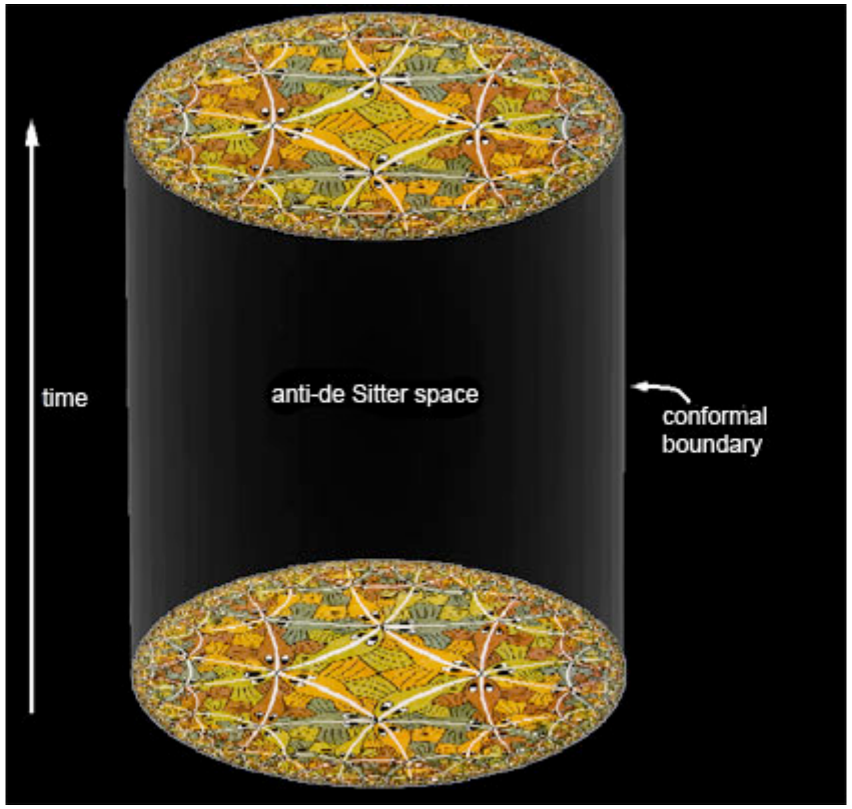
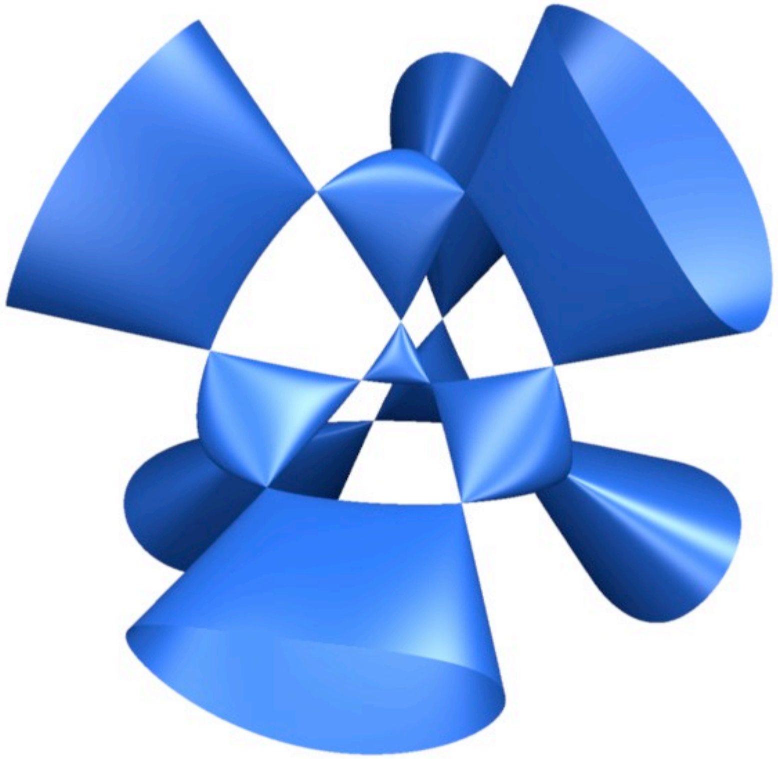
Some comments on equivariant Gromov-Witten theory and 3d gravity (a talk in two parts)





Shamit Kachru
(Stanford & SLAC)

Much of the talk will be review of things well known to various parts of the audience.

To the extent anything very new appears, it is based on two recent collaborative papers:

* arXiv : 1507.00004 with Benjamin, Dyer, Fitzpatrick

* arXiv : 1508.02047 with Cheng, Duncan, Harrison

Some slightly older work with other (also wonderful) collaborators makes brief appearances.

Today, I'd like to talk about two distinct topics. Each is related to the general subjects of this symposium, but neither has been central to any of the talks we've heard yet.

I. Extremal CFTs and quantum gravity

...where we see how special chiral CFTs with sparse spectrum may be important in gravity...

II. Equivariant Gromov-Witten & K3

...where we see how certain enumerative invariants of K3 may be related to subjects we've heard about...

I. Extremal CFTs and quantum gravity

A fundamental role in our understanding of quantum gravity is played by the holographic correspondence between conformal field theories and AdS gravity.

In the basic dictionary between these subjects

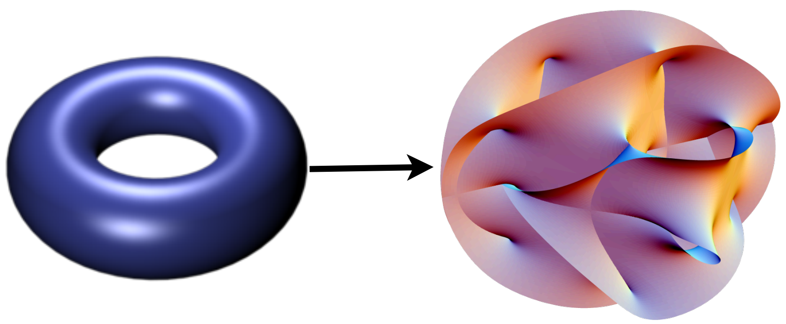
$$conformal\ symmetry \ \leftrightarrow\ AdS\ isometries$$

$$primary\ field\ of\ dimension\ \Delta \ \leftrightarrow\ bulk\ quantum\ field\ of\ mass\ m(\Delta)$$

$$\cdots\cdots$$

There are famous examples where the correspondence can be made very precise. One, relevant to this conference, relates the sigma model with target Hilb(K3) to superstrings compactified on AdS3xS3xK3.

$$\phi : \Sigma \to K3$$



$$L = 2t \int d^2 z \left(\frac{1}{2} g_{IJ}(\Phi) \partial_z \phi^I \partial_{\bar{z}} \phi^J + \frac{i}{2} g_{IJ} \psi_-^I D_z \psi_-^J + \frac{i}{2} g_{IJ} \psi_+^I D_{\bar{z}} \psi_+^J + \frac{1}{4} R_{IJKL} \psi_+^I \psi_+^J \psi_-^K \psi_-^L \right).$$

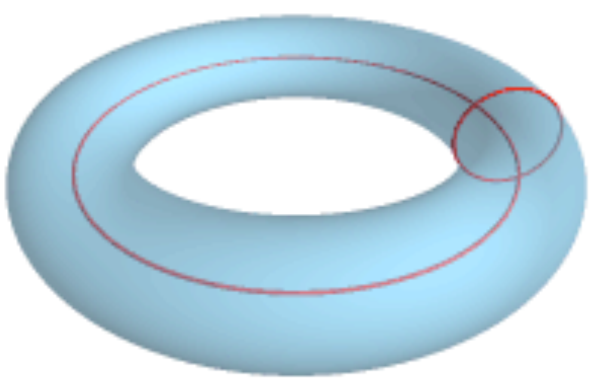
Here is a natural question: given a desired spectrum of bulk AdS fields ("set of elementary particles"), does there exist a dual CFT which defines it non-perturbatively?

This question is far beyond our reach. But we can explore simple examples. Perhaps the simplest is pure 3d quantum gravity.

Do there exist conformal field theory duals to pure 3d (super)gravity?

The spectrum of pure 3d gravity would be just the graviton multiplet and the associated multiparticle states.

We can use the spectrum to read off the partition function of the conjectural dual CFT:



$$Z = \sum_n c_n q^n \qquad q = e^{2\pi i \tau}$$

$c_n = \#$ of states at mass level n

$$Z(\tfrac{a\tau+b}{c\tau+d}) \sim Z(\tau)$$

You would quickly see that the spectrum of a theory with just graviton and its descendants is not consistent. This would give a partition function:

$$Z_0(q) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n}.$$

where c = 24k is the central charge of the theory, and determines the AdS3 cosmological term.

This alone is not modular. Can we fix it up?

Yes! We forgot that in gravity, we expect also black holes. The spectrum should contain:

$$E << c \;:\; graviton \;+\; descendants$$

$$E \geq c \;:\; black\ holes$$

Witten's criterion (too strong??):

Require the polar pieces in Z are graviton + descendants, and "anything goes" after that.

Happily, the "anything goes" portion of the spectrum is then determined uniquely by modularity!

$$Z(q) = \sum_{r=0}^{k} f_r J^r,$$

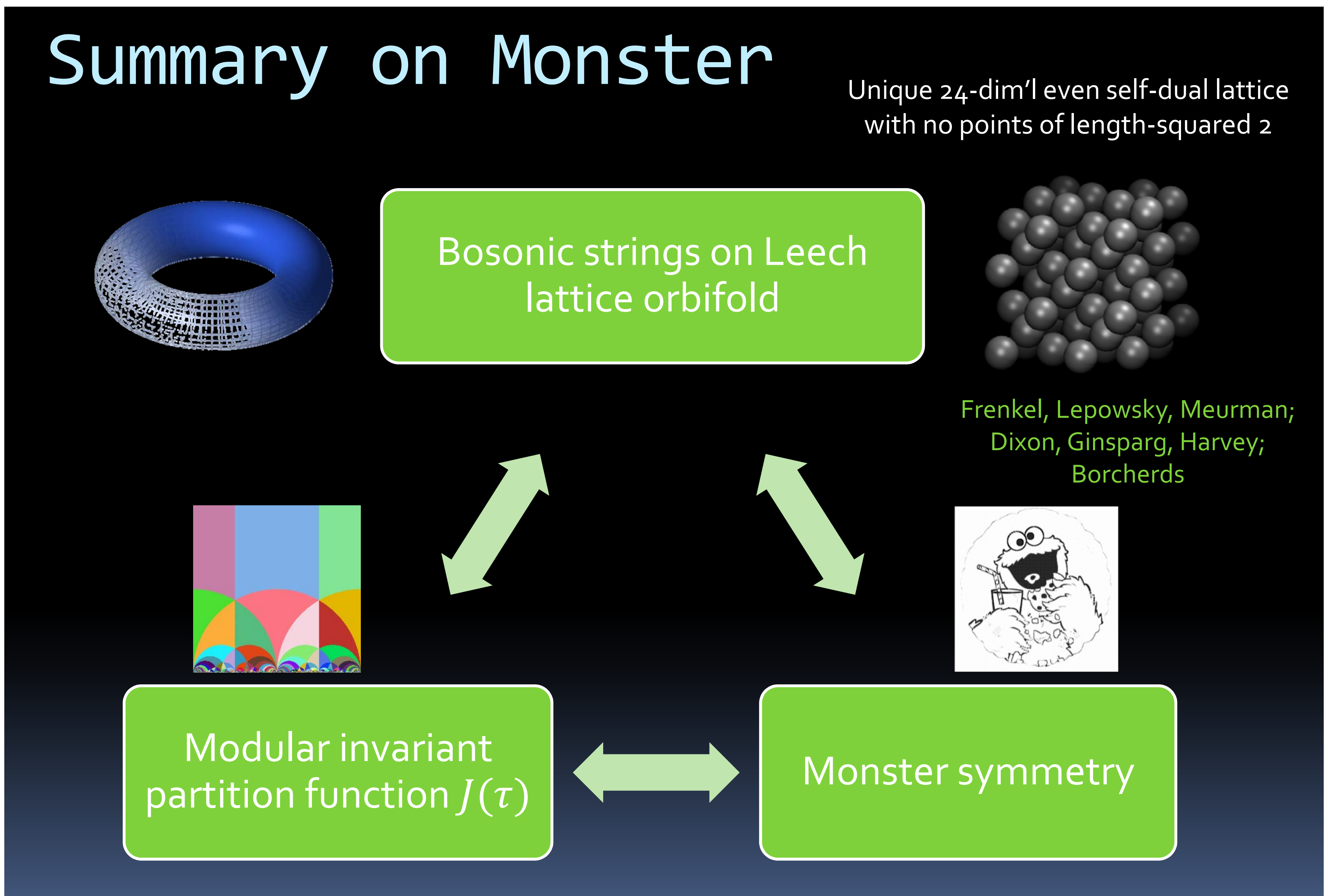
The k+1 conditions on the polar terms determine the function uniquely.

The first case, k=1, yields the J function. A few more:

$$\begin{aligned}
Z_2(q) =& J(q)^2 - 393767 \\
=& q^{-2} + 1 + 42987520q + 40491909396q^2 + \dots \\
Z_3(q) =& J(q)^3 - 590651J(q) - 64481279 \\
=& q^{-3} + q^{-1} + 1 + 2593096794q + 12756091394048q^2 + \dots \\
Z_4(q) =& J(q)^4 - 787535J(q)^2 - 8597555039J(q) - 644481279 \\
=& q^{-4} + q^{-2} + q^{-1} + 2 + 81026609428q + 1604671292452452276q^2 + \dots.
\end{aligned}$$

no CFTs
with such Z
are known to
exist

Interestingly, the k=1 case yields as a candidate dual to pure gravity with deep negative cosmological constant, precisely the FLM theory!



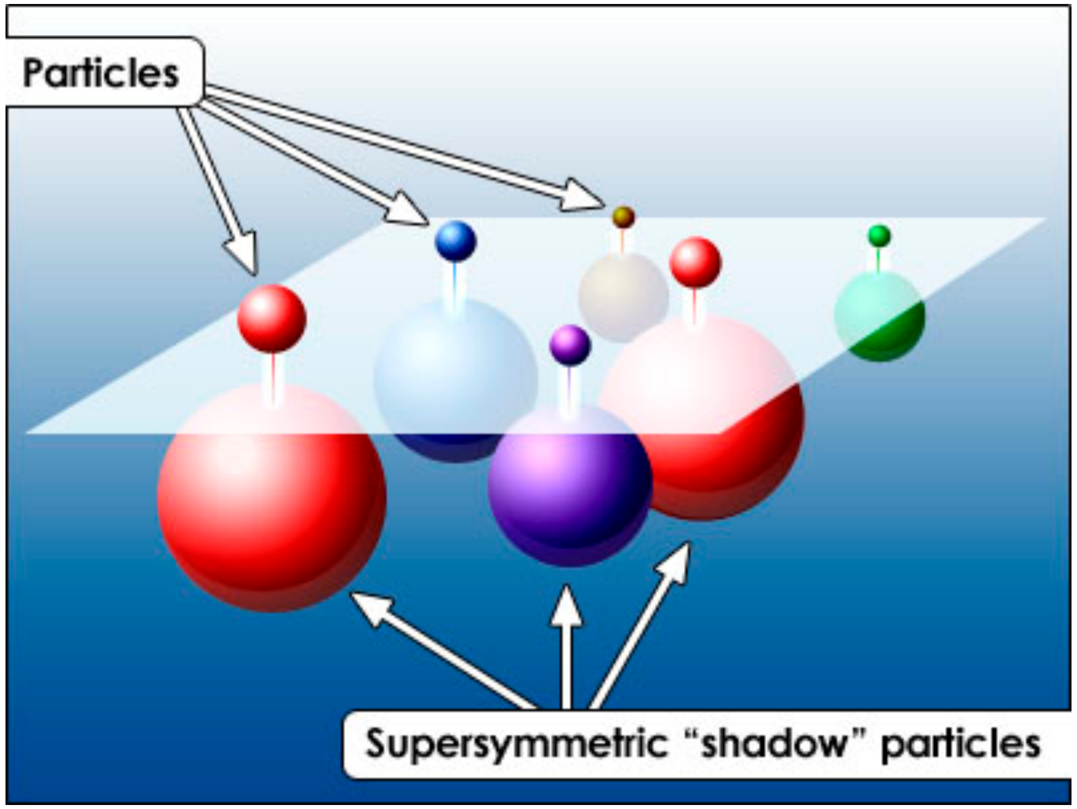
Caveats:

* k=1 is, at most, barely geometry -- but note

$$\log(196,883) \sim 4\pi$$

Bekenstein-Hawking entropy of lightest black holes is close to working...

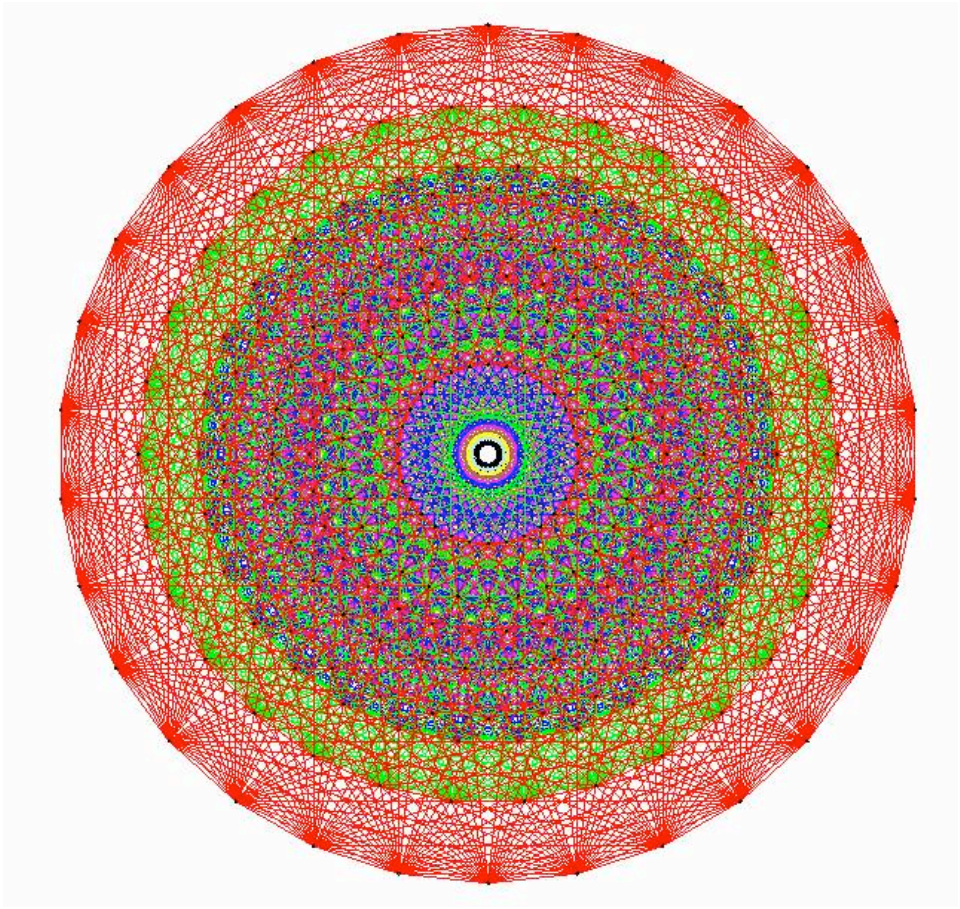
* There is a subtlety with holomorphic factorization. Perhaps this is best thought of as a duality between "chiral CFT" and "chiral gravity." I will not discuss this further here.



One can run the same story with supersymmetry.

Now, c=12k is accessible.

With minimal SUSY, at k=1, again a candidate CFT exists!



The supersymmetrized theory of bosons propagating on the E8 lattice, realizes moonshine for Conway's largest sporadic group!

It has an equivalent description as the orbifold of the theory of 24 free chiral fermions:

$$\psi_i(z), \quad \psi \to -\psi$$

FLM; Duncan

The partition function is:

$$
\begin{aligned}
Z_{NS,E8}(\tau) &= \mathrm{tr}_{NS}\, q^{L_0 - c/24} = \frac{1}{2}\left(\frac{E_4\theta_3^4}{\eta^{12}} + 16\frac{\theta_4^4}{\theta_2^4} + 16\frac{\theta_2^4}{\theta_4^4}\right)(\tau) \\
&= q^{-1/2} + 0 + 276\, q^{1/2} + 2048\, q + 11202\, q^{3/2} + \cdots ,
\end{aligned}
$$

with simple Co_0 decompositions, e.g. 276 = 276 and:

$$2048 = 24 + 2024$$

$$11202 = 1 + 276 + 299 + 1771 + 8855$$

It will play a role in part II of the talk.

Again, at k>2, no extremal CFTs with minimal supersymmetry are known. (For k=2, wait).

Increasing the supersymmetry has always been a strategy that string theorists find useful. Let's do that here also.

Gaberdiel, Gukov, Keller, Moore and Ooguri asked a natural question in the context of 2d N=(2,2) supersymmetry.

To an N=2 theory, we can associate a Jacobi form:

$$Z_{EG}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}(-1)^{F_L} e^{2\pi i z J_0} q^{L_0 - \frac{c}{24}} (-1)^{F_R} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \ ,$$

the elliptic genus (we could use the partition function if the theory is chiral).

We can declare that a theory is extremal if the elliptic genus is "as close as possible to that determined by the vacuum character of the N=2 algebra."

Given a Fourier expansion for a weak Jacobi form of index m and weight 0:

$$\phi(\tau, z) = \sum_{n \geq 0, \ell \in \mathbb{Z}} c(n, \ell) q^n y^\ell$$

it can be useful to define the polar part:

$$\mathcal{P}^{(m)} := \{(\ell, n) : 1 \leq \ell \leq m, \quad 0 \leq n, \quad p = 4mn - \ell^2 < 0\} \ .$$

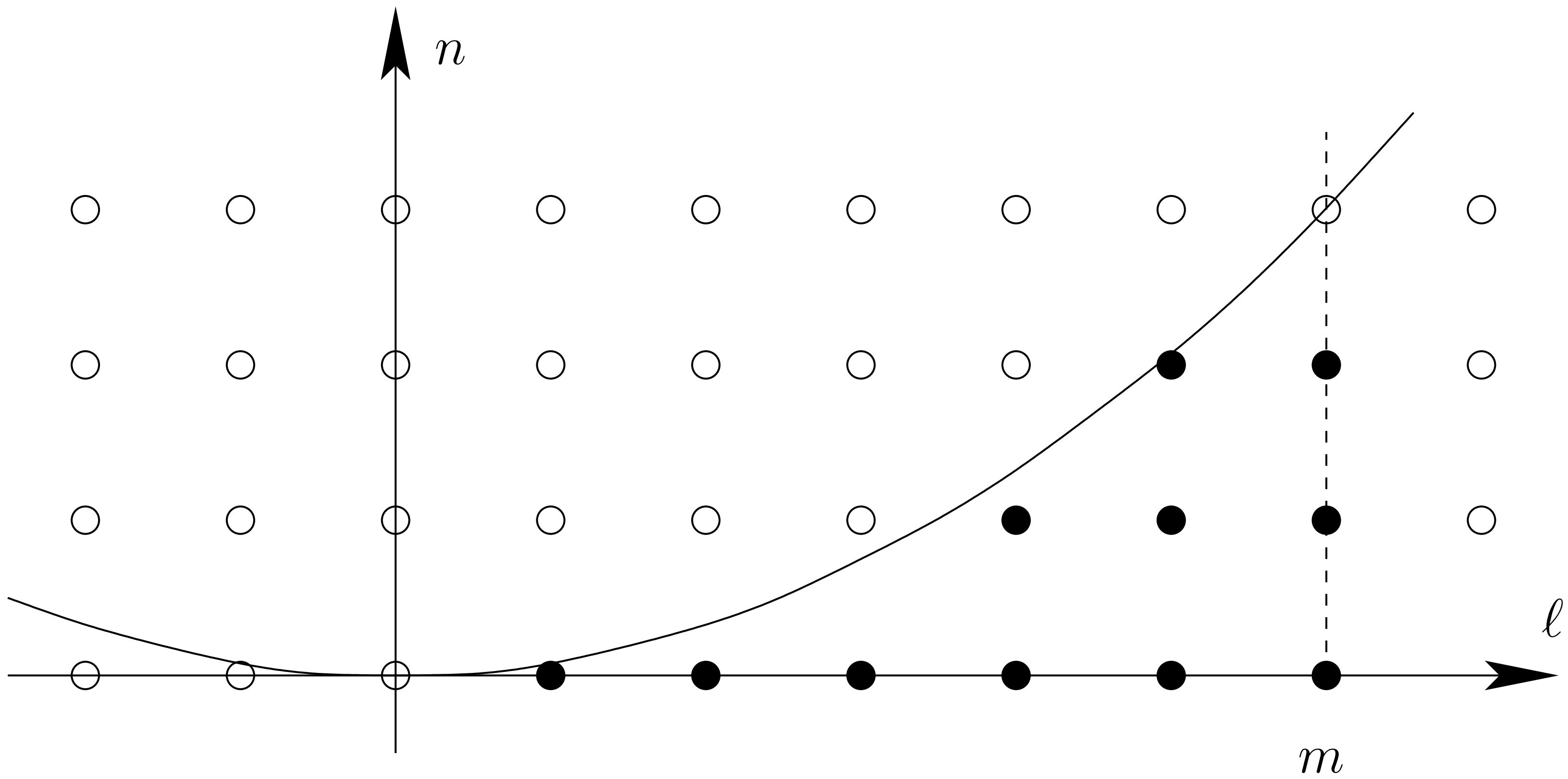


Figure 1: A cartoon showing polar states (represented by "•") in the region $\mathcal{P}^{(m)}$. Spectral flow by $\theta = \frac{1}{2}$ relates these states to particle states in the NS sector of an $\mathcal{N} = 2$ superconformal field theory which are holographically dual to particle states in AdS_3.

Using facts from Eichler and Zagier, GGKMO were able to prove that for large index m, no extremal SCFT (by this definition) can exist.

However, there could be theories at

$$c = 6, 12, 18, 24, 30, 42, 48, 66, 78$$

We can provide explicit constructions for chiral CFTs realizing the preferred Jacobi forms at c=12,24. This doubles the list of known "extremal CFTs".

The construction at c=12 arose "by accident" in a project attempting to do something else.

One can view the super-E8 theory as a theory with a variety of different supersymmetry algebras.

Superalgebra	Geometrical Representation	Global symmetry group
$\mathcal{N}=0$	$\mathbb{R}^{24}$	$Spin(24)$
$\mathcal{N}=1$	Λ_{Leech}	Co_0
Spin(7)	Λ_{Leech}, fixed 1-plane	M_{24}
$\mathcal{N}=2$	Λ_{Leech}, fixed 2-plane	M_{23}
$\mathcal{N}=4$	Λ_{Leech}, fixed 3-plane	M_{22}

The fixed planes give one, two, or three fermions out of which one can write currents to enhance the N=1 algebra to other choices.

Cheng, Dong, Duncan, Harrison, SK, Wrase;
Benjamin, Harrison, Paquette, SK, Whalen;
Cheng, Harrison, SK, Whalen

The first few "extremal Jacobi forms" are:

$$
\begin{aligned}
Z_{EG}^{m=1} &= \varphi_{0,1} \\
Z_{EG}^{m=2} &= \frac{1}{6}\varphi_{0,1}^2 + \frac{5}{6}\varphi_{-2,1}^2 E_4 \\
Z_{EG}^{m=3} &= \frac{1}{48}\varphi_{0,1}^3 + \frac{7}{16}\varphi_{0,1}\varphi_{-2,1}^2 E_4 + \frac{13}{24}\varphi_{-2,1}^3 E_6 \\
Z_{EG}^{m=4} &= \frac{67}{144}\varphi_{-2,1}^4 E_4^2 + \frac{11}{27}\varphi_{-2,1}^3\varphi_{0,1} E_6 + \frac{1}{8}\varphi_{-2,1}^2\varphi_{0,1}^2 E_4 + \frac{1}{432}\varphi_{0,1}^4 \ .
\end{aligned}
$$

The m=2 case precisely coincides the with chiral partition function of the N=2 formulation of this c=12 theory. It has M23 symmetry.

We have also been able to engineer, with Benjamin, Dyer and Fitzpatrick, an N=2 theory with c=24 matching the m=4 case above.

The m=4 case has character expansion:

$$
\begin{aligned}
Z_{EG}^{m=4} &= \mathrm{ch}^{N=2}_{\frac{7}{2};1,4} + 47\ \mathrm{ch}^{N=2}_{\frac{7}{2};1,0} \\
&+ (23 + 61984q + \cdots)\mathrm{ch}^{N=2}_{\frac{7}{2};2,4} \\
&+ (2024 + 485001q + \cdots)(\mathrm{ch}^{N=2}_{\frac{7}{2};2,3} + \mathrm{ch}^{N=2}_{\frac{7}{2};2,-3}) \\
&+ (14168 + 1659174q + \cdots)(\mathrm{ch}^{N=2}_{\frac{7}{2};2,2} + \mathrm{ch}^{N=2}_{\frac{7}{2};2,-2}) \\
&+ (32890 + 2969208q + \cdots)(\mathrm{ch}^{N=2}_{\frac{7}{2};2,1} + \mathrm{ch}^{N=2}_{\frac{7}{2};2,-1})\ .
\end{aligned}
$$

suggestive of Mathieu group

Because of the small size of the space of Jacobi forms of index 4 and weight 0, it is fixed just by a few coeffs:

$$Z_{EG}^{m=4,RR} = \frac{1}{y^4} + 46 + y^4 + \mathcal{O}(q)\ .$$

In fact, a familiar friend from recent talks allows us to construct this theory.

The c=24 theories are conjecturally classified. The simplest examples are associated with Niemeier lattices.

From these a simple construction follows:

* The theory associated to the A_1^{24} Niemeier lattice can be orbifolded by the canonical $\mathbb{Z}_2$ symmetry.

* The resulting theory has a supersymmetry, by analogy with the Dixon-Ginsparg-Harvey construction.

* In fact, by choosing a U(1) generator in one of the 24 copies of A_1, one can promote this to N=2 supersymmetry.

* The 72 currents associated with the 24 copies of SU(2) current algebra split into 48 which have half-integral singularities with the supercharge, and 24 which have integral singularities.

* These facts immediately allow one to prove that this theory has chiral partition function given by

$$Z_{EG}^{m=4} \quad = \quad \frac{67}{144}\varphi^4_{-2,1}E_4^2 + \frac{11}{27}\varphi^3_{-2,1}\varphi_{0,1}E_6 + \frac{1}{8}\varphi^2_{-2,1}\varphi^2_{0,1}E_4 + \frac{1}{432}\varphi^4_{0,1} \ .$$

It also has an M23 symmetry: the M24 of the A_1^{24} Niemeier lattice is broken to M23 by the choice of N=2 R-current.

We are currently pursuing various ideas to find analogous N=2 constructions at higher values of c, perhaps (by slightly relaxing the criteria) even values "not allowed" by GGKMO.

More generally, I think connections between the new objects playing a role in moonshine and natural objects in string theory or quantum gravity, are worth pursuing.

II. Enumerative invariants of K3 surfaces

...and now for something completely different.

Happily, we have built up a stock of objects and facts which I can now use to simply explain observations in our paper which appeared yesterday on the arXiv.

Motivation: In summer 2014, three prominent mathematicians (Katz, Klemm, Pandharipande) wrote a paper "On the motivic stable pairs invariants of K3 surfaces," whose abstract states: "Numerical data suggest the motivic invariants are linked to the Mathieu M24 phenomena."

Let me try to explain the relevant invariants.

There is a very easy way to count BPS states in type IIA string theory on K3. This theory enjoys a string duality, relating it to heterotic strings on T4.

In the heterotic theory, the string has right-moving supersymmetry. Any strings in the right-moving ground state but with arbitrary left excitations, are BPS.

$$\text{BPS partition function} = \frac{1}{\eta(q)^{24}}$$

One might, however, want an intrinsically type II accounting. This was first provided by Yau-Zaslow.

Let $\mathcal{M}_n^H$ denote the moduli space of holomorphic curves of genus n equipped with a flat U(1)-bundle, living inside K3.

Then if we define

$$d_n = \chi(\mathcal{M}_n^H)$$

$$\sum_{n\geq 0} d_n q^{n-1} \;=\; \frac{1}{\Delta(\tau)} \;=\; q^{-1}(1 + 24q + 324q^2 + 3200q^3 + \cdots)$$

A more intuitive version of this formula comes in yet another duality frame. Instead of imagining D2-branes wrapping curves, we can imagine n D0-branes inside one D4-brane.

Then the same counting function arises from

$$\sum_{n\geq 0}\chi(K3^{[n]})q^{n-1} = \frac{1}{\Delta(\tau)},$$

Gottsche

and the construction with the Hilbert scheme of n points on a K3 surface becomes manifest.

Now, we could hope to gain more information than that provided by this function. For instance, the space-time physics of type II strings on K3 (times a circle) has SU(2) x SU(2) symmetry.

Can't we write a more refined generating function which keeps track of the SU(2) Cartan quantum numbers?

$$\sum_{n\geq 0} \chi_{-y}(K3^{[n]}) y^{-n} q^{n-1} = q^{-1} \prod_{k>0} (1-yq^k)^{-2}(1-q^k)^{-20}(1-y^{-1}q^k)^{-2}$$

$$= (-y+2-y^{-1}) \frac{\eta(\tau)^6}{\theta_1^2(\tau,z)} \frac{1}{\Delta(\tau)}.$$

Moving to the χ_y genus gives the "KKV invariants."

But we expect more refinement from both sides -- two Cartan U(1)s exist, and K3 cohmology admits two gradings. This is the role of the "KKP invariants."

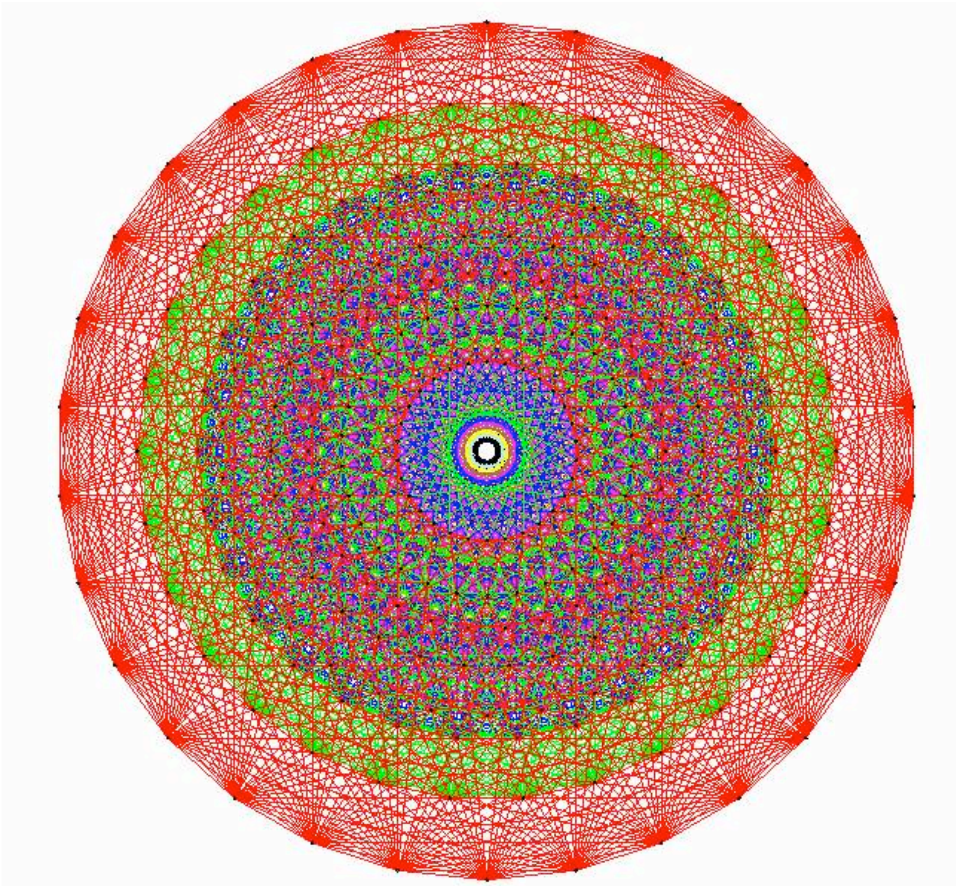
$$
\begin{aligned}
&\sum_{n\geq 0} \chi_{\mathrm{Hodge}}(K3^{[n]})q^{n-1} \\
&= q^{-1}\prod_{k>0}(1-uyq^k)^{-1}(1-u^{-1}yq^k)^{-1}(1-q^k)^{-20}(1-uy^{-1}q^k)^{-1}(1-u^{-1}y^{-1}q^k)^{-1} \\
&= (u-y-y^{-1}+u^{-1})\frac{\eta(\tau)^6}{\theta_1(\tau,z+w)\theta_1(\tau,z-w)}\frac{1}{\Delta(\tau)},
\end{aligned}
$$

From the q-expansions of these functions, via formulae a bit too elaborate for me to reproduce here, one can obtain various famous enumerative invariants ("curve counting formulae") associated to K3.

Trace functions in the Moonshine module $V^{s\natural}$

The interesting (but, perhaps, unsurprising) thing, is that all of these can be obtained from trace functions in an a priori unconnected Moonshine module.

Let us return to our friend:



Its Ramond ground states span a 24 of the Conway group.

As Duncan explained in his talk, there are states which are related by the operator-state correspondence to generators of a c=6 N=4 algebra:

c.f. Duncan, Mack-Crane

$$
\begin{aligned}
\jmath^3 &:= \frac{1}{4}(a_X^-(-\tfrac{1}{2})a_X^+(-\tfrac{1}{2})\mathbf{v} + a_Z^-(-\tfrac{1}{2})a_Z^+(-\tfrac{1}{2})\mathbf{v}),\\
\jmath^\pm &:= \frac{i}{2}a_X^\pm(-\tfrac{1}{2})a_Z^\pm(-\tfrac{1}{2})\mathbf{v},\\
\tau_1^\pm &:= \sqrt{2}\left(a_X^-(0) \pm a_X^+(0) \pm a_X^\mp(0)a_Z^-(0)a_Z^+(0)\right)\tau_{\mathrm{tw}},\\
\tau_2^\pm &:= \mp i\sqrt{2}\left(a_Z^-(0) \pm a_Z^+(0) \pm a_Z^\mp(0)a_X^-(0)a_X^+(0)\right)\tau_{\mathrm{tw}}
\end{aligned}
$$

With these choices, the U(1)-graded partition function of this chiral theory turns out to be:

$$Z^{s\natural}(\tau,z) := \mathrm{Tr}((-1)^F q^{L(0)-\frac{c}{24}} y^{J(0)} | V^{s\natural}_{\mathrm{tw}})$$
$$= \frac{1}{2}\frac{1}{\eta^{12}(\tau)}\sum_{i=2}^{4}(-1)^{i+1}\theta_i^2(\tau,z)\theta_i^{10}(\tau,0)$$

And happily,

$$Z^{s\natural}(\tau,z) = Z_{\mathrm{EG}}(\tau,z;K3).$$

Now, the symmetries of $V^{s\natural}$ are under excellent control.

Our choices equipping it with an N=4 structure here leave symmetries preserving 4-planes in the 24 of Co_0 .

c.f. Gaberdiel, Hohenegger, Volpato

Let G_Π be the subgroup preserving 4-plane Π.

Then for any $g \in G_\Pi$, we can define the twined partition function

$$Z_g^{s\natural}(\tau, z) := \mathrm{Tr}(g(-1)^F q^{L(0)-\frac{c}{24}} y^{J(0)} | V_{\mathrm{tw}}^{s\natural}),$$

and compare with twinings of K3 sigma models:

$$Z_{\mathrm{EG}}(\tau, z; \Pi, g) := \mathrm{Tr}_{\mathrm{RR}}(g(-1)^{F+\bar{F}} q^{L(0)-\frac{c}{24}} \bar{q}^{\bar{L}(0)-\frac{\bar{c}}{24}} y^{J(0)})$$

Natural question:

$$Z_g^{s\natural}(\tau, z) \overset{??}{=} Z_{\mathrm{EG}}(\tau, z; \Pi, g)$$

In many cases yes. Universally? There may be a role for other Umbral modules.

Now, this allows us to define twined enumerative invariants as well. Because although we have only compared the elliptic genus of K3 to the partition function of this Moonshine module, all of the previous generating functions are determined by (special limits of) the elliptic genus!

This may not be obvious, as they involve Euler characters (and generalizations) of Hilbert schemes. But recall that via the formula of Dijkgraaf-Moore-Verlinde-Verlinde,

$$Z_{\mathrm{EG}}(\tau, z; K3) = \sum_{\substack{n,\ell \in \mathbb{Z} \\ n \geq 0}} c(4n - \ell^2)\, q^n y^\ell = 2y + 2y^{-1} + 20 + O(q)$$

determines

$$\sum_{n \geq 0} Z_{\mathrm{EG}}(\tau, z; K3^{[n]}) p^{n-1} = p^{-1} \prod_{\substack{r,s,t \in \mathbb{Z} \\ r>0, s \geq 0}} (1 - q^s y^t p^r)^{-c(4rs - t^2)}.$$

The formula of KKV is just the $\tau \to i\infty$ limit of DMVV.

Then, via the relationship between K3 sigma models and $V^{s\natural}$ we can produce equivariant curve counts.

To get to the KKP invariants, which have an additional grading, one simply needs to grade the partition function in $V^{s\natural}$ by an additional U(1).

This succeeds in relating the KKP (and more primitive) invariants to sporadic groups and moonshine, though perhaps not by the route they expected.

Last but not least:

THANKS TO THE ORGANISERS FOR AN EXCELLENT PROGRAM!!