Soler model: stability, bi-frequency solitons, and SU(1,1)

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Einstein: $E^2 = p^2 + m^2$

Schrödinger: $(i\partial_t)^2\psi = (-i\nabla)^2\psi + m^2\psi$

$$E = \sqrt{m^2 + p^2} \approx m + \frac{p^2}{2m}$$

[Schrödinger[26]]: $i\partial_t\psi = \frac{1}{2m}(-i\nabla)^2\psi$

[Dirac[28]]: $E = \sqrt{p^2 + m^2} = \alpha \cdot p + \beta m,$

$$i\partial_t\psi = \underbrace{(-i\alpha\cdot\nabla + \beta m)}_{D_m}\psi, \qquad \psi(x,t)\in\mathbb{C}^4, \quad x\in\mathbb{R}^3$$

Dirac matrices α_j $(1\le j\le 3)$ and β: self-adjoint, $(-i\alpha\cdot\nabla + \beta m)^2 = -\Delta + m^2$

Standard choice: $\alpha_j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix}$ (Pauli matrices), $\beta = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$

Dirac–Maxwell system

$\bar{\psi} := \psi^* \beta$

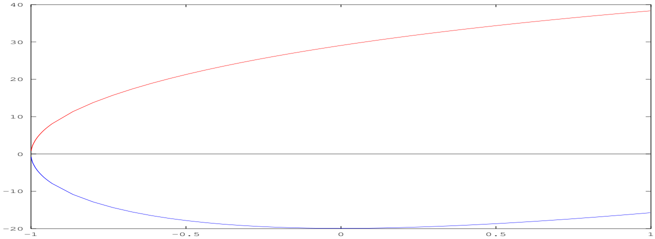
$$\gamma^\mu (i\partial_\mu - eA_\mu)\psi = m\psi, \qquad \Box A^\mu = e\bar{\psi}\gamma^\mu\psi, \qquad \partial_\mu A^\mu = 0.$$

[*Esteban et al.*[96], *Abenda*[98]]: solitary waves $\psi(x,t) = \phi(x)e^{-i\omega t}$, $\quad \omega \in (-m, m)$

Dirac–Maxwell and Dirac–Coulomb systems

$Q(\phi_\omega e^{-i\omega t})$

$E(\phi_\omega e^{-i\omega t})$



$$-m < \omega \leq m$$

[Comech[15]]: solitary waves with $\omega = m$ for Dirac–Coulomb; $\phi(x) \sim e^{-\mathrm{const}\sqrt{|x|}}$.

$\gamma^0 = \beta, \ \ \gamma^j = \beta\alpha^j, \ 1 \leq j \leq 3$

Self-interacting spinors

$$\bar{\psi} := \psi^* \beta$$

Models of self-interacting spinor field:
[*Ivanenko*[38], *Finkelstein et al.*[51], *Finkelstein et al.*[56], *Heisenberg*[57]] [...]

Soler model [*Ivanenko*[38], *Soler*[70]], scalar self-interaction:

$$\mathscr{L}_{\text{Soler}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + |\bar{\psi}\psi|^{k+1} \qquad k > 0$$

in (1+1)D: massive Gross-Neveu model [*Gross & Neveu*[74]]

Massive Thirring model [*Thirring*[58]], vector self-interaction:

$$\mathscr{L}_{\text{MTM}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \left|\bar{\psi}\gamma_\mu\psi \; \bar{\psi}\gamma^\mu\psi\right|^{\frac{k+1}{2}} \qquad k > 0$$

$\mathscr{J}^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$: "charge-current density"

Soler model: NLD with scalar self-interaction

$$i\partial_t\psi = \underbrace{(-i\alpha\cdot\nabla + m\beta)}_{D_m}\psi - (\bar\psi\psi)^k\beta\psi, \qquad \psi(x,t)\in\mathbb{C}^N, \quad x\in\mathbb{R}^n$$

• [*Soler*[70], *Cazenave & Vázquez*[86]]: existence of solitary waves in $\mathbb{R}^3$,

$$\psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \qquad \omega\in(0,m), \qquad \phi_\omega\in H^1(\mathbb{R}^3)$$

• Attempts at stability: [*Bogolubsky*[79], *Alvarez & Soler*[86], *Strauss & Vázquez*[86]] ...

• Numerics suggest that (all?) solitary waves in 1D cubic Soler model are stable: [*Alvarez & Carreras*[81], *Alvarez & Soler*[83], *Cooper et al.*[10], *Berkolaiko & Comech*[12]], [*Mertens et al.*[12], *Shao et al.*[14], *Cuevas-Maraver et al.*[14]]

• Assuming linear stability, one tries to prove asymptotic stability [*Pelinovsky & Stefanov*[12]] [*Boussaid & Cuccagna*[12]] [*Comech, Phan, Stefanov*[14]]

Nonrelativistic limit of NLD: $\omega \lesssim m$

[*Ounaies*[00], *Guan*[08]]

Solitary wave: $\psi(x,t) = \begin{bmatrix} v(x) \\ u(x) \end{bmatrix} e^{-i\omega t}; \qquad v,\ u \in \mathbb{C}^2 \qquad x \in \mathbb{R}^3$

$$i\dot{\psi} = \left\{ -i \begin{bmatrix} 0 & \sigma\cdot\nabla \\ \sigma\cdot\nabla & 0 \end{bmatrix} + \left(m - (\bar{\psi}\psi)^k\right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \psi,$$

$$\omega \begin{bmatrix} v \\ u \end{bmatrix} \approx -i\sigma\cdot\nabla \begin{bmatrix} u \\ v \end{bmatrix} + \left(m - |v|^{2k}\right) \begin{bmatrix} v \\ -u \end{bmatrix}$$

If $\omega \lesssim m$, $|u| \ll |v| \ll 1$: $\quad 2mu \approx -i\sigma\cdot\nabla v, \quad v$ satisfies NLS:

$$-(m-\omega)v = -\frac{1}{2m}\Delta v - |v|^{2k} v$$

Scaling: $v(x) = \epsilon^{1/k}\Phi(\epsilon x), \quad \epsilon = \sqrt{m-\omega},$

$$-\Phi = -\frac{1}{2m}\Delta\Phi - |\Phi|^{2k}\Phi, \qquad \Phi(x) \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

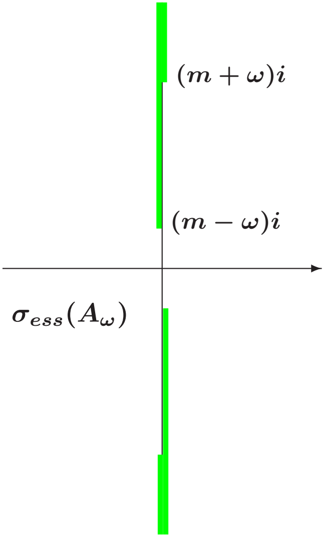
NLD: linearization at a solitary wave

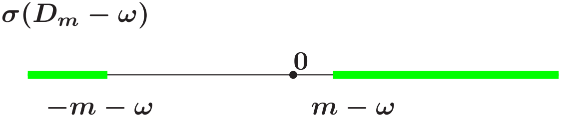
Given $\phi_\omega(x)e^{-i\omega t}$, $\omega \in (-m, m)$, consider $\psi(x,t) = \big(\phi_\omega(x) + r(x,t)\big)e^{-i\omega t}$

Linearized eqn on $r(x,t)$, $\quad i\partial_t r = D_m r - \omega r + \dots$

$$\partial_t \begin{bmatrix} \mathrm{Re}\, r \\ \mathrm{Im}\, r \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & D_m - \omega + \dots \\ -D_m + \omega + \dots & 0 \end{bmatrix}}_{A_\omega} \begin{bmatrix} \mathrm{Re}\, r \\ \mathrm{Im}\, r \end{bmatrix}$$

$\partial_t R = A_\omega R$; $\sigma(A_\omega) \subset i\mathbb{R}$ ("spectral stability") ???





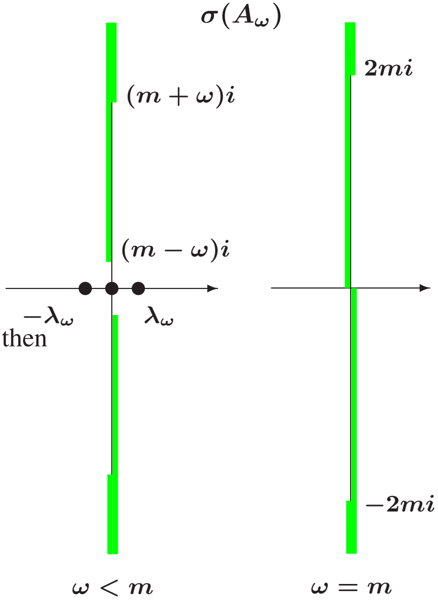
Linear instability of NLD

$$i\partial_t\psi = D_m\psi - (\bar\psi\psi)^k\beta\psi, \qquad x\in\mathbb{R}^n$$

$$\psi(x,t) = \phi_\omega(x)e^{-i\omega t}$$

Theorem 1 ([*Comech, Guan, Gustafson*[12]]).

> If $k > 2/n$, so that NLS_k is linearly unstable, then for $\omega \lesssim m$, $\exists \ \pm\lambda_\omega \in \sigma_d(A_\omega)$,
>
> $$\mathrm{Re}\,\lambda_\omega > 0, \quad \lambda_\omega \xrightarrow[\omega\to m]{} 0$$



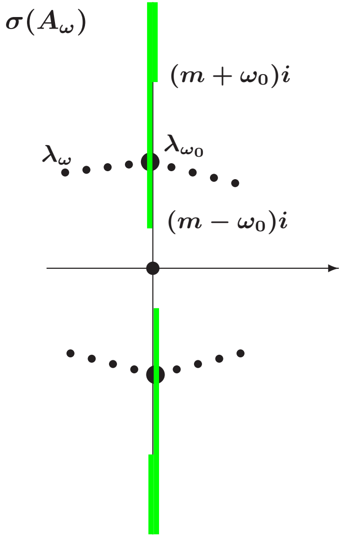
Unstable for $n = 1$, $k > 2$ (above quintic)

Unstable for $n = 2$, $k > 1$ (above cubic)

Unstable for $n = 3$, $k > 2/3$

Proof: Rescale; use Schur reduction and Rayleigh-Schrödinger perturbation theory. □

Bifurcations from σ_{ess}



Let $0 \le \omega_0 \le m$

Theorem 2 ([*Boussaid & Comech*[16]]).

Assume: $\lambda_\omega \in \sigma_p(A_\omega)$, $\operatorname{Re}\lambda_\omega \ne 0$,

$$\lambda_\omega \underset{\omega\to\omega_0}{\longrightarrow} \lambda_{\omega_0} \in i\mathbb{R}$$

Then:

$$|\lambda_{\omega_0}| \le m + \omega_0,$$

$$\lambda_{\omega_0} \in \sigma_p(A_{\omega_0}) \cup \{0\} \cup \{i(m+\omega_0)\}$$

LAP methods of [*Agmon*[75]], [*Berthier & Georgescu*[87]]:

$$\|u\|_{H^1_{-s}} \le c_{z,s,\delta} \|(D_m - z)u\|_{L^2_s}, \quad z \in \mathbb{C}\setminus[-m,m], \quad s > 1/2, \quad |z \pm m| > \delta$$

$$\|u\|_{H^1_s} := \|(1+|x|)^s u)\|_{H^1}$$

Linear stability of NLD

$$i\partial_t\psi = D_m\psi - (\bar\psi\psi)^k\beta\psi, \qquad x\in\mathbb{R}^n$$

$$\psi(x,t) = \phi_\omega(x)e^{-i\omega t}$$

Theorem 3 ([*Comech*[11]]).

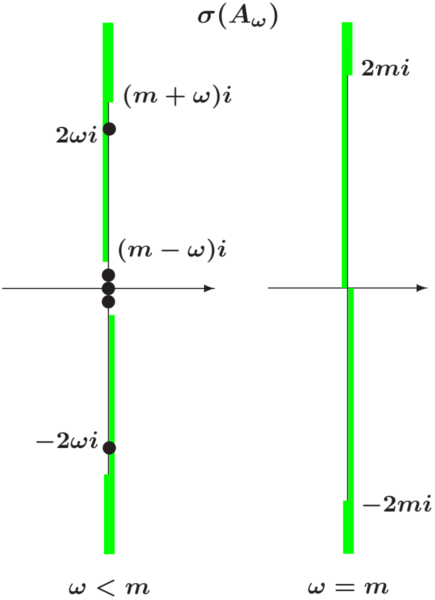
If $n\le 3$, then $\lambda = \pm 2\omega i \in \sigma_p(A_\omega)$.

Theorem 4 ([*Boussaid & Comech*[16]]).

$n\le 3$, $\quad k\lesssim 2/n$:

$\phi_\omega e^{-i\omega t}$ are linearly stable for $\omega\lesssim m$

Including the "charge-critical" case $k=2/n$



Eigenvalues $\lambda = \pm 2\omega i$ in the Soler model

$\boldsymbol{\lambda = \pm 2\omega i}$ is bad for proving asymptotic stability!

[*Boussaid & Cuccagna*[12]], [*Comech, Phan, Stefanov*[14]]

Asymptotic stability of solitons in Soler model for "radially symmetric" case:

1. perturbations orthogonal to translations;
2. perturbations orthogonal to $\boldsymbol{\pm 2\omega i}$ eigenvectors

SU(1,1) invariance of the Soler model and the Soler charge

Theorem 5 ([*Boussaid & Comech*[16]]).

1. Soler model hamiltonian is invariant under transformations

$$\psi(x,t) \quad \longrightarrow \quad (a - ib\gamma^2\mathbb{C})\psi(x,t), \qquad a,\, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1$$

2. These transformations form the group $\mathbf{SU(1,1)} \supset \mathrm{U(1)}$ $\qquad \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$

3. Conserved charges: $Q = \int \psi^*\psi \, dx > 0$

$$\Lambda = \int \psi^t(-i\gamma^2)\psi \, dx \in \mathbb{C}$$

4. Bi-frequency solutions: if $\phi(x)e^{-i\omega t}$ is a solitary wave, so is

$$\psi(x,t) = a e^{-i\omega t}\phi(x) - b e^{i\omega t} i\gamma^2\mathbb{C}(\phi(x))$$

Note: if $|b| \ll 1$, then $\psi(x,t) \approx e^{-i\omega t}\Big(\phi(x) - b e^{2i\omega t} i\gamma^2\mathbb{C}(\phi(x))\Big)$

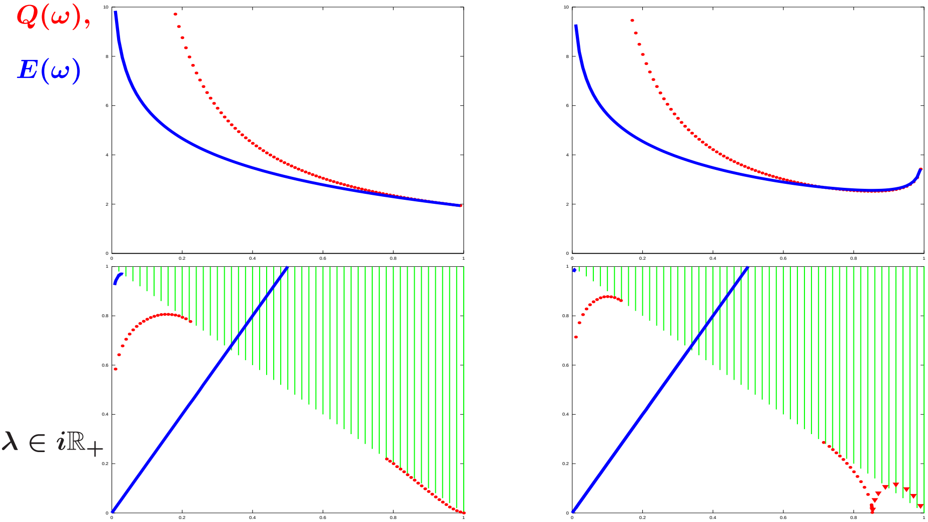


Figure 1: (1+1)D Soler model. LEFT: $k = 2$ ("charge-critical"); RIGHT: $k = 3$.
TOP ROW: charge and energy of the solitary waves as functions of $\omega \in (0, 1)$.
BOTTOM ROW: Spectrum on the upper half of the imaginary axis. Note the exact eigenvalue $\lambda = 2\omega i$.

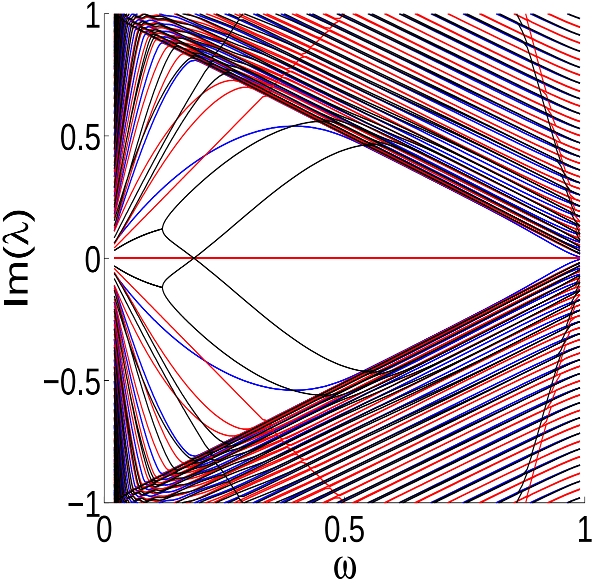


Figure 2: (2+1)D cubic "charge-critical" Soler model [*Cuevas-Maraver et al.*[16]]

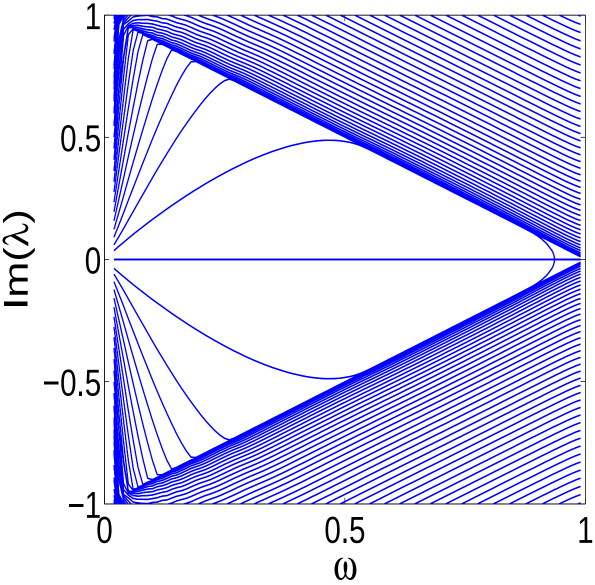


Figure 3: (3+1)D cubic "charge-supercritical" Soler model [*Cuevas-Maraver et al.*[16]]

Theorem 6 ([*Berkolaiko, Comech, Sukhtayev*[15]]).

> Both $Q'(\omega) = 0$ and $E(\omega) = 0$ indicate collision of eigenvalues at $\lambda = 0$

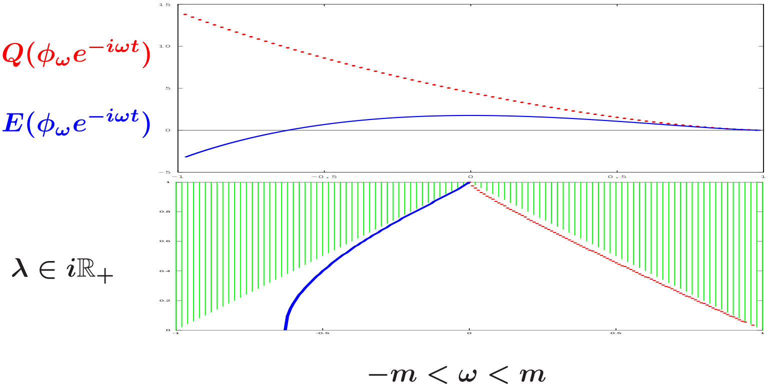


Figure 4: "Quadratic" massive Thirring model, $k = 1/2$.
TOP: energy and charge as functions of $\omega \in (-m, m)$.
BOTTOM: The spectrum of the linearization at a solitary wave.
Dotted eigenvalue collides with its opposite at the origin when $\omega_\star \approx -0.6276m$, where $E(\omega_\star) = 0$

References

[*Abenda*98] S. Abenda, *Solitary waves for Maxwell-Dirac and Coulomb-Dirac models*, Ann. Inst. H. Poincaré Phys. Théor., 68 (1998), pp. 229–244.

[*Agmon*75] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 2 (1975), pp. 151–218.

[*Alvarez & Carreras*81] A. Alvarez & B. Carreras, *Interaction dynamics for the solitary waves of a nonlinear Dirac model*, Phys. Lett. A, 86 (1981), pp. 327–332.

[*Alvarez & Soler*83] A. Alvarez & M. Soler, *Energetic stability criterion for a nonlinear spinorial model*, Phys. Rev. Lett., 50 (1983), pp. 1230–1233.

[*Alvarez & Soler*86] A. Alvarez & M. Soler, *Stability of the minimum solitary wave of a nonlinear spinorial model*, Phys. Rev. D, 34 (1986), pp. 644–645.

[*Berkolaiko & Comech*12] G. Berkolaiko & A. Comech, *On spectral stability of solitary waves of nonlinear Dirac equation in 1D*, Math. Model. Nat. Phenom., 7 (2012), pp. 13–31.

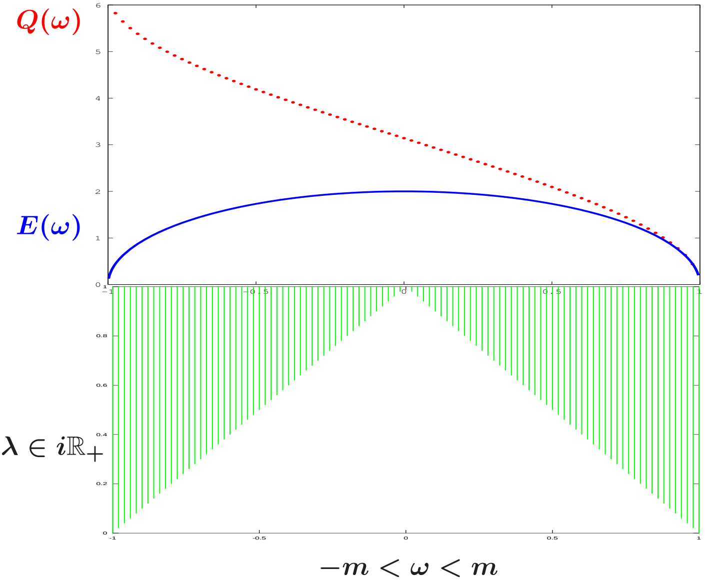


Figure 5: Massive Thirring model with $k = 1$.
TOP: energy and charge as functions of $\omega \in (-1, 1)$.
BOTTOM: The spectrum of the linearization at a solitary wave on the upper half of the imaginary axis.
No nonzero eigenvalues in completely integrable model.

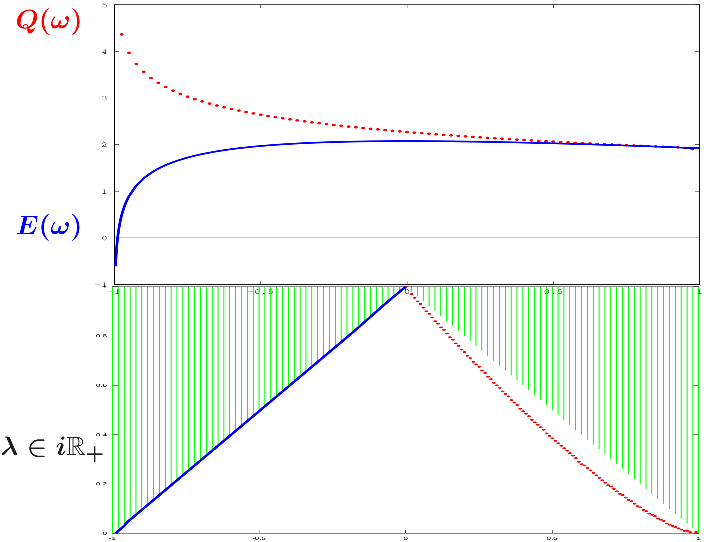


Figure 6: Massive Thirring model with $k = 2$, "charge critical".
TOP: energy and charge as functions of $\omega \in (-1, 1)$.
BOTTOM: The spectrum of the linearization at a solitary wave.

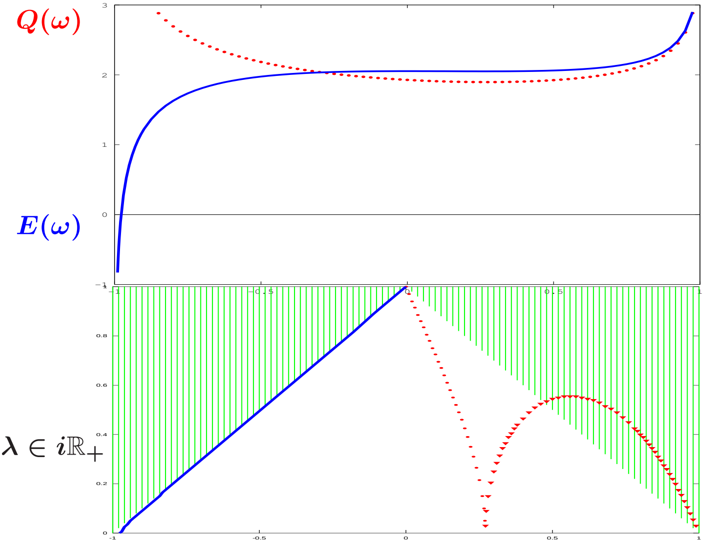


Figure 7: Massive Thirring model with $k = 3$.
TOP: energy and charge as functions of $\omega \in (-1, 1)$.
BOTTOM: The spectrum of the linearization at a solitary wave.

[*Berkolaiko et al.*[15]] G. Berkolaiko, A. Comech, & A. Sukhtayev, *Vakhitov-Kolokolov and energy vanishing conditions for linear instability of solitary waves in models of classical self-interacting spinor fields*, Nonlinearity, 28 (2015), pp. 577–592.

[*Berthier & Georgescu*[87]] A. Berthier & V. Georgescu, *On the point spectrum of Dirac operators*, J. Funct. Anal., 71 (1987), pp. 309–338.

[*Bogolubsky*[79]] I. L. Bogolubsky, *On spinor soliton stability*, Phys. Lett. A, 73 (1979), pp. 87–90.

[*Boussaid & Comech*[16]] N. Boussaid & A. Comech, *On spectral stability of nonlinear Dirac equation*, J. Funct. Anal., (2016).

[*Boussaid & Cuccagna*[12]] N. Boussaid & S. Cuccagna, *On stability of standing waves of nonlinear Dirac equations*, Comm. Partial Differential Equations, 37 (2012), pp. 1001–1056.

[*Cazenave & Vázquez*[86]] T. Cazenave & L. Vázquez, *Existence of localized solutions for a classical nonlinear Dirac field*, Comm. Math. Phys., 105 (1986), pp. 35–47.

[*Comech*[11]] A. Comech, *On the meaning of the Vakhitov-Kolokolov stability criterion for the nonlinear Dirac equation*, ArXiv e-prints, (2011).

[*Cooper et al.*[10]] F. Cooper, A. Khare, B. Mihaila, & A. Saxena, *Solitary waves in the nonlinear Dirac equation with arbitrary nonlinearity*, Phys. Rev. E, 82 (2010), p. 036604.

[*Cuevas-Maraver et al.*[14]] J. Cuevas-Maraver, P. G. Kevrekidis, & A. Saxena, *Solitary waves in a discrete nonlinear Dirac equation*, ArXiv e-prints, (2014).

[*Cuevas-Maraver et al.*[16]] J. Cuevas-Maraver, P. G. Kevrekidis, A. Saxena, A. Comech, & R. Lan, *Stability of solitary waves and vortices in a 2D nonlinear Dirac model*, Phys. Rev. Lett., 116 (2016), p. 214101.

[*Dirac*[28]] P. Dirac, *The quantum theory of the electron*, Proc. R. Soc. A, 117 (1928), pp. 610–624.

[*Esteban et al.*[96]] M. J. Esteban, V. Georgiev, & É. Séré, *Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations*, Calc. Var. Partial Differential Equations, 4 (1996), pp. 265–281.

[*Finkelstein et al.*[56]] R. Finkelstein, C. Fronsdal, & P. Kaus, *Nonlinear spinor field*, Phys. Rev., 103 (1956), pp. 1571–1579.

[*Finkelstein et al.*[51]] R. Finkelstein, R. LeLevier, & M. Ruderman, *Nonlinear spinor fields*, Phys. Rev., 83 (1951), pp. 326–332.

[*Gross & Neveu*74] D. J. Gross & A. Neveu, *Dynamical symmetry breaking in asymptotically free field theories*, Phys. Rev. D, 10 (1974), pp. 3235–3253.

[*Guan*08] M. Guan, *Solitary wave solutions for the nonlinear Dirac equations*, ArXiv e-prints, (2008).

[*Heisenberg*57] W. Heisenberg, *Quantum theory of fields and elementary particles*, Rev. Mod. Phys., 29 (1957), pp. 269–278.

[*Ivanenko*38] D. D. Ivanenko, *Notes to the theory of interaction via particles*, Zh. Éksp. Teor. Fiz, 8 (1938), pp. 260–266.

[*Mertens et al.*12] F. G. Mertens, N. R. Quintero, F. Cooper, A. Khare, & A. Saxena, *Nonlinear Dirac equation solitary waves in external fields*, Phys. Rev. E, 86 (2012), p. 046602.

[*Ounaies*00] H. Ounaies, *Perturbation method for a class of nonlinear Dirac equations*, Differential Integral Equations, 13 (2000), pp. 707–720.

[*Pelinovsky & Stefanov*12] D. E. Pelinovsky & A. Stefanov, *Asymptotic stability of small gap solitons in nonlinear Dirac equations*, J. Math. Phys., 53 (2012), pp. 073705, 27.

[*Schrödinger*26] E. Schrödinger, *Quantisierung als Eigenwertproblem*, Ann. Phys., 386 (1926), pp. 109–139.

[*Shao et al.*[14]] S. Shao, N. R. Quintero, F. G. Mertens, F. Cooper, A. Khare, & A. Saxena, *Stability of solitary waves in the nonlinear Dirac equation with arbitrary nonlinearity*, ArXiv e-prints, (2014).

[*Soler*[70]] M. Soler, *Classical, stable, nonlinear spinor field with positive rest energy*, Phys. Rev. D, 1 (1970), pp. 2766–2769.

[*Strauss & Vázquez*[86]] W. A. Strauss & L. Vázquez, *Stability under dilations of nonlinear spinor fields*, Phys. Rev. D (3), 34 (1986), pp. 641–643.

[*Thirring*[58]] W. E. Thirring, *A soluble relativistic field theory*, Ann. Physics, 3 (1958), pp. 91–112.