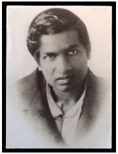
Rogers-Ramanujan and Umbral Moonshine

Ken Ono (Emory University)

Closely Related "Modular" Topics

I. Rogers-Ramanujan type modular units



II. Monstrous Moonshine and Umbral Moonshine

Ramanujan's continued fraction

Famous Fact

The **golden ratio** *is the* **algebraic integral unit**

$$\phi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

as a root of $x^2 - x - 1$.

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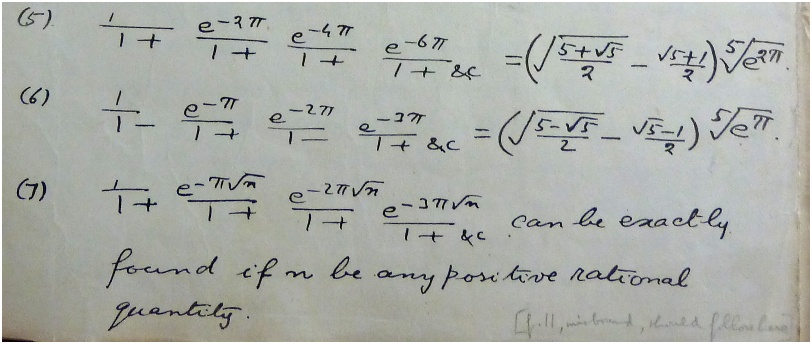
Question

Is there a theory of **special values** *for*

$$R(q) := \frac{1}{1+\frac{q}{1+\frac{q^2}{1+\frac{q^3}{1+\cdots}}}}?$$

Ramanujan's first letter to Hardy


(5) $$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+\&c} = \left(\sqrt{\frac{5+\sqrt5}{2}} - \frac{\sqrt5+1}{2}\right)\sqrt[5]{e^{2\pi}}.$$

(6) $$\frac{1}{1-}\,\frac{e^{-\pi}}{1+}\,\frac{e^{-2\pi}}{1-}\,\frac{e^{-3\pi}}{1+\&c} = \left(\sqrt{\frac{5-\sqrt5}{2}} - \frac{\sqrt5-1}{2}\right)\sqrt[5]{e^{\pi}}.$$

(7) $\frac{1}{1+}\,\frac{e^{-\pi\sqrt n}}{1+}\,\frac{e^{-2\pi\sqrt n}}{1+}\,\frac{e^{-3\pi\sqrt n}}{1+\&c}$ can be exactly found if n be any positive rational quantity.

Hardy's reaction

"[These formulas] defeated me completely. . . . they could only be written down by a mathematician of the highest class. They must be true because no one would have the imagination to invent them."

G. H. Hardy 

Rogers-Ramanujan

Theorem (Rogers, Ramanujan)

We have that

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$
$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})},$$

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and $R(q) = H(q)/G(q)$.

Ubiquity of the RR Identities

- Number theory
- Conformal field theory
- K-theory
- Kac-Moody Lie algebras
- Knot theory
- Probability theory
- Statistical mechanics
- ...

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Remark

RR identities $\Longrightarrow$ *Lepowsky-Wilson program*
$\ldots \Longrightarrow$ **vertex operator theory** $\Longrightarrow$ **Moonshine**.

Ramanujan's Claim

Theorem (Berndt-Chan-Zhang (1996), Cais-Conrad (2006))

If τ is a CM point, then

$$e^{2\pi i\tau/5} \cdot R(e^{2\pi i\tau})$$

is an **algebraic integral unit.**

Fundamental Problems

Problem 1

Is there a larger (and conceptual) framework of identities:

$$\text{“Summatory } q\text{-series”} \quad = \quad \text{“Infinite product modular function”?}$$

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Problem 2

If so, do **natural ratios** *generalize $R(q)$ to give* **integral units***?*

Problem 1

"Theorem" (Griffin-O-Warnaar)

There are four triples (a, b, c) *such that* **for all** $m, n \geq 1$ *we have*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^{bn+c})$$

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Remark

RR identities when $m = n = 1$ *and* $(a, b, c) = (1, 2, -1), (2, 2, -1)$.

Integer Partitions

Definition

A **partition** is a nonincreasing sequence of positive integers

$$\lambda := (\lambda_1, \lambda_2, \dots)$$

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- For positive i we let $m_i :=$ "multiplicity" of size i parts.
- For $n \geq l(\lambda)$ we let $m_0 := n - l(\lambda)$.

Hall-Littlewood symmetric polynomials

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If λ is a partition with $l(\lambda) \leq n$, then let

$$x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n},$$

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The **Hall-Littlewood polynomial** is

$$P_\lambda(x;q) = \frac{1}{v_\lambda(q)} \sum_{w \in S_n} w\left(x^\lambda \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \right).$$

Example 1

For $n \geq 1$ we have

$$P_{(2)}(x_1, x_2, \ldots, x_n; q) = \frac{(1-q)^{n-1}}{(q)_{n-1}} \cdot \sum_{w \in S_n} w\left(x_1^2 \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \right).$$

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We find that

$$P_{(2)}(x_1; q) = x_1^2$$

$$P_{(2)}(x_1, x_2; q) = x_1^2 + x_2^2 + (1-q)x_1x_2$$

$$P_{(2)}(x_1, x_2, x_3; q) = x_1^2 + x_2^2 + x_3^2 + (1-q)(x_1x_2 + x_1x_3 + x_2x_3)$$

$$\vdots \quad = \quad \vdots$$

Example 1 (Continued)

Letting $x_1 = 1, x_2 = q, x_3 = q^2, \ldots,$ *we obtain*

$$\begin{aligned} P_{(2)}(1;q) &= 1 \\ P_{(2)}(1,q;q) &= 1+q \\ P_{(2)}(1,q,q^2;q) &= 1+q+q^2 \\ \vdots \qquad & \qquad \vdots \end{aligned}$$

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More generally, for every $n \geq 1$ *we have*

$$P_{(2)}(1, q, q^2, \ldots, q^n; q) = 1 + q + q^2 + \cdots + q^n.$$

Example 1 (Continued)

- *For each* $n \geq 1$ *we have*

$$P_{(2)}(x_1, \ldots, x_n; q) = \frac{1+q}{2}\left(x_1^2 + \cdots + x_n^2\right) + \frac{1-q}{2}\left(x_1 + \cdots + x_n\right)^2 .$$

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- *Make the identifications*

$$\begin{aligned} (x_1, x_2, \ldots) \quad &\longleftrightarrow \quad (1, q, q^2, \ldots) \\ x_1^r + x_2^r + \cdots + x_n^r \quad &\longleftrightarrow \frac{1}{1-q^r} \end{aligned}$$

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- *This gives us*

$$P_{(2)}(1, q, q^2, \ldots; q) = \frac{(1+q)}{2(1-q^2)} + \frac{1-q}{2(1-q)^2} = \frac{1}{1-q}.$$

Example 2

For $n \geq 2$ find that

$$P_{(2,2)}(x_1 \ldots, x_n; q) = -\frac{q^3-q}{4}(x_1+\cdots+x_n)^2(x_1^2+\cdots+x_n^2) + \frac{q^3-3q+2}{24}(x_1+\cdots+x_n)^4 + \frac{q^3+q+2}{8}(x_1^2+\cdots+x_n^2)^2 + \frac{q^3-1}{3}(x_1+\cdots+x_n)(x_1^3+\cdots+x_n^3) - \frac{q^3+q}{4}(x_1^4+\cdots+x_n^4).$$

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$$P_{(2,2)}(x_1 \ldots, x_n; q) = -\frac{q^3 - q}{4}(x_1 + \cdots + x_n)^2(x_1^2 + \cdots + x_n^2)$$
$$+ \frac{q^3 - 3q + 2}{24}(x_1 + \cdots + x_n)^4 + \frac{q^3 + q + 2}{8}(x_1^2 + \cdots + x_n^2)^2$$
$$+ \frac{q^3 - 1}{3}(x_1 + \cdots + x_n)(x_1^3 + \cdots + x_n^3) - \frac{q^3 + q}{4}(x_1^4 + \cdots + x_n^4).$$

Arguing as before gives:

$$P_{(2,2)}(1, q, q^2, \ldots; q) = \frac{q^2}{(1-q)(1-q^2)}.$$

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$$x_1^r + \cdots + x_n^r.$$

2. *Obtain $P_\lambda(1, q, q^2, \ldots; q^T)$ by replacing*

$$x_1^r + \cdots + x_n^r \quad \longmapsto \quad 1 + q^r + q^{2r} + \cdots = \frac{1}{1 - q^r}.$$

Problem 1

"Theorem" (Griffin-O-Warnaar)

There are four triples (a, b, c) *such that* **for all** $m, n \geq 1$ *we have*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots ; q^{bn+c})$$

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Remark

RR identities when $m = n = 1$ *and* $(a, b, c) = (1, 2, -1), (2, 2, -1)$.

Notation

Definition (Pochammer)

$$(a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}),$$

and

$$\theta(a;q) := (a;q)_\infty (q/a;q)_\infty.$$

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Remark

The $\theta(a;q)$ are "modular functions" studied by Kubert and Lang.

Theorem 1 (Griffin-O-Warnaar)

If $m, n \geq 1$ *and* $\kappa := 2m + 2n + 1$, *then*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^{2n-1})$$

$$= \frac{(q^\kappa; q^\kappa)_\infty^n}{(q)_\infty^n} \cdot \prod_{i=1}^{n} \theta(q^{i+m}; q^\kappa) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa)$$

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{2|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^{2n-1})$$

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Easy to use Theorem 1

Example

If $m = n = 2$, then we obtain **Dyson's favorite**

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 2}} q^{|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^3) = \prod_{n=1}^{\infty} \frac{(1-q^{9n})}{(1-q^n)},$$

and

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 2}} q^{2|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^3)$$
$$= \prod_{n=1}^{\infty} \frac{(1-q^{9n})(1-q^{9n-1})(1-q^{9n-8})}{(1-q^n)(1-q^{9n-4})(1-q^{9n-5})}.$$

Normalizations

Definition

For each of the four families, if $m, n \geq 1$, then let

$$\Phi_{a,b,c}(m,n;\tau) := q^{\kappa_{a,b,c}(m,n)} \sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^{bn+c}).$$

Integrality properties

Theorem 2 (Griffin-O-Warnaar)

If τ is a CM point, then the following are true:

1. *The singular value $\Phi_*(m, n; \tau)$ is a unit over $\mathbb{Z}[1/\kappa]$.*

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1. *The singular value $\Phi_*(m,n;\tau)$ is a unit over $\mathbb{Z}[1/\kappa]$.*
2. *The ratio $\Phi_{1,2,-1}(m,n;\tau)/\Phi_{2,2,-1}(m,n;\tau)$ is an integral* **unit**.

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Remark

Theorem 2 (2) is the $q^{1/5}R(q)$ result when $m=n=1$.

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- For $\tau = i/3$ the first 100 terms give:

$$\Phi_{1,2,-1}(2,2;i/3) = 0.577350\cdots \stackrel{?}{=} \frac{1}{\sqrt{3}}$$
$$\Phi_{2,2,-1}(2,2;i/3) = 0.217095\ldots$$

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- They **are not** algebraic integers, but are roots of:

$$3x^2 - 1$$
$$3^9x^{18} - 3^7 \cdot 37x^{12} - 2 \cdot 3^9x^9 + 2^3 \cdot 3^4 \cdot 17x^6 - 2 \cdot 3^5x^3 - 1.$$

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- By Theorem 2 (1), **both** $\sqrt{3}\Phi_{1*}(2,2;i/3)$ are integral units.

Example when $m = n = 2$ continued.

- Which gives Theorem 3 (3) that

$$\Phi_{1,2,-1}(2,2; i/3)/\Phi_{2,2,-1}(2,2; i/3) = 4.60627\ldots$$

is an algebraic integral unit.

Example when $m = n = 2$ continued.

- Which gives Theorem 3 (3) that

$$\Phi_{1,2,-1}(2,2;i/3)/\Phi_{2,2,-1}(2,2;i/3) = 4.60627\ldots$$

is an algebraic integral unit.

- Indeed, $\Phi_{1,2,-1}(2,2;i/3)/\Phi_{2,2,-1}(2,2;i/3)$ is a root of

$$x^{18} - 102x^{15} + 420x^{12} - 304x^9 - 93x^6 + 6x^3 + 1.$$

Classical proof of RR

Theorem (G. N. Watson (1929))

$$\frac{(aq, aq/bc)_N}{(aq/b, aq/c)_N} \sum_{r=0}^{N} \frac{(b, c, aq/de, q^{-N})_r}{(q, aq/d, aq/e, bcq^{-N}/a)_r} q^r$$
$$= \sum_{r=0}^{N} \frac{1-aq^{2r}}{1-a} \cdot \frac{(a, b, c, d, e, q^{-N})_r}{(q, aq/b, aq/c, aq/d, aq/e)_r} \cdot \left(\frac{a^2 q^{N+2}}{bcde}\right)^r.$$

Proof of the RR identities

- Letting $b, c, d, e, N \to \infty$ suitably gives...

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Corollary (Rogers-Selberg Identity)

$$\sum_{r=0}^{\infty} \frac{a^r q^{r^2}}{(q;q)_r} = \frac{1}{(aq;q)_\infty} \sum_{r=0}^{\infty} \frac{1-aq^{2r}}{1-a} \cdot \frac{(a;q)_r}{(q;q)_r} \cdot (-1)^r a^{2r} q^{5\binom{r}{2}+2r}.$$

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- Letting $a = 1, q$ on the LHS gives RR.
- What is the RHS when $a = 1, q$?

Proof of the RR identities continued

Lemma (Jacobi Triple Product)

$$\sum_{r=-\infty}^{\infty} (-1)^r x^r q^{\binom{r}{2}} = (q;q)_\infty \cdot \theta(x;q),$$

Proof of the RR identities continued

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$$\sum_{r=-\infty}^{\infty} (-1)^r x^r q^{\binom{r}{2}} = (q;q)_\infty \cdot \theta(x;q),$$

- Rogers-Selberg + JTP $\implies$ RR. $\square$

Obtaining the framework

"Theorem" (Bartlett-Warnaar (2013))

There are "crazier" transformation, arising from Lie algebra root systems, where

$$a \quad \longleftrightarrow \quad (x_1, x_2, \ldots, x_n).$$

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There are “crazier” transformation, arising from Lie algebra root systems, where

$$a \quad \longleftrightarrow \quad (x_1, x_2, \ldots, x_n).$$

Remark

Their transformation laws make use of

$$\Delta_{\mathbb{C}}(x) := \prod_{i=1}^{n}(1-x_i^2) \prod_{1\leq i<j\leq n}(x_i-x_j)(x_ix_j-1).$$

Bartlett-Warnaar Transformation Law

Theorem 4.2 (C_n Andrews transformation). *For m a nonnegative integer and $N \in \mathbb{Z}_+^n$,*

$$\begin{aligned}(4.3)\quad & \sum_{0 \subseteq r \subseteq N} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n} \Bigg[\prod_{\ell=1}^{m+1} \frac{(b_\ell x_i, c_\ell x_i)_{r_i}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i}} \Big(\frac{q}{b_\ell c_\ell} \Big)^{r_i} \\ & \qquad \times \prod_{j=1}^{n} \frac{(q^{-N_j} x_i/x_j, x_i x_j)_{r_i}}{(qx_i/x_j, q^{N_j+1} x_i x_j)_{r_i}} q^{N_j r_i} \Bigg] \\ & = \prod_{i,j=1}^{n} (qx_i x_j)_{N_i} \prod_{1 \le i < j \le n} \frac{1}{(qx_i x_j)_{N_i+N_j}} \\ & \quad \times \sum_{r^{(1)}, \dots, r^{(m)} \in \mathbb{Z}_+^n} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{N_i}}{(qx_i/x_j)_{N_i - r_j^{(1)}}} \prod_{\ell=1}^{m} f^{(0)}_{r^{(\ell)}, r^{(\ell+1)}}(x; q) \\ & \quad \times \prod_{\ell=1}^{m+1} \Bigg[(q/b_\ell c_\ell)_{|r^{(\ell-1)}| - |r^{(\ell)}|} \Big(\frac{q}{b_\ell c_\ell} \Big)^{|r^{(\ell)}|} \prod_{i=1}^{n} \frac{(b_\ell x_i, c_\ell x_i)_{r_i^{(\ell)}}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i^{(\ell-1)}}} \Bigg],\end{aligned}$$

where $r^{(0)} := N$ and $r^{(m+1)} := 0$.

What next?

- Make use of the added flexibility.

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What next?

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- Let parameters $\to \infty$ and take a nonterminating limit.
- Analyze the RHS....using definition of Hall-Littlewood polynomials.

Theorem (Higher Rogers-Selberg Identity)

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}(x;q) = L_m^{(0)}(x;q),$$

where

$$L_m^{(0)}(x;q) := \sum_{r \in \mathbb{Z}_+^n} \frac{\Delta_{\mathbb{C}}(xq^r)}{\Delta_{\mathbb{C}}(x)} \times \prod_{i=1}^n x_i^{2(m+1)r_i} q^{(m+1)r_i^2 + n\binom{r_i}{2}} \cdot \prod_{i,j=1}^n \left(-\frac{x_i}{x_j}\right)^{r_i} \frac{(x_i x_j)_{r_i}}{(q x_i/x_j)_{r_i}}.$$

Obtaining the framework

- It is easy to modify LHS for each theorem.

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- Many pages of reformulations involving Macdonald identities for

$$D_{n+1}^{(2)}, \quad B_n^{(1)}, \quad D_n^{(1)},$$

Weyl-Kac denominator formulas, and of course JTP. $\square$

II. Monstrous and Umbral Moonshine

Hint of moonshine

John McKay observed that

$$196884 = 1 + 196883$$

John Thompson's generalizations

Thompson further observed:

$$\underbrace{\begin{array}{r} 196884 \\ 21493760 \\ 864299970 \end{array}}_{\text{Coefficients of } j(\tau)} \begin{array}{c} = \\ = \\ = \end{array} \underbrace{\begin{array}{l} 1 + 196883 \\ 1 + 196883 + 21296876 \\ 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \end{array}}_{\text{Dimensions of irreducible representations of the Monster } \mathbb{M}}$$

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Definition

Klein's j-function

$$j(\tau)-744=\sum_{n=-1}^{\infty} c(n)q^n$$
$$= q^{-1}+196884q+21493760q^2+864299970q^3+\dots.$$

The Monster characters

The character table for $\mathbb{M}$ (ordered by size) gives dimensions:

$$\begin{aligned}
\chi_1(e) &= 1 \\
\chi_2(e) &= 196883 \\
\chi_3(e) &= 21296876 \\
\chi_4(e) &= 842609326 \\
&\vdots \\
\chi_{194}(e) &= 258823477531055064045234375.
\end{aligned}$$

Monster module

Conjecture (Thompson)

There is an infinite-dimensional graded module

$$V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$$

with

$$\dim(V_n^\natural) = c(n).$$

The McKay-Thompson Series

Definition (Thompson)

Assuming the conjecture, if $g \in \mathbb{M}$, then define the **McKay–Thompson series**

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{tr}(g|V_n^{\natural})q^n.$$

Conway and Norton

Conjecture (Monstrous Moonshine)

For each $g \in \mathbb{M}$ there is an explicit genus 0 discrete subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$ for which $T_g(\tau)$ is the unique modular function with

$$T_g(\tau) = q^{-1} + O(q).$$

Borcherds' work

Theorem (Frenkel–Lepowsky–Meurman)

The moonshine module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *is a vertex operator algebra of central charge* 24 *whose graded dimension is given by* $j(\tau) - 744$*, and whose automorphism group is* $\mathbb{M}$.

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Theorem (Borcherds)

The Monstrous Moonshine Conjecture is true.

The Monster and Supersingular elliptic curves

Theorem (Griess (1982))

The Monster group $\mathbb{M}$ *exists. It has order*

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

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Theorem (Ogg, 1974)

Toutes les valuers supersingulières de j sont $\mathbb{F}_p$ si, et seulement si

$$p \in Ogg_{ss} := \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

Ogg's Jack Daniels Problem

Remarque 1. – Dans sa leçon d'ouverture au Collège de France, le 14 janvier 1975, J. TITS mentionna le groupe de Fischer, "le monstre", qui, s'il existe, est un groupe simple "sporadique" d'ordre

$$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71\ ,$$

i. e. divisible exactement par les quinze nombres premiers de la liste du corollaire. Une bouteille de Jack Daniels est offerte à celui qui expliquera cette coïncidence.

Ogg's Problem

Problem 1

Do order p elements in $\mathbb{M}$ know the $\overline{\mathbb{F}}_p$ supersingular j-invariants?

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Theorem (Dwork's Generating Function)

If $p \geq 5$ is prime, then

$$(j(\tau) - 744) \mid U(p) \equiv - \sum_{\alpha \in SS_p} \frac{A_p(\alpha)}{j(\tau) - \alpha} - \sum_{g(x) \in SS_p^*} \frac{B_p(g) j(\tau) + C_p(g)}{g(j(\tau))} \pmod{p}.$$

Answer to Problem 1

- If $g \in \mathbb{M}$ and p is prime, then **Moonshine implies** that

$$T_g + pT_g \mid U(p) = T_{g^p}.$$

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- ....giving us Dwork's generating function

$$T_g \mid U(p) \equiv (j - 744) \mid U(p) \pmod{p}. \qquad \square$$

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Answer

- *By Ogg, if $p \notin Ogg_{ss}$, then $X_0^+(p)$ has positive genus, and there is no hauptmodul.*

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- *But Serre implies $j'(h_p \mid U(p)) \in S_2(p) \pmod{p}$.*
- *Pizer proved Θ's from quaternion alg's suffice iff $p \in Ogg_{ss}$.*

Recent moonshine

Observation (Eguchi, Ooguri, Tachikawa (2010))

Using the K3 surface elliptic genus, there is a **mock modular form**

$$H(\tau) = q^{-\frac{1}{8}} \left(-2 + 45q + 231q^2 + 770q^3 + 2277q^4 + 5796q^5 + \ldots\right).$$

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The degrees of the irreducible repn's of the Mathieu group M_{24} are:

$$1, 23, 45, 231, 252, 253, 483, 770, 990, 1035,$$
$$1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395.$$

Mathieu Moonshine

Theorem (Gannon (2013))

There is an infinite dimensional graded M_{24}-module whose McKay-Thompson series are specific mock modular forms.

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Remark

There are well known connections with even unimodular positive definite rank 24 lattices:

$$M_{24} \quad \longleftrightarrow \quad A_1^{24} \text{ lattice.}$$

Conjecture (Cheng, Duncan, Harvey (2013))

Let L^X (up to isomorphism) be an even unimodular positive-definite rank 24 lattice, and let :

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- *For each $g \in G^X$ let $H_g^X(\tau)$ be a specific* **mock modular form** *with "minimal principal parts".*

Then there is an infinite dimensional graded G^X module K^X for which $H_g^X(\tau)$ is the McKay-Thompson series for g.

What are mock modular forms?

Notation. Throughout, let

$$\tau = x + iy \in \mathbb{H} \quad \text{with} \quad x, y \in \mathbb{R}.$$

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Hyperbolic Laplacian.

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Harmonic Maass forms

Definition

A *harmonic Maass form of weight k on a subgroup* $\Gamma \subset SL_2(\mathbb{Z})$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying:

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1. For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathbb{H}$, we have

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2. We have that $\Delta_k M = 0$.

Fourier expansions

Fundamental Lemma

If $M \in H_{2-k}$ and $\Gamma(a, x)$ is the incomplete Γ-function, then

$$M(\tau) = \sum_{n \gg -\infty} c^+(n) q^n + \sum_{n<0} c^-(n) \Gamma(k-1, 4\pi|n|y) q^n.$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

Mock modular form M^+ **Nonholomorphic part M^-**

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Mock modular form M^{+} **Nonholomorphic part** M^{-}

Remark

If $\xi_{2-k} := 2iy^{2-k}\overline{\frac{\partial}{\partial \overline{\tau}}}$, then the **shadow of** *M is $\xi_{2-k}(M^{-}) \in S_{k}$.*

Our results....

Theorem (Duncan, Griffin, Ono)

The Umbral Moonshine Conjecture is true.

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The Umbral Moonshine Conjecture is true.

Remark

This result is a "numerical proof". It is analogous to the work of Atkin, Fong and Smith in the case of monstrous moonshine.

Beautiful examples

Example

For M_{12} the MT series include Ramanujan's mock thetas:

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

$$\phi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$

$$\chi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}$$

Strategy of Proof

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For each X we compute non-negative integers $\mathbf{m}_i^X(n)$ for which

$$K^X = \sum_{n=-1}^{\infty} \sum_{\chi_i} \mathbf{m}_i^X(n) V_{\chi_i}.$$

$T_\chi^X(\tau)$

- Define the weight 1/2 harmonic Maass form

$$T_{\chi_i}^X(\tau) := \frac{1}{|G^X|} \sum_{g \in G^X} \overline{\chi_i(g)} H_g^X(\tau).$$

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- Method of **holomorphic projection** gives:

$$\pi_{hol} : H_{\frac{1}{2}} \longrightarrow \widetilde{M}_2 = \{\text{wgt 2 quasimodular forms}\}.$$

Holomorphic projection

Holomorphic projection

Definition

Let f be a wgt $k \geq 2$ (not necessarily holomorphic) modular form

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_f(n, y) q^n.$$

Then its **holomorphic projection** is

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$$(\pi_{hol} f)(\tau) := (\pi_{hol}^k f)(\tau) := c_0 + \sum_{n=1}^{\infty} c(n) q^n,$$

where for $n > 0$ we have

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^{\infty} a_f(n, y) e^{-4\pi n y} y^{k-2} dy.$$

Holomorphic projection continued

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Remark

Holomorphic projections appeared earlier in works of Sturm, and Gross-Zagier, and work of Imamoglu, Raum, and Richter, Mertens, and Zwegers in connection with mock modular forms.

Sketch of the proof of umbral moonshine

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- Check the finitely many (less than 400) cases directly.

$\square$

Executive Summary

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1. Framework of RR identities arising from Hall-Littlewood symmetric functions with nice algebraic properties.
2. The Monster knows about supersingular elliptic curves.
3. Umbral Moonshine Conjecture is true.