K$.

Theorem (S.–Tsankov, 2015)

Let $F_1, F_2 \subseteq \mathbb{P}$ be finite sets whose points have minimal types and whose points p are such that $R(p) = \{p\}$.

Define relations S and T on $\mathbb{P}$ as follows

$S(x, y)$ if and only if $x, y \in K$ for some substructure $K \subsetneq \mathbb{P}$;

$T(x, y, z)$ if and only if $x, y \in K$ and $z \notin K$ for some substructure $K \subseteq \mathbb{P}$ with $R(K) = K$.

Theorem (S.–Tsankov, 2015)

Let $F_1, F_2 \subseteq \mathbb{P}$ be finite sets whose points have minimal types and whose points p are such that $R(p) = \{p\}$. Let $f\colon F_1 \to F_2$ be a bijection preserving S and T.

Define relations S and T on $\mathbb{P}$ as follows

$S(x, y)$ if and only if $x, y \in K$ for some substructure $K \subsetneq \mathbb{P}$;

$T(x, y, z)$ if and only if $x, y \in K$ and $z \notin K$ for some substructure $K \subseteq \mathbb{P}$ with $R(K) = K$.

Theorem (S.–Tsankov, 2015)

Let $F_1, F_2 \subseteq \mathbb{P}$ be finite sets whose points have minimal types and whose points p are such that $R(p) = \{p\}$. Let $f\colon F_1 \to F_2$ be a bijection preserving S and T. Then f extends to an element of $\mathrm{Aut}(\mathbb{P})$.

Is there a maximal homogeneity of $\mathbb{P}$?

Is there a maximal homogeneity of $\mathbb{P}$? More precisely: extend the theorem above to (appropriate) compact F and G.

Is there a maximal homogeneity of $\mathbb{P}$? More precisely: extend the theorem above to (appropriate) compact F and G.

Is every element of $\mathrm{Homeo}(\mathbb{P}/R)$ conjugate to an element of $\mathrm{Aut}(\mathbb{P})$?

Is there a maximal homogeneity of $\mathbb{P}$? More precisely: extend the theorem above to (appropriate) compact F and G.

Is every element of $\mathrm{Homeo}(\mathbb{P}/R)$ conjugate to an element of $\mathrm{Aut}(\mathbb{P})$?

Can orbits of the natural action of $\mathrm{Aut}(\mathbb{P})$ on $\mathbb{P}$ be characterized by types or sequences of types?

Is there a maximal homogeneity of $\mathbb{P}$? More precisely: extend the theorem above to (appropriate) compact F and G.

Is every element of $\mathrm{Homeo}(\mathbb{P}/R)$ conjugate to an element of $\mathrm{Aut}(\mathbb{P})$?

Can orbits of the natural action of $\mathrm{Aut}(\mathbb{P})$ on $\mathbb{P}$ be characterized by types or sequences of types?

Can $t_{(M,p)}(f)$ be viewed as actual types?

Menger compacta

Menger compacta

$\mathbb{N} \cup \{\infty\} \ni n \to \mu_n$ a compact, second countable space

Menger compacta

$$\mathbb{N} \cup \{\infty\} \ni n \to \mu_n \text{ a compact, second countable space}$$

$\mu_0 =$ Cantor set

$\mu_\infty =$ Hilbert cube

Menger compacta

$$\mathbb{N} \cup \{\infty\} \ni n \to \mu_n \text{ a compact, second countable space}$$

$\mu_0 =$ Cantor set

$\mu_\infty =$ Hilbert cube

μ_n is n-dimensional, universal for n-dimensional second countable spaces, highly homogeneous

Joint with Panagiotopoulos

Joint with Panagiotopoulos

There exists a projective Fraïssé family $\mathcal{M}_1$ such that if $\mathbb{M}_1 = \varprojlim \mathcal{M}_1$ is taken with the binary relation R_1, then

Joint with Panagiotopoulos

There exists a projective Fraïssé family $\mathcal{M}_1$ such that if $\mathbb{M}_1 = \varprojlim \mathcal{M}_1$ is taken with the binary relation R_1, then

$\mathbb{M}_1 / R_1 = \mu_1$

Joint with Panagiotopoulos

There exists a projective Fraïssé family $\mathcal{M}_1$ such that if $\mathbb{M}_1 = \varprojlim \mathcal{M}_1$ is taken with the binary relation R_1, then

$\mathbb{M}_1/R_1 = \mu_1$

$\mathbb{M}_1$ is highly homogeneous.

In fact, given $n \in \mathbb{N}$, there exists a projective Fraïssé family $\mathcal{M}_n$ analogous to $\mathcal{M}_1$.

Is it the case that $\mathbb{M}_n/R_n = \mu_n$?

In fact, given $n \in \mathbb{N}$, there exists a projective Fraïssé family $\mathcal{M}_n$ analogous to $\mathcal{M}_1$.

Is it the case that $\mathbb{M}_n / R_n = \mu_n$?

For an answer, we need appropriate homology groups for $\mathbb{M}_n$.