Symmetries of K3 sigma models

Matthias Gaberdiel
ETH Zürich

LMS Symposium
New Moonshines, Mock Modular Forms and String Theory

Durham, 5 August 2015

K3 sigma models

Consider CFT sigma model with target K3.

Spectrum:

$$\mathcal{H} = \bigoplus_{i,j} N_{ij}\, \mathcal{H}_i \otimes \bar{\mathcal{H}}_j \ .$$

repr. of N=4 superconformal algebra

Full spectrum complicated --- only known explicitly at special points in moduli space.

K3 Moduli Space

Moduli space of K3 sigma models is 80 real-dimensional, and explicitly given as

[Aspinwall, Morrison], [Aspinwall]
[Nahm,Wendland]

$$\mathcal{M} = \mathrm{O}(4,20;\mathbb{Z}) \backslash \underbrace{\mathrm{O}(4,20;\mathbb{R})/\mathrm{O}(4,\mathbb{R}) \times \mathrm{O}(20,\mathbb{R})}$$

discrete identifications

Grassmannian, describing pos. def. 4d subspace

$$\Pi \subset \mathbb{R}^{4,20}$$

Elliptic genus

Instead of full partition function consider
`partial index' = elliptic genus:

$$\phi_{\mathrm{K3}}(\tau,z) = \mathrm{Tr}_{\mathrm{RR}}\Big(q^{L_0-\frac{c}{24}}\, y^{J_0}\, (-1)^F\, \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\, (-1)^{\bar{F}}\Big) \equiv \phi_{0,1}(\tau,z)\ .$$

$$(q = e^{2\pi i\tau},\ y = e^{2\pi i z})$$

- constant over moduli space

- defines weak Jacobi form of weight w=0 and index m=1.

[Kawai et. al.]

BPS states

Define BPS states as subspace of the RR states

BPS states = right-moving ground states

(These are the states that contribute to elliptic genus.)

Then, w.r.t left-moving N=4 have decomposition

$$\mathcal{H}_{\mathrm{BPS}} = 20 \cdot \mathcal{H}_{\frac{1}{4},j=0} \oplus 2 \cdot \mathcal{H}_{\frac{1}{4},j=\frac{1}{2}} \oplus \bigoplus_{n=1}^{\infty} D_n \otimes \mathcal{H}_{n+\frac{1}{4},j=\frac{1}{2}}$$

multiplicity spaces --- not constant over moduli space.

Indices

However, since elliptic genus is constant over moduli space, the indices

$$A_n = \mathrm{Tr}_{D_n} (-1)^{\bar{F}}$$

are constant. Explicitly:

$$\begin{aligned} A_1 &= 90 &&= \mathbf{45} + \overline{\mathbf{45}} \\ A_2 &= 461 &&= \mathbf{231} + \overline{\mathbf{231}} \\ A_3 &= 1540 &&= \mathbf{770} + \overline{\mathbf{770}} \end{aligned}$$

dims of irreps of M24!

[Eguchi, Ooguri, Tachikawa]

Virtual representations

Note: similar decomposition for the lowest coefficients

$$\mathcal{H}_{\mathrm{BPS}} = 20 \cdot \mathcal{H}_{\frac{1}{4},j=0} \oplus 2 \cdot \mathcal{H}_{\frac{1}{4},j=\frac{1}{2}} \oplus \bigoplus_{n=1}^{\infty} D_n \times \mathcal{H}_{n+\frac{1}{4},j=\frac{1}{2}}$$

They involve virtual representations:

$$\begin{aligned} 20 &= \mathbf{23} - 3 \cdot \mathbf{1} \\ -2 &= -2 \cdot \mathbf{1} \ . \end{aligned}$$

Evidence

There is by now overwhelming evidence for this structure: the `twining genera' were guessed independently by [MRG, Hohenegger, Volpato] and [Eguchi, Hikami], and from them decomposition of the multiplicity spaces into M24 representations can be deduced.

For the first 1000 or so multiplicity spaces, consistent decomposition was found; argument of [Gannon] then implies that this will be true for all multiplicity spaces.

Why Mathieu?

Mukai Theorem: any finite group of symplectic automorphisms ($\longrightarrow$ fixes all three complex structures) of a K3 surface is isomorphic to a subgroup of the Mathieu group M23.

[The Mathieu group M23 is the subgroup of M24 (as a subgroup of the permutation group) that consists of the permutations with at least one 1-cycle.]

However, the symplectic automorphisms of K3 have at least 5 orbits on the set of 24 points: form proper subgroups of M23.

Conjugacy classes

Because of this, useful to separate the conjugacy classes of M24 into two classes:

16 classes with representative in M23

1A, 2A, 3A, 4B, 5A, 6A, 7A, 7B, 8A,
11A, 14A, 14B, 15A, 15B, 23A, 23B

geometric

non-geometric

10 classes with no representative in M23

2B, 3B, 4A, 4C, 6B, 10A, 12A, 12B, 21A, 21B.

Geometrical explanation?

This geometrical point of view therefore only explains small part of M24 Mathieu Moonshine.

Natural idea: generalise Mukai Theorem to K3 sigma-models.

see also [Taormina, Wendland]

Moduli space

Recall structure of moduli space:

$$\mathcal{M} = O(4,20;\mathbb{Z}) \backslash \underbrace{O(4,20;\mathbb{R})/O(4,\mathbb{R}) \times O(20,\mathbb{R})}_{\Pi \subset \mathbb{R}^{4,20}}$$

discrete autos of fixed
`charge lattice'

$\Gamma^{4,20} \subset \mathbb{R}^{4,20}$

Grassmannian, describing
choice of 4-plane

[Think of $\mathbb{R}^{4,20}$ as the even real homology; the sigma-model is determined by choosing a Ricci flat metric and B-field on K3 manifold — corresponds to choice of 4-plane Π .]

K3 symmetries

In string theory, the real homology can be identified with the space of RR ground states.

Furthermore, the lattice $\Gamma^{4,20} \subset \mathbb{R}^{4,20}$ is the RR charge lattice of the D-branes of the theory (integer homology).

> The symmetries of a given K3 sigma model are those automorphisms of the RR charge lattice, i.e., elements in $\mathrm{O}(4,20;\mathbb{Z})$, that leave the 4-plane Π invariant.

Susy symmetries

The N=(4,4) superconformal algebra contains the R-symmetry

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$$

With respect to this symmetry, the 24 RR ground states decompose as

$$20 \cdot (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{2})$$

as follows from

$$\mathcal{H}_{\mathrm{BPS}} = 20 \cdot \mathcal{H}_{\frac{1}{4}, j=0} \oplus 2 \cdot \mathcal{H}_{\frac{1}{4}, j=\frac{1}{2}} \oplus \bigoplus_{n=1}^{\infty} D_n \times \mathcal{H}_{n+\frac{1}{4}, j=\frac{1}{2}}$$

Mathieu symmetries

The subspace Π can be identified with the space spanned by the 4 RR ground states that transform as (**2**,**2**).

In the context of the Mathieu symmetries, the (**2**,**2**) states are singlets

$$-2 = -2 \cdot \mathbf{1}$$

Thus we are interested in K3 sigma model symmetries for which the automorphisms leave Π pointwise invariant.

Susy symmetries

Because these 4 RR ground states correspond to the 4 spacetime supercharges, the physics interpretation of the above condition is that these are the symmetries that preserve

spacetime supersymmetry

Analysis of groups

For a given sigma model characterised by Π
the susy symmetry group is therefore

$$G_\Pi \subset \mathrm{O}(4,20;\mathbb{Z}) \ \text{ that fixes } \Pi \text{ pointwise}$$

To study these groups, define $(G \equiv G_\Pi, L \equiv \Gamma^{4,20})$

$$\begin{aligned} L^G &= \{v \in \Gamma^{4,20} \ : \ g \cdot v = v \ \ \forall g \in G\} \\ L_G &= \left(L^G\right)^\perp \end{aligned}$$

cf. [Kondo's proof of Mukai Thm]

Analysis of groups

Π has signature (4,0) — thus L_G has negative signature, and is of rank less or equal to 20.

Regularity of sigma model: L_G does not contain any roots (vectors of length squared -2).

Then can embed $L_G(-1) \hookrightarrow \Lambda_L$ [Nikulin]

such that action of G extends to whole Leech lattice:

$$G \subseteq \mathrm{Co}_1 \subset \mathrm{Co}_0 \equiv \mathrm{Aut}(\Lambda_L)$$

Precise description

In order to give more precise description of the possible symmetry groups study the subgroups of the Conway group that fix a sublattice of rank at least 4 in the Leech lattice.

The detailed analysis is quite technical, but using results of Curtis, Conway et.al., and Allcock one arrives at the following list:

Classification of symmetries

The possible symmetry groups of K3 sigma models are given by:

[MRG,Hohenegger,Volpato]

$$\begin{aligned}
&\text{(i)} && G = G'.G'', \ \text{where } G' \text{ is a subgroup of } \mathbb{Z}_2^{11}, \\
& && \qquad\qquad\qquad \text{and } G'' \text{ is a subgroup of } \mathbb{M}_{24} \\
&\text{(ii)} && G = 5^{1+2} : \mathbb{Z}_4 \\
&\text{(iii)} && G = \mathbb{Z}_3^4 : A_6 \\
&\text{(iv)} && G = 3^{1+4} : \mathbb{Z}_2.G'', \ \text{where } G'' \text{ is either trivial,} \\
& && \qquad\qquad\qquad \mathbb{Z}_2 \text{ or } \mathbb{Z}_2^2.
\end{aligned}$$

extra-special group

semi-direct product

[. normal subgroup]

Observations

(1) None of the K3 sigma-models has M24 as symmetry group.

[In case (i) only those elements of M24 appear which have at least 4 orbits when acting as a permutation. Thus 12B, 21A, 21B, 23A, 23B never arise.]

(2) Some K3 sigma-models have symmetries that are not contained in M24.

[In particular, this is the case for (ii), (iii) and (iv), as well as (i) with non-trivial G'.]

(3) All symmetry groups fit inside the Conway group

[but there is no evidence for `Conway Moonshine' in the elliptic genus.]

Existence

Provided that for any `regular' 4-plane Π, a non-singular K3 sigma model exists, the above analysis also implies that all of these cases are realised by some K3 sigma model.

We have subsequently shown (see also below) that at least all cases (ii) — (iv) are indeed realised. [In case (i) obviously only those combinations should be realised that come from a suitable subspace Π.]

Exceptional cases

Call K3 sigma model exceptional if symmetry group is not contained in M24.

Observation: [MRG,Volpato]

- If sigma model is cyclic torus orbifold, then it it is always exceptional.

- Every sigma model corresponding to case (ii) is equal to $\mathbb{T}^4/\mathbb{Z}_5$

- Every sigma model corresponding to case (iii) & (iv) is equal to $\mathbb{T}^4/\mathbb{Z}_3$

Quantum Symmetry

Basic idea of argument: K3 sigma model is cyclic torus orbifold if and only if it has `quantum symmetry' whose orbifold is a torus.

$$\mathrm{K3} = \mathbb{T}^4/\mathbb{Z}_n \ \leftrightarrow \ \mathbb{T}^4 \cong \mathrm{K3}/\tilde{\mathbb{Z}}_n$$

Suppose that K3 has a symmetry g of order n, which satisfies level matching (= trivial multiplier phase) so that orbifold is consistent.

Orbifold K3

Suppose that K3 has a symmetry g of order n, which satisfies level matching (= trivial multiplier phase) so that orbifold is consistent.

By usual orbifold rules, elliptic genus of orbifold equals

$$\tilde{\phi}(\tau, z) = \frac{1}{n} \sum_{i,j=1}^{n} \phi_{g^i, g^j}(\tau, z)$$

twisted twining genus -- can be obtained from twining genus of g^d where d=gcd(i,j,n) by suitable modular transformation.

Orbifold theory

The orbifold is a torus theory if and only if its elliptic genus vanishes for z=0.

But for z=0 we have simply

$$\phi_{g^d}(\tau, 0) = \mathrm{Tr}_{24}(g^d) = \mathrm{constant}$$

and hence

$$\phi_{g^i, g^j}(\tau, 0) = \mathrm{Tr}_{24}(g^{\mathrm{gcd}(i,j,n)})$$

Orbifold theory

Thus

$$\tilde{\phi}(\tau, 0) = \frac{1}{n} \sum_{i,j=1}^{n} \mathrm{Tr}_{24}(g^{\mathrm{gcd}(i,j,n)}) \ .$$

coincides with trace in standard 24d rep of Conway group.

For each conjugacy class of Conway can hence decide whether orbifold (if consistent) will lead to torus or not.

Conway classes

Conway group has 167 conjugacy classes, but only 42 contain symmetries that are realised by some K3.

Of these, 31 have necessarily trivial multiplier phase (since trace over 24d rep non-trivial):

- 21 lead to K3, i.e. $\tilde{\phi}(\tau,0)=24$

- 10 lead to T4, i.e. $\tilde{\phi}(\tau,0)=0$

none of them has representative in M24

Quantum symmetry of T4-orbifold is always exceptional!

Other classes

The remaining 11 classes all lead to inconsistent orbifolds (=level matching not satisfied), i.e.

$$\tilde{\phi}(\tau, 0) \neq 0, 24$$

except for one case that can be identified with a $\mathbb{Z}_2$ torus orbifold. (Its quantum symmetry is also not inside M24.)

[MRG, Taormina, Volpato, Wendland]

- If sigma model is cyclic torus orbifold, then its quantum symmetry is always exceptional. ✓

Other cases

To prove:

- Every sigma model corresponding to case (ii) is equal to $\mathbb{T}^4/\mathbb{Z}_5$ ✓

note that case (ii) contains 5C class of Conway (not in M24) whose orbifold is a torus.

We have also constructed the asymmetric $\mathbb{Z}_5$ orbifold explicitly and checked that it has the symmetry in case (ii).

Similarly for:

- Every sigma model corresponding to case (iii) & (iv) is equal to $\mathbb{T}^4/\mathbb{Z}_3$ ✓

Exceptions

However, there are also exceptional K3 sigma-models in case (i). Some of them are cyclic torus orbifolds, but some of them are not

--- maybe non-abelian orbifolds?

More fundamentally, it would be important to understand what the significance of being a (cyclic) torus orbifolds is (if any)….

Some comments I

The above analysis applies to susy preserving symmetries of full CFT sigma model — however, elliptic genus only sees (BPS) part of the spectrum.

The symmetries of the BPS spectrum could be larger or smaller…

larger: not every symmetry needs to lift to full theory
smaller: symmetry generators may not define consistent operator on BPS cohomology

[e.g. for $\mathbb{T}^4/\mathbb{Z}_2$, the twining genus of quantum symmetry has formally $\mathrm{Tr}_{\mathbf{90}}(Q) = -102\ldots$]

Some comments I (ctd)

As a vector space, this cohomology is essentially the \`BPS algebra' of [Harvey, Moore] — which is not just a quotient space, but also carries algebra structure.

Thus a natural explanation of Mathieu Moonshine would be if the automorphism group of the \`BPS algebra' was M24….

[This would also tie in nicely with the observation that Generalised Mathieu Moonshine behaves exactly like a holomorphic CFT…]

[MRG, Persson, Ronellenfitsch, Volpato]

Some comments II

One difficulty with trying to work this out explicitly is that at a generic point in moduli space, where $\dim(D_n) = |A_n|$, the theory does not seem to have any (?) non-trivial symmetries.

On the other hand, all known CFT descriptions of K3s (e.g. orbifolds) sit at rather special points in moduli space where most multiplicity spaces are non-minimal, i.e.

$$\dim(D_n) > |A_n|$$

cf. [Taormina, Wendland]

How does one get rid of these additional states?

Some comments III

The BPS spectrum involves virtual representations of M24.

It is striking though that the `virtualness' is restricted to the `massless' states.

Is this the analogue of the constant term in Monstrous Moonshine?

[Gannon]

Some comments III (ctd)

So should we get rid of these massless states, just as the $\mathbb{Z}_2$ orbifold of the Leech lattice theory does for the case of Monstrous Moonshine?

Incidentally, in effect this is what happens when one considers Mathieu Moonshine from the perspective of mock modular forms….

Some comments IV

From the viewpoint of K3 sigma models, however, there is no natural reason to discard these massless states — in fact, they are needed for correct modular behaviour…

So K3 perspective is really about Jacobi forms, not mock modular forms…

On the other hand, mock interpretation of Mathieu Moonshine will probably not involve K3 (but may be the natural analogue of Monstrous Moonshine)…

The BIG question

The big question thus remains:

> Is Mathieu Moonshine really a K3 phenomenon, or is it just a property of some other structure associated to the weak Jacobi form $\phi_{0,1}$, e.g. the corresponding mock modular form.