Spectral Approach to Homogenization of Periodic Differential Operators

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Mathematical and Computational Aspects of Maxwell's Equations
Durham, July 2016

Plan

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- Statement of the problem
- The effective operator
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Introduction

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Homogenization theory studies differential equations with rapidly oscillating coefficients. One is interested in the behavior of the solutions in the small period limit. A broad literature is devoted to homogenization problems. First of all, we mention the books

- A. Bensoussan, J.-L. Lions, G. Papanicolaou. Asymptotic analysis for periodic structures, 1978.
- E. Sanchez-Palencia. Nonhomogeneous media and vibration theory, 1980.
- N. S. Bakhvalov, G. P. Panasenko. Homogenization: averaging of processes in periodic media, 1984.
- V. V. Zhikov, S. M. Kozlov, O. A. Oleinik. Homogenization of differential operators, 1993.

Introduction

One of the methods in homogenization theory is a spectral approach based on the Floquet-Bloch theory and the spectral perturbation theory.
Mention the following papers where the spectral method was used:

- E. V. Sevost'yanova, *Asymptotic expansion of the solution of a second-order elliptic equation with periodic rapidly oscillating coefficients*, Math. USSR-Sbornik, 1982.
- V. V. Zhikov, *Spectral approach to asymptotic diffusion problems*, Diff. Equations, 1989.
- C. Conca, R. Orive, M. Vanninathan, *Bloch approximation in homogenization and applications*, SIAM J. Math. Anal., 2002.

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The main idea of our approach is that the homogenization can be studied as a spectral threshold effect at the bottom of the spectrum of a periodic elliptic operator.
Consider the simplest homogenization problem. Let $\varepsilon > 0$ be a small parameter. In $L_2(\mathbb{R}^d)$, consider the operator

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Here $g(\mathbf{x})$ is positive definite and bounded $(d \times d)$-matrix-valued function, periodic with respect to some lattice of periods. The operator A_ε is the acoustics operator, it describes a periodic acoustical medium with rapidly oscillating parameters.

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From physical point of view the convergence $u_\varepsilon \to u_0$ means homogenization of the medium: a medium with rapidly oscillating parameters in the small period limit behaves like a homogeneous medium with constant effective parameters. Mathematicians are interested in the character of convergence and estimates of the error $u_\varepsilon - u_0$.

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Such estimates are called operator error estimates in homogenization theory. In order to obtain estimate (2), it is useful to apply the scaling transformation. We have the following identity:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2\to L_2} = \varepsilon^2\|(A + \varepsilon^2 I)^{-1} - (A^0 + \varepsilon^2 I)^{-1}\|_{L_2\to L_2},$$

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$$\|(A + \varepsilon^2 I)^{-1} E_\delta^\perp\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \delta^{-1}.$$

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Statement of the problem

Let Γ be a lattice in $\mathbb{R}^d$, let Ω be the cell of Γ.
By $\widetilde{\Gamma}$ we denote the dual lattice. Let $\widetilde{\Omega}$ be the Brillouin zone of $\widetilde{\Gamma}$.

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Now we introduce the class of second order DOs. Let A be the operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ admitting the following factorization

$$A = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x}), \quad \mathbf{D} = -i\nabla.$$

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Here $g(\mathbf{x})$ is an $(m \times m)$-matrix, $f(\mathbf{x})$ is an $(n \times n)$-matrix, and $b(\mathbf{D})$ is an $(m \times n)$-matrix DO. It is assumed that $m \geqslant n$.

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The operator $b(\mathbf{D})$ is given by

$$b(\mathbf{D}) = \sum_{j=1}^{d} b_j D_j,$$

where b_j are constant $(m \times n)$-matrices. We assume that the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^{d} b_j \xi_j$ satisfies

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This condition is equivalent to

$$\alpha_0 \mathbf{1}_n \leqslant b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leqslant \alpha_1 \mathbf{1}_n, \quad |\boldsymbol{\theta}| = 1, \quad 0 < \alpha_0 \leqslant \alpha_1 < \infty. \tag{3}$$

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The precise definition of A is given in terms of the quadratic form

$$a[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle g(\mathbf{x}) b(\mathbf{D})(f(\mathbf{x})\mathbf{u}(\mathbf{x})), b(\mathbf{D})(f(\mathbf{x})\mathbf{u}(\mathbf{x}))\rangle \, d\mathbf{x},$$

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Main object

Our main objects are the operators

$$\widehat{A}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}), \quad A_\varepsilon = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}).$$

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Problem

Our goal is to study the behavior of the resolvents $(\widehat{A}_\varepsilon + I)^{-1}$ *and* $(A_\varepsilon + I)^{-1}$ *for small* ε.

The effective matrix. The effective operator

Definition of the effective matrix

Let $\Lambda(\mathbf{x})$ *be the* $(n \times m)$*-matrix-valued* Γ*-periodic solution of the problem*

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0.$$

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Then the effective matrix g^0 *is an* $(m \times m)$*-matrix given by*

$$g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}, \quad \widetilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m).$$

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It turns out that the matrix g^0 is positive definite.

Proposition

The effective matrix satisfies the estimates (known as the Voight-Reuss bracketing)

$$\underline{g} \leqslant g^0 \leqslant \overline{g}.$$

Here

$$\overline{g} = |\Omega|^{-1} \int_\Omega g(\mathbf{x})\, d\mathbf{x}, \quad \underline{g} = \left(|\Omega|^{-1} \int_\Omega g(\mathbf{x})^{-1}\, d\mathbf{x} \right)^{-1}.$$

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The operator

$$\widehat{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$$

is called the effective operator for $\widehat{A}$.

Results: approximation for the resolvent

Now we formulate the main results. We start with the principal term of approximation for the resolvent of $\widehat{A}_\varepsilon$.

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Now we formulate the main results. We start with the principal term of approximation for the resolvent of $\widehat{A}_\varepsilon$.

Theorem 1 [M. Birman and T. Suslina]

Let $\widehat{A}_\varepsilon = b(\mathbf{D})^* g^\varepsilon b(\mathbf{D})$, *and let* $\widehat{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ *be the effective operator. Then*

$$\|(\widehat{A}_\varepsilon + I)^{-1} - (\widehat{A}^0 + I)^{-1}\|_{L_2 \to L_2} \leqslant C\varepsilon, \quad 0 < \varepsilon \leqslant 1. \tag{4}$$

The constant C *depends only on the norms* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *on* α_0, α_1, *and the parameters of the lattice* Γ.

Results: approximation for the resolvent

If $f \neq \mathbf{1}$, the resolvent of the operator $A_\varepsilon = (f^\varepsilon)^* b(\mathbf{D})^* g^\varepsilon b(\mathbf{D}) f^\varepsilon$ cannot be approximated by the resolvent of some operator with constant coefficients.

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Theorem 2 [M. Birman and T. Suslina]

We have

$$\|(A_\varepsilon + I)^{-1} - (f^\varepsilon)^{-1}(\widehat{A}^0 + \overline{Q})^{-1}((f^\varepsilon)^*)^{-1}\|_{L_2 \to L_2} \leqslant C\varepsilon, \quad 0 < \varepsilon \leqslant 1. \quad (5)$$

Here

$$\overline{Q} = |\Omega|^{-1} \int_\Omega (f(\mathbf{x}) f(\mathbf{x})^*)^{-1} \, d\mathbf{x}.$$

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Estimates (4) and (5) are order-sharp, and the constants in estimates are well controlled.

Results: more accurate approximation for the resolvent

In order to describe more accurate approximation of the resolvent, we need to introduce a corrector $K(\varepsilon)$:

$$K(\varepsilon) = K_1(\varepsilon) + K_1(\varepsilon)^* + K_3. \tag{6}$$

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and Π_ε is the auxiliary smoothing operator

$$(\Pi_\varepsilon \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x}, \boldsymbol{\xi}\rangle} \widehat{\mathbf{u}}(\boldsymbol{\xi})\, d\boldsymbol{\xi},$$

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$\widehat{\mathbf{u}}(\boldsymbol{\xi})$ is the Fourier image of $\mathbf{u}(\mathbf{x})$. The operator K_3 is given by

$$K_3 = -(\widehat{A}^0 + I)^{-1} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})(\widehat{A}^0 + I)^{-1},$$

where $L(\mathbf{D})$ is the first order differential operator with the symbol

$$L(\boldsymbol{\xi}) = |\Omega|^{-1} \int_\Omega (\Lambda(\mathbf{x})^* b(\boldsymbol{\xi})^* \widetilde{g}(\mathbf{x}) + \widetilde{g}(\mathbf{x})^* b(\boldsymbol{\xi}) \Lambda(\mathbf{x}))\, d\mathbf{x}.$$

Results: more accurate approximation for the resolvent

Theorem 3 [M. Birman and T. Suslina]

We have

$$\|(\widehat{A}_\varepsilon + I)^{-1} - (\widehat{A}^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2 \to L_2} \leqslant C\varepsilon^2, \quad 0 < \varepsilon \leqslant 1. \qquad (8)$$

The constant C depends on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, α_0, α_1, and the parameters of the lattice Γ.

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In particular, this is possible in the following cases: a) if $d \leqslant 4$, b) for the scalar operator $\widehat{A} = \mathbf{D}^* g(\mathbf{x})\mathbf{D} = -\operatorname{div} g(\mathbf{x})\nabla$, where $g(\mathbf{x})$ has real entries.

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A similar but more complicated result is true for operators A_ε (we will not dwell on this).

Results: approximation of the resolvent in the energy norm

Now we approximate the resolvent in the energy norm.

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The form of the corrector depends on the type of the operator norm.
The analog of Theorem 4 is true for more general operator A_ε.

Method: the scaling transformation

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Scaling transformation. Let T_ε be the unitary scaling operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$:

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$$\|(\widehat{A}_\varepsilon + I)^{-1} - (\widehat{A}^0 + I)^{-1}\|_{L_2\to L_2} = \varepsilon^2 \|(\widehat{A} + \varepsilon^2 I)^{-1} - (\widehat{A}^0 + \varepsilon^2 I)^{-1}\|_{L_2\to L_2}.$$

Method: the scaling transformation

Consequently, the required estimate

$$\|(\widehat{A}_\varepsilon + I)^{-1} - (\widehat{A}^0 + I)^{-1}\|_{L_2 \to L_2} \leqslant C\varepsilon$$

is equivalent to

$$\|(\widehat{A} + \varepsilon^2 I)^{-1} - (\widehat{A}^0 + \varepsilon^2 I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C\varepsilon^{-1}. \tag{10}$$

Method: the direct integral expansion

Applying the Floquet-Bloch theory, we decompose $\widehat{A}$ in the direct integral of the operators $\widehat{A}(\mathbf{k})$ acting in $L_2(\Omega; \mathbb{C}^n)$ and depending on the parameter $\mathbf{k} \in \mathbb{R}^d$ called the *quasimomentum*.

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$$(\mathcal{U}\mathbf{f})(\mathbf{k},\mathbf{x}) = |\widetilde{\Omega}|^{-1/2}\sum_{\mathbf{a}\in\Gamma} e^{-i\langle \mathbf{k},\mathbf{x}+\mathbf{a}\rangle}\mathbf{f}(\mathbf{x}+\mathbf{a}),\quad \mathbf{x}\in\Omega,\ \mathbf{k}\in\widetilde{\Omega}.$$

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Next, $\mathcal{U}$ extends by continuity to a unitary mapping

$$\mathcal{U}: L_2(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\widetilde{\Omega}\times\Omega;\mathbb{C}^n) = \int_{\widetilde{\Omega}}\oplus L_2(\Omega;\mathbb{C}^n)\,d\mathbf{k}.$$

Method: the direct integral expansion

The operator $\widehat{A}(\mathbf{k})$ acts in $L_2(\Omega;\mathbb{C}^n)$ and is given by

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with periodic boundary conditions. Precisely, $\widehat{A}(\mathbf{k})$ is a selfadjoint operator in $L_2(\Omega;\mathbb{C}^n)$ corresponding to the closed nonnegative quadratic form

$$\widehat{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] = \int_\Omega \langle g(\mathbf{x}) b(\mathbf{D}+\mathbf{k})\mathbf{u}, b(\mathbf{D}+\mathbf{k})\mathbf{u}\rangle\, d\mathbf{x}, \quad \mathbf{u} \in H^1_{\mathrm{per}}(\Omega;\mathbb{C}^n).$$

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Under our assumptions the form $\widehat{a}(\mathbf{k})$ satisfies the two-sided estimates

$$c_0 \int_\Omega |(\mathbf{D}+\mathbf{k})\mathbf{u}|^2 \, d\mathbf{x} \leqslant \widehat{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] \leqslant c_1 \int_\Omega |(\mathbf{D}+\mathbf{k})\mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in H^1_{\mathrm{per}}(\Omega;\mathbb{C}^n). \tag{11}$$

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The direct integral expansion for $\widehat{A}$ is given by the formula

$$\mathcal{U}\widehat{A}\mathcal{U}^{-1} = \int_{\widetilde{\Omega}} \oplus \widehat{A}(\mathbf{k}) \, d\mathbf{k}. \tag{12}$$

Method: the spectral properties

By (12), the required estimate (10) is equivalent to the following estimate which must be uniform in $\mathbf{k}$:

$$\|(\widehat{A}(\mathbf{k}) + \varepsilon^2 I)^{-1} - (\widehat{A}^0(\mathbf{k}) + \varepsilon^2 I)^{-1}\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C\varepsilon^{-1}, \quad \mathbf{k} \in \widetilde{\Omega}. \quad (13)$$

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Spectral properties. The operators $\widehat{A}(\mathbf{k})$ have discrete spectrum. By $E_j(\mathbf{k})$, $j \in \mathbb{N}$, we denote the consecutive eigenvalues of $\widehat{A}(\mathbf{k})$:

$$E_1(\mathbf{k}) \leqslant E_2(\mathbf{k}) \leqslant \dots \leqslant E_j(\mathbf{k}) \leqslant \dots$$

The functions $E_j(\mathbf{k})$ are called band functions.

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The functions $E_j(\mathbf{k})$ are called band functions. Band functions $E_j(\mathbf{k})$ are continuous and periodic with respect to $\widetilde{\Gamma}$. The spectrum of the initial operator $\widehat{A}$ has a band structure:

$$\operatorname{spec} \widehat{A} = \bigcup_{j\in\mathbb{N}} \operatorname{Ran} E_j.$$

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The functions $E_j(\mathbf{k})$ are called band functions. Band functions $E_j(\mathbf{k})$ are continuous and periodic with respect to $\widetilde{\Gamma}$. The spectrum of the initial operator $\widehat{A}$ has a band structure:

$$\operatorname{spec} \widehat{A} = \bigcup_{j\in\mathbb{N}} \operatorname{Ran} E_j.$$

Spectral bands can overlap; there may be gaps in the spectrum.

Method: the spectral properties

From estimates (11) with $\mathbf{k} = 0$ it is clear that

$$\mathfrak{N} := \mathrm{Ker}\, \widehat{A}(0) = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}, \quad \dim \mathfrak{N} = n.$$

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From (11), by simple variational arguments it follows that

$$\min_{\mathbf{k}} E_j(\mathbf{k}) = E_j(0) = 0, \quad j = 1, \dots, n,$$

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$$E_j(\mathbf{k}) \geqslant c_* |\mathbf{k}|^2, \quad \mathbf{k} \in \widetilde{\Omega}, \quad j = 1, \dots, n, \quad c_* > 0. \qquad (14)$$

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$$\mathbf{k} = t\boldsymbol{\theta}, \quad t = |\mathbf{k}|, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1},$$

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This operator family is studied by means of the analytic perturbation theory with respect to the one-dimensional parameter t. But we have to make estimates uniform in $\boldsymbol{\theta}$. The unperturbed operator is $\widehat{A}(0)$, and the perturbed operator is $\widehat{A}(\mathbf{k}) = A(t, \boldsymbol{\theta})$ (with small $t = |\mathbf{k}|$).

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The family $A(t, \boldsymbol{\theta})$ is studied in the framework of an abstract operator-theoretic scheme. For this scheme, it is important that this operator family admits a factorization of the form

$$A(t, \boldsymbol{\theta}) = X(t, \boldsymbol{\theta})^* X(t, \boldsymbol{\theta}), \quad X(t, \boldsymbol{\theta}) = X_0 + tX_1(\boldsymbol{\theta}).$$

Here X_0 is given by

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So, the point $\lambda = 0$ is an eigenvalue of multiplicity n for the unpertured operator $\widehat{A}(0)$. Then for $t \leqslant t^0$ the perturbed operator $A(t, \boldsymbol{\theta})$ has exactly n eigenvalues on the interval $[0, \delta]$, while the interval $(\delta, 3\delta)$ is free of the spectrum. We control δ and t^0 explicitly.

Method: analytic perturbation theory

By the Kato-Rellich theorem, for $t \leqslant t^0$ there exist real-analytic branches of the eigenvalues $\lambda_l(t, \boldsymbol{\theta})$ and real-analytic branches of the eigenvectors $\varphi_l(t, \boldsymbol{\theta})$ of the operator $A(t, \boldsymbol{\theta})$:

$$A(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}) = \lambda_l(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}), \quad l = 1, \dots, n,$$

such that the vectors $\varphi_l(t, \boldsymbol{\theta})$, $l = 1, \dots, n$, form an orthonormal basis in the eigenspace of $A(t, \boldsymbol{\theta})$ corresponding to the interval $[0, \delta]$.

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$$\lambda_l(t, \boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \mu_l(\boldsymbol{\theta})t^3 + \dots, \quad l = 1, \dots, n,$$
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We have $\gamma_l(\boldsymbol{\theta}) \geqslant c_* > 0$. The vectors $\omega_l(\boldsymbol{\theta})$, $l = 1, \dots, n$, form an orthonormal basis in $\mathfrak{N}$. The coefficients $\gamma_l(\boldsymbol{\theta})$ and the vectors $\omega_l(\boldsymbol{\theta})$, $l = 1, \dots, n$, are called threshold characteristics of $A(t, \boldsymbol{\theta})$.

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Definition of the spectral germ

The selfadjoint operator $S(\boldsymbol{\theta}) : \mathfrak{N} \to \mathfrak{N}$ *such that*

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where g^0 is the effective matrix. It turns out that the operator family $A^0(t, \boldsymbol{\theta}) = \widehat{A}^0(\mathbf{k})$ which corresponds to the effective operator has the same spectral germ as $A(t, \boldsymbol{\theta}) = \widehat{A}(\mathbf{k})$.

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$$\|F(t,\boldsymbol{\theta}) - P\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C_1 t,$$

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Such approximations are proved by integration of the resolvent $(A(t, \boldsymbol{\theta}) - zI)^{-1}$ over the contour in $\mathbb{C}$ which envelopes the interval $[0, \delta]$ equidistantly at the distance δ.

Method: analytic perturbation theory

Next, with the help of Theorem 5, we find a finite rank approximation of the resolvent $(A(t,\boldsymbol{\theta}) + \varepsilon^2 I)^{-1}$ in terms of the spectral germ.

Theorem 6 [M. Birman and T. Suslina]

Let P *be the orthogonal projection of* $L_2(\Omega;\mathbb{C}^n)$ *onto* $\mathfrak{N}$. *Let* $S(\boldsymbol{\theta}) : \mathfrak{N} \to \mathfrak{N}$ *be the spectral germ of* $A(t,\boldsymbol{\theta})$. *Then*

$$\|(A(t,\boldsymbol{\theta}) + \varepsilon^2 I)^{-1} - (t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C\varepsilon^{-1},$$
$$0 < \varepsilon \leqslant 1, \quad t \leqslant t^0.$$

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Since the effective operator family has the same germ, from Theorem 6 we deduce the required estimate (13):

$$\|(\widehat{A}(\mathbf{k}) + \varepsilon^2 I)^{-1} - (\widehat{A}^0(\mathbf{k}) + \varepsilon^2 I)^{-1}\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C\varepsilon^{-1}, \quad \mathbf{k} \in \widetilde{\Omega}.$$

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This completes the proof of Theorem 1.

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- For the study of more general operator A_ε, we use the identity $A_\varepsilon = (f^\varepsilon)^* \widehat{A}_\varepsilon f^\varepsilon$. It follows that the resolvent of the operator A_ε is related to the generalized resolvent of $\widehat{A}_\varepsilon$ by the identity

$$(A_\varepsilon + I)^{-1} = (f^\varepsilon)^{-1}(\widehat{A}_\varepsilon + Q^\varepsilon)^{-1}((f^\varepsilon)^*)^{-1}. \qquad (15)$$

Here $Q = (ff^*)^{-1}$. We study the generalized resolvent of the operator $\widehat{A}_\varepsilon$ and then use the identity (15).

Applications

Operators of the form $\widehat{A}_\varepsilon$:

- The acoustics operator $\widehat{A}_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$. Then $n = 1$, $m = d$, $b(\mathbf{D}) = \mathbf{D}$.

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  Example. Let $d = 2$. Then

$$b(\mathbf{D}) = \begin{pmatrix} D_1 & 0 \\ \frac{1}{2}D_2 & \frac{1}{2}D_1 \\ 0 & D_2 \end{pmatrix},$$

  and $g(\mathbf{x})$ is a symmetric (3×3)-matrix-valued function with real entries; it is bounded, positive definite and periodic. In the isotropic case $g(\mathbf{x})$ is expressed in terms of the Lame parameters.

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$$\widehat{A}_\varepsilon = \mathrm{curl}\, h^\varepsilon(\mathbf{x})\mathrm{curl} - \nabla \nu^\varepsilon(\mathbf{x})\mathrm{div}.$$

Here $h(\mathbf{x})$ is a symmetric (3×3)-matrix-valued function with real entries, and $\nu(\mathbf{x})$ is a real-valued function. Both $h(\mathbf{x})$ and $\nu(\mathbf{x})$ are periodic, bounded and positive definite.

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$$b(\mathbf{D}) = \begin{pmatrix} -i\operatorname{curl} \\ -i\operatorname{div} \end{pmatrix}, \quad g(\mathbf{x}) = \begin{pmatrix} h(\mathbf{x}) & 0 \\ 0 & \nu(\mathbf{x}) \end{pmatrix}.$$

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Such operator with $\nu(\mathbf{x}) = 1$ arises in the study of the Maxwell equations in the case where the magnetic permeability is constant.

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- The Schrödinger operator

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Suppose that the bottom of the spectrum of the operator $A = \mathbf{D}^* g(\mathbf{x})\mathbf{D} + V(\mathbf{x})$ is the point $\lambda = 0$. Then there exists a positive Γ-periodic solution $\omega(\mathbf{x})$ of the equation $A\omega = 0$. The operator A_ε admits the following factorization

$$A_\varepsilon = (\omega^\varepsilon)^{-1}\mathbf{D}^*(\omega^\varepsilon)^2 g^\varepsilon \mathbf{D}(\omega^\varepsilon)^{-1}.$$

Applications

- The two-dimensional Pauli operator

$$A_\varepsilon = \begin{pmatrix} P_{-,\varepsilon} & 0 \\ 0 & P_{+,\varepsilon} \end{pmatrix}, \quad P_{\pm,\varepsilon} = (\mathbf{D} - \varepsilon^{-1}\mathbf{a}^\varepsilon(\mathbf{x}))^2 \pm \varepsilon^{-2} b^\varepsilon(\mathbf{x}).$$

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Here the magnetic potential $\mathbf{a}(\mathbf{x})$ is Γ-periodic Lipschitz $\mathbb{R}^2$-valued function such that $\operatorname{div} \mathbf{a} = 0$, $\int_\Omega \mathbf{a}\, d\mathbf{x} = 0$. Next, $b = \partial_1 a_2 - \partial_2 a_1$.

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$$A_\varepsilon = f^\varepsilon b(\mathbf{D}) g^\varepsilon b(\mathbf{D}) f^\varepsilon,$$

with $m = n = 2$, $g(\mathbf{x}) = f(\mathbf{x})^2$,

$$b(\mathbf{D}) = \begin{pmatrix} 0 & D_1 - iD_2 \\ D_1 + iD_2 & 0 \end{pmatrix}, \quad f(\mathbf{x}) = \begin{pmatrix} e^{\varphi(\mathbf{x})} & 0 \\ 0 & e^{-\varphi(\mathbf{x})} \end{pmatrix}.$$

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