Introduction to reconstructing the topological monoid of endomorphisms of the rationals.

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Presenting joint work with...

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Outline

Topological Monoids.

Automatic homeomorphicity

Clones

Reconstruction of Topology

Whether we can reconstruct the canonical topology of an endomorphism monoid $\mathrm{End}(\mathbb{A})$ from its underlying abstract monoid structure?

Automatic continuity

In which situations are homomorphisms or isomorphisms between transformation monoids automatically continuous?

Reconstruction of Topology

Whether we can reconstruct the canonical topology of a polymorphism clone $\mathrm{Pol}(\mathbb{A})$ from its underlying abstract clone structure?

Automatic continuity

In which situations are homomorphisms or isomorphisms between function clones automatically continuous?

Transformation monoids

- For a set A, we denote by $\mathrm{O}_A^{(1)} := A^A$ the set of all unary functions on A and by
$$\mathrm{Tr}\,(A)$$
the full transformation monoid on A.
- The submonoids
$$M \leq \mathrm{Tr}\,(A)$$
are transformation monoids on A.

If we equip A with the discrete topology, then $\mathrm{Tr}(A)$ is a product space of A equipped with the Tychonoff topology.

Pointwise convergence topology

Let I be an index set. For every finite $J \subseteq I$ and $u : J \to A$:

$$U(J,u) := \{f\colon I \to A \mid f\restriction_J = u\}.$$

A basis for the topology of A^I can be expressed as

$$B_{\text{pwc}} = \{\emptyset\} \cup \left\{U(J,u) \mid J \subseteq I \text{ finite} \wedge u \in A^J\right\}.$$

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Special case $I = A$, $J = \left\{a_1^1, \ldots, a_1^m\right\}$, and we fix m elements $a_0^j = u\left(a_1^j\right) \in A$ for $1 \leq j \leq m$.

Topology on $\mathrm{Tr}(A)$

A non-empty basic open set is:

$$U(J,u) = \left\{ f\colon A \to A \mid \forall\, 1 \le j \le m\colon f\left(a_1^j\right) = u\left(a_1^j\right) = a_0^j \right\}.$$

- Topological monoids are abstract monoids which carry a topology under which the composition is continuous.
- A transformation monoid $M \le \mathrm{Tr}(A)$ is considered as a topological subspace of $\mathrm{Tr}(A)$.

Given a relational structure $\mathbb{A} = \left(A, \left(R^{\mathbb{A}}\right)_{\underline{R}\in\Sigma}\right)$, where $R^{\mathbb{A}} \subseteq A^{\mathrm{ar}(\underline{R})}$ for each $\underline{R} \in \Sigma$.

Endomorphism monoids

A function $f \in \mathrm{O}_A^{(1)}$ is called an endomorphism of $\mathbb{A}$ if

$$f : \mathbb{A} \xrightarrow{\text{homo}} \mathbb{A}.$$

The set of all endomorphisms on $\mathbb{A}$ is denoted by

$$\mathrm{End}\left(\mathbb{A}\right).$$

Given a relational structure $\mathbb{A} = \left(A, \left(R^{\mathbb{A}}\right)_{\underline{R}\in\Sigma}\right)$, where $R^{\mathbb{A}} \subseteq A^{\mathrm{ar}(\underline{R})}$ for each $\underline{R} \in \Sigma$.

Polymorphism

A function $f \in \mathrm{O}_A^{(k)} := A^{A^k}$ is called a polymorphism of $\mathbb{A}$ if

$$f : \mathbb{A}^k \xrightarrow{\text{homo}} \mathbb{A}.$$

The set of all polymorphisms on $\mathbb{A}$ is denoted by

$$\mathrm{Pol}\left(\mathbb{A}\right) = \bigcup_{k\in\mathbb{N}_+} \mathrm{Pol}^{(k)}\left(\mathbb{A}\right).$$

$f \in \mathrm{O}_A^{(k)}, \mathrm{ar}\left(R^{\mathbb{A}}\right) = m$

$$f \circ \left(\underset{\underset{R^{\mathbb{A}}}{\Cap}}{\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}}, \cdots, \underset{\underset{R^{\mathbb{A}}}{\Cap}}{\begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix}} \right) = \underset{\underset{R^{\mathbb{A}}}{\Cap}}{\begin{pmatrix} f(a_{11} & \dots & a_{1k}) \\ & \ddots & \\ f(a_{m1} & \dots & a_{mk}) \end{pmatrix}}$$

Topological closure

Remark

The submonoid $M \leq \operatorname{Tr}(A)$ is closed $\iff$ $M = \operatorname{End}(\mathbb{A})$ for some relational structure $\mathbb{A}$ with domain A.

Outline

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Clones

Definition (M.Bodirsky, M.Pinsker, A.Pongrácz)

A closed monoid $M \leq \operatorname{Tr}(A)$ has reconstruction $:\Longleftrightarrow$ for every other closed monoid $M' \leq \operatorname{Tr}(B)$, if there exists a monoid isomorphism between M and M', then there also exists a monoid isomorphism between M and M' which is a homeomorphism.

Definition

A closed monoid $M \leq \operatorname{Tr}(A)$ has automatic continuity $:\Longleftrightarrow$ every monoid homomorphism from M into $\operatorname{Tr}(A)$ is continuous.

Corollary (D. Lascar (1991))

Any continuous isomorphism between closed subgroups of $\mathbb{S}_A$ is a homeomorphism.

Definition (M.Bodirsky, M.Pinsker, A.Pongrácz)

A closed monoid $M \leq \mathrm{Tr}\,(A)$ has automatic homeomorphicity $:\Longleftrightarrow$ every monoid isomorphism from M to a closed $M' \leq \mathrm{Tr}\,(B)$ is a homeomorphism.

Some monoids with automatic homeomorphicity:

$$\mathrm{Emb}\,(\mathbb{N}, =)\,,\mathrm{Emb}\,(\Gamma)\,,\mathrm{End}\,(\Gamma)$$

where $\Gamma =$ Random graph.

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Some monoids with automatic homeomorphicity:

$$\mathrm{Emb}(\mathbb{N}, =), \mathrm{Emb}(\Gamma), \mathrm{End}(\Gamma), \mathrm{End}(\mathbb{Q}, <), \mathrm{End}(\mathbb{Q}, \leq)$$

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For groups, automatic continuity implies automatic homeomorphicity

Let $\mathbb{A}, \mathbb{B}$ be countable. If $\mathrm{Aut}(\mathbb{A})$ has S.I.P., then

$$\xi : \mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$$

is a homeomorphism.

We want to investigate the automatic homeomorphicity of

$$\mathrm{End}\,(\mathbb{Q}, \leq) \qquad \mathrm{End}\,(\mathbb{Q}, <)$$

We want to investigate the automatic homeomorphicity of

$$\mathrm{End}\,(\mathbb{Q},\leq) \qquad \mathrm{End}\,(\mathbb{Q},<) \qquad \mathrm{Pol}\,(\mathbb{Q},<) \qquad \& \qquad \mathrm{Pol}\,(\mathbb{Q},\leq)$$

Constants

- For $d \in \mathbb{Q}$
$$c_d \in E := \mathrm{End}\,(\mathbb{Q},\leq)$$
where $c_d(x) := d$.
- An element $c \in M \leq \mathrm{O}_A^{(1)}$ is called a constant $:\iff$
$$\forall x, y \in A : c\,(x) = c\,(y)\,.$$
- The set $C = \{g \in E : (\forall f \in E)\ \ gf = g\}$ is a definable subset of E.

Proposition (M.Bodirsky, M.Pinsker, A.Pongrácz)

Let $\mathbb{A}$ be a structure such that $\mathrm{Pol}(\mathbb{A})$ *contains all constant functions, and $\xi : \mathrm{Pol}(\mathbb{A}) \to \mathcal{D}$ be a clone isomorphism to a clone of functions $\mathcal{D}$. Then ξ is open.*

Proposition

Let $\mathbb{A}$ be a structure such that $\mathcal{M}_A := \mathrm{End}(\mathbb{A})$ contains all unary constant operations, and $\xi : \mathcal{M}_A \to \mathcal{M}_B := \mathrm{End}(\mathbb{B})$ be a monoid isomorphism. Then ξ is open.

Proposition

Let $\mathbb{A}$ be a structure such that $\mathcal{M}_A := \operatorname{End}(\mathbb{A})$ contains all unary constant operations, and $\xi : \mathcal{M}_A \to \mathcal{M}_B := \operatorname{End}(\mathbb{B})$ be a monoid isomorphism. Then ξ is open.

Let $a, b \in A$, and $E_{a,b} = \{f \in \mathcal{M}_A \mid f(a) = b\} = \{f \in \mathcal{M}_A \mid f \circ c_a = c_b\}$ be a basic open set. Then, we show that $\xi(E_{a,b})$ is open

$$\begin{aligned}
\xi(E_{a,b}) &= \{\xi(f) \mid f \in \mathcal{M}_A \wedge f \circ c_a = c_b\} \\
&= \{\xi(f) \mid f \in \mathcal{M}_A \wedge \xi(f \circ c_a) = \xi(c_b)\} && \text{(since } \xi \text{ is inj.)} \\
&= \{\xi(f) \mid f \in \mathcal{M}_A \wedge \xi(f) \circ \xi(c_a) = \xi(c_b)\} && \text{(since } \xi \text{ is a hom.)} \\
&= \{g \in \mathcal{M}_B \mid g \circ \xi(c_a) = \xi(c_b)\} && \text{(since } \xi \text{ is surj.)}
\end{aligned}$$

Example

$$A = \{0, 1\}$$

$$\mathcal{M}_A := \{\mathrm{id}_A, c_0, c_1\}$$

$$c_0(x) = 0$$

$$c_1(x) = 1$$

$$B = \mathbb{N}$$

$$\mathcal{M}_B := \{\mathrm{id}_B, e_0, e_1\}$$

$$e_0(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod 2 \\ 1 & \text{if } x \equiv 1 \pmod 2 \end{cases}$$

$$e_1(x) = \begin{cases} 2 & \text{if } x \equiv 0 \pmod 2 \\ 3 & \text{if } x \equiv 1 \pmod 2 \end{cases}$$

- $\xi : \mathcal{M}_A \to \mathcal{M}_B$ does not map constants to constants.
- $U = \{g \in \mathcal{M}_A \mid g(0) = 0\} = \{\mathrm{id}_A, c_0\}$ and $\xi[U] = \{\mathrm{id}_B, e_0\}$ are basic open sets in $\mathcal{M}_A$ and $\mathcal{M}_B$, respectively.

Lemma

Let $S \leq \left\langle A^A, \circ \right\rangle$ and $T \leq \left\langle B^B, \circ \right\rangle$ be transformation semigroups and $\xi\colon S \to T$ be a semigroup homomorphism, whose $\operatorname{im}(\xi) \leq T$ acts transitively on B (by evaluation). That is, for all $x, y \in B$ there exists some $f_{x,y} \in S$ such that $\xi(f_{x,y})(x) = y$. In these circumstances ξ maps any constant operation $c \in S$ to a constant operation on B.

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Proof.

- If $c \in S$ is constant, $\implies c \circ f = c$ for all $f \in S$.
- For $x, y \in B$: $\quad c \circ f_{x,y} = c$.
- For $x, y \in B$: $\quad \xi(c) \circ \xi\left(f_{x,y}\right) = \xi\left(c \circ f_{x,y}\right) = \xi(c)$.
- Evaluating at $x \in B$: $\xi(c)(x) = \xi(c)\,\xi\left(f_{x,y}\right)(x) = \xi(c)(y)$,
  $\implies \xi(c)$ is a constant function.

$\square$

Corollary

Let $S \leq \left\langle A^A, \circ \right\rangle$, $T \leq \left\langle B^B, \circ \right\rangle$ and $\xi \colon S \to T$ be a semigroup homomorphism. Suppose S contains at least one constant operation, then the following facts are equivalent:

1. *$im\,(\xi) \leq T$ acts transitively on B (by evaluation).*
2. *$im\,(\xi)$ contains all unary constants on B.*

Lemma

Assume,

- $\mathcal{C} \leq \left\langle A^A, \circ \right\rangle$ *contains all constant operations,*
- $\mathcal{D} \leq \langle B^B, \circ \rangle$ *acts transitively,*
- $\xi : \mathcal{C} \to \mathcal{D}$ *semigroup isomorphism,*

then the image of any open subset of $\mathcal{C}$ *under* ξ *is open in* B^B.

Proof.

Let $a, b \in A$, and $E_{a,b} = \{f \in \mathcal{C} \mid f(a) = b\} = \{f \in \mathcal{C} \mid f \circ c_a = c_b\}$ be a basic open set. Then, we show that $\xi(E_{a,b})$ is open

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\begin{aligned}
\xi(E_{a,b}) &= \{\xi(f) \mid f \in \mathcal{C} \wedge f \circ c_a = c_b\} \\
&= \{\xi(f) \mid f \in \mathcal{C} \wedge \xi(f \circ c_a) = \xi(c_b)\} && \text{(since } \xi \text{ is inj.)} \\
&= \{\xi(f) \mid f \in \mathcal{C} \wedge \xi(f) \circ \xi(c_a) = \xi(c_b)\} && \text{(since } \xi \text{ is a hom.)} \\
&= \{g \in \mathcal{D} \mid g \circ \xi(c_a) = \xi(c_b)\} && \text{(since } \xi \text{ is surj.)} \\
&= \{g \in \mathcal{D} \mid g \circ (c_p) = c_q\}. \\
&\qquad \text{(for some constants } p, q \in B \text{, according to Lemma 5)} \\
&= E_{p,q}.
\end{aligned}
$$

$\square$

Lemma

Assume,

- $\mathrm{Const}^1_A \subseteq \mathcal{C} \leq \langle A^A, \circ \rangle$
- $\mathcal{D} \leq \langle B^B, \circ \rangle$ *acts transitively*
- $\xi : \mathcal{C} \to \mathcal{D}$ *semigroup isomorphism,*

then ξ *is continuous.*

Corollary

Assume,

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then ξ *is a homeomorphism, moreover, both* $\mathcal{C}$ *and* $\mathcal{D}$*, contain all constant respective unary operations.*

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Clones

Clone

$F \subseteq \mathrm{O}_A := \bigcup_{k\in\mathbb{N}_+} \mathrm{O}_A^{(k)}$ is a clone (of operations) on A iff

1. $J_A \subseteq F$
2. F is closed w.r.t. composition.

Definition

A function $\xi : F \to F'$ is a clone isomorphism iff

1. ξ is a bijection,
2. ξ respects the arities, i.e. $\mathrm{ar}\,(\xi(f)) = \mathrm{ar}\,(f)$ for all $f \in F$.
3. $\forall\ 1 \leq j \leq n;\ \ \xi\left(\pi_j^{(n)}\right) = \pi_j^{(n)} \in F'$,
4. for $f \in F^{(k)}, g_1, \ldots, g_k \in F^{(m)}$ we have

$$\xi\,(f \circ (g_1, \ldots, g_k)) = \xi\,(f) \circ (\xi\,(g_1)\,, \ldots, \xi\,(g_k))$$

Examples

1. J_A Clone of all projections.
2. O_A Clone of all operations.
3. Arbitratry intersections of clones are clones again.
   Let $F \subseteq \mathrm{O}_A$. The clone generated by F is

   $$\langle F \rangle_{\mathrm{O}_A} := \bigcap \{C \text{ is clone } \mid F \subseteq C\}$$

   and it is the smallest clone containing F.
4. $\mathrm{Pol}\,(\mathbb{A})$ for some relational structure $(\mathbb{A})$.

If we equip A with the discrete topology, then $\mathrm{O}_A^{(n)}$ is a product space of A equipped with the product topology.

Pointwise convergence topology

Let I be an index set. For every finite $J \subseteq I$ and $u : J \to A$:

$$U(J, u) := \{f \colon I \to A \mid f \restriction_J = u\} .$$

A basis for the topology of A^I can be expressed as

$$B_{\text{pwc}} = \{\emptyset\} \cup \left\{ U(J, u) \mid J \subseteq I \text{ finite } \wedge \ u \in A^J \right\} .$$

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Special case $I = A^n$, $J = \left\{\left(a_1^1, \ldots, a_n^1\right), \ldots, (a_1^m, \ldots, a_n^m)\right\}$, and we fix m elements $a_0^j = u\left(a_1^j, \ldots, a_n^j\right) \in A$ for $1 \leq j \leq m$.

Lemma (ϕ is open)

Assume,

- $\mathrm{Const}_A^1 \subseteq \mathcal{C} \leq \mathrm{O}_A$,
- $\mathcal{D} \leq \mathrm{O}_B$,
- $\phi \colon \mathcal{C} \to \mathcal{D}$ *clone isomorphism,*
- $\xi := \phi \restriction_{\mathcal{C}^{(1)}}$ *semigroup homomorphism, such that* $im(\xi)$ *acts transitively on* B.

Then, for all $n > 0$ $\phi[U]$ *is open in* B^{B^n} *for all open* $U \subseteq \mathcal{C}^{(n)}$.

Lemma (ϕ maps any n-ary constant to an n-ary constant)

Assume,

- $\mathrm{Const}_A^1 \subseteq \mathcal{C} \leq \mathrm{O}_A$,
- $\mathcal{D} \leq \mathrm{O}_B$, *clone*
- $\phi \colon \mathcal{C} \to \mathcal{D}$ *clone isomorphism*,

Then, the restriction $\xi := \phi \restriction_{\mathcal{C}^{(1)}}$ maps unary constants to unary constants and ϕ maps any n-ary constant to an n-ary constant

Proof

$f \in \mathrm{O}_A^{(1)}$ constant,

$\iff \forall x, y \in A,\ f(x) = f(y) \iff f \circ \pi_1^{(2)} = f \circ \pi_2^{(2)}$. Hence,

$\xi(f) \circ \pi_1^{(2)} = \xi(f) \circ \xi\left(\pi_1^{(2)}\right) = \xi\left(f \circ \pi_1^{(2)}\right) = \xi\left(f \circ \pi_2^{(2)}\right) = \xi(f) \circ \pi_2^{(2)}$.

$\implies \xi(f)$ is constant on B.

Lemma (ϕ is open)

Assume,

- $\mathrm{Const}_A^1 \subseteq \mathcal{C} \leq \mathrm{O}_A$,
- $\mathcal{D} \leq \mathrm{O}_B$,
- $\phi\colon \mathcal{C} \to \mathcal{D}$ *clone isomorphism*,
- $\xi : \mathcal{C}^{(1)} \to \mathcal{D}^{(1)}$ *is the restriction of ϕ to the unary part of the clones and monoid isomorphism.*

Then, ξ is open and ϕ is open.

From last lemma we know

$$\phi : \mathrm{Pol}\,(\mathbb{Q}, \leq) \xrightarrow{\text{clone iso.}} \mathcal{D} \quad \text{is open}$$

We can apply the automatic homeomorphicity of End $(\mathbb{Q}, <)$ and following proposition to show that

$$\xi : \text{Pol}\,(\mathbb{Q}, <) \xrightarrow{\text{clone iso.}} \mathcal{C}' \quad \text{is open}$$

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Proposition (32 BPP)

- *Let $\mathcal{C}$ be a topological clone on A (with the product topology) such that $\mathcal{C}^{(1)}$ acts transitively on A,*
- *let ξ be an injective clone homomorphism from $\mathcal{C}$ to a topological clone $\mathcal{C}'$ (on another set B).*

Suppose that the restriction $\xi \restriction_{\mathcal{C}^{(1)}}^{\mathcal{C}'^{(1)}} : \mathcal{C}^{(1)} \to \mathcal{C}'^{(1)}$ is open, then so is ξ.

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$$\xi : \text{Pol}\,(\mathbb{Q}, <) \xrightarrow{\text{clone iso.}} \mathcal{C}' \quad \text{is open}$$

Proposition (32 BPP)

- *Let $\mathcal{C}$ be a topological clone on A (with the product topology) such that $\mathcal{C}^{(1)}$ acts transitively on A,*
- *let ξ be a clone isomorphism from $\mathcal{C}$ to a topological clone $\mathcal{C}'$ (on another set B).*

Suppose that the restriction $\xi \restriction_{\mathcal{C}^{(1)}}^{\mathcal{C}'^{(1)}} : \mathcal{C}^{(1)} \to \mathcal{C}'^{(1)}$ is open, then so is ξ.

Lemma (ϕ is continuous)

Assume,

- $\mathrm{Const}_A^1 \subseteq \mathcal{C} \leq \mathrm{O}_A$,
- $\mathcal{D} \leq \mathrm{O}_B$,
- $\phi \colon \mathcal{C} \to \mathcal{D}$ *clone isomorphism*,
- $\xi := \phi\restriction_{\mathcal{C}^{(1)}}$ *semigroup isomorphism, and suppose* $\mathcal{D}^{(1)} = im(\xi)$ *acts transitively on B.*

Then, for all $n > 0$ $\phi^{-1}[U]$ *is open in* A^{A^n} *for all open* $U \subseteq \mathcal{D}^{(n)}$, *i.e.* ϕ *is continuous.*

John K. Truss talk

$$\theta : E := \mathrm{End}\,(\mathbb{Q}, \leq) \longrightarrow \mathrm{Tr}\,(\Omega)$$

may be viewed as semigroup action of E on Ω.

$$\Omega = \bigcup_{i \in I} \Omega_i$$

where $\Omega_i = \left\{ a^i_B \mid B \in [\mathbb{Q}]^{n_i} \right\}$, $[\mathbb{Q}]^{n_i} := \{ A \subseteq \mathbb{Q} \mid |A| = n_i \}$

- $\theta\,(g)\left(a^i_B\right) = a^i_{gB}$ if $g \in G := \mathrm{Aut}\,(\mathbb{Q}, \leq)$
- $\theta\,(f)\left(a^i_B\right) = a^i_{fB}$ if $f \in M := \mathrm{Emb}\,(\mathbb{Q}, \leq)$
  $f \in E := \mathrm{End}\,(\mathbb{Q}, \leq)$ with $|fB| = B$.

θ is continuous and open

θ : End$\,(\mathbb{Q}, <) \xrightarrow{\text{inj.}} M' \leq \mathrm{Tr}\,(\Omega)$ is homeomorphism.

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Thank you :)