Higher order transmission conditions for the homogenization of interface problems

Sonia Fliss
POEMS UMR CNRS/ENSTA/INRIA

joint work with Xavier Claeys (LJLL, Paris 6)
and Valentin Vinoles (POEMS)

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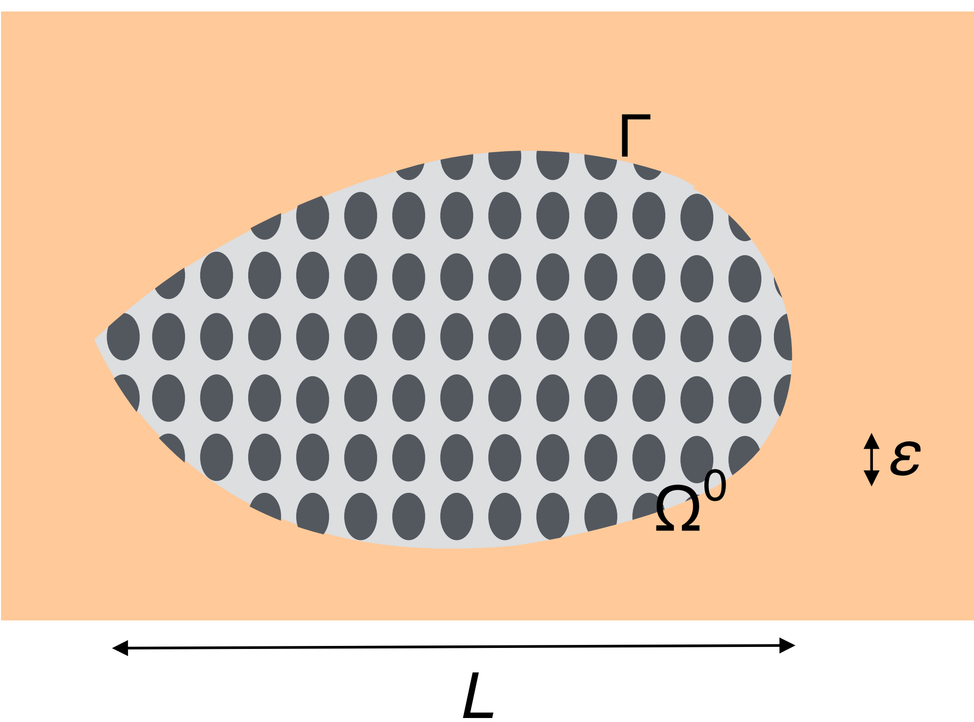
Model problem

Transmission problem between a homogeneous and a periodic media

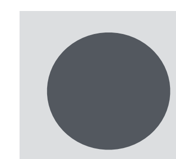
$$-\nabla \cdot \left[a(\frac{\mathbf{x}}{\varepsilon}) \nabla u_\varepsilon(\mathbf{x}) \right] - \omega^2 u_\varepsilon(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$



where ✓ $\mathrm{Im}(\omega^2) > 0$

✓ $a(\frac{\mathbf{x}}{\varepsilon}) = \begin{cases} a_p(\frac{\mathbf{x}}{\varepsilon}), & \mathbf{x} \in \Omega^0 \\ a_0, & \mathbf{x} \notin \Omega^0 \end{cases}$

with a_p 1-periodic



$Y = (0,1)^2$

✓ $0 < a_- \leq a(\frac{\mathbf{x}}{\varepsilon}) \leq a_+, \qquad \forall \mathbf{x} \in \mathbb{R}^2$

✓ $\varepsilon << \lambda \sim L$

To reduce the computational cost, a natural idea is to replace the periodic medium by an **effective** homogeneous one. This process is justified by the **homogenization theory**.

Bensoussan-Lions-Papanicolaou 1978, Sánchez-Palencia 1980, Bakhalov&Panasenko 1990, Zhikov-Kozlov-Oleinik 1994,

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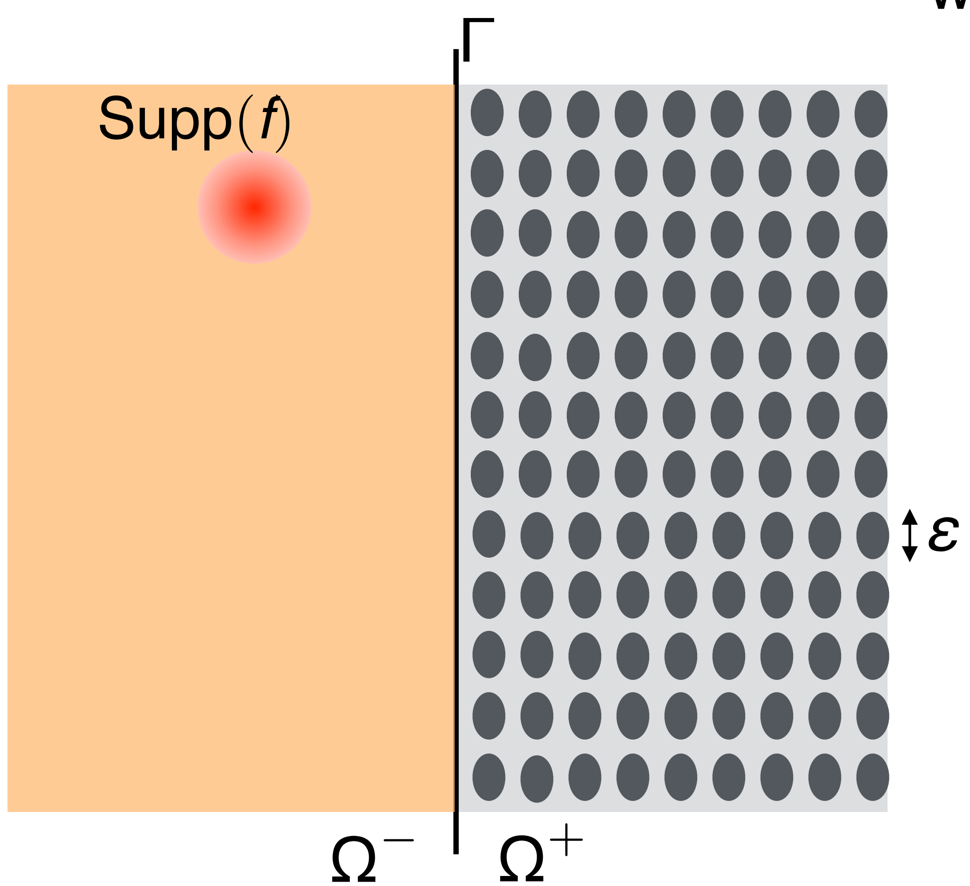
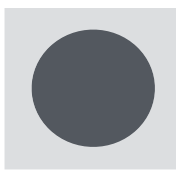


where ✓ $\mathrm{Im}(\omega^2) > 0$

✓ $a(\frac{\mathbf{x}}{\varepsilon}) = \begin{vmatrix} a_0, & \mathbf{x} \in \Omega^- = \mathbb{R}^- \times \mathbb{R} \\ a_p(\frac{\mathbf{x}}{\varepsilon}), & \mathbf{x} \in \Omega^+ = \mathbb{R}^+ \times \mathbb{R} \end{vmatrix}$

with a_p 1-periodic



$Y = (0, 1)^2$

✓ $0 < a_- \leq a(\frac{\mathbf{x}}{\varepsilon}) \leq a_+, \qquad \forall \mathbf{x} \in \mathbb{R}^2$

✓ $\varepsilon << \lambda \sim L$

✓ $\mathrm{Supp}\, f \subset \Omega^-$

To reduce the computational cost, a natural idea is to replace the periodic medium by an **effective** homogeneous one. This process is justified by the **homogenization theory**.

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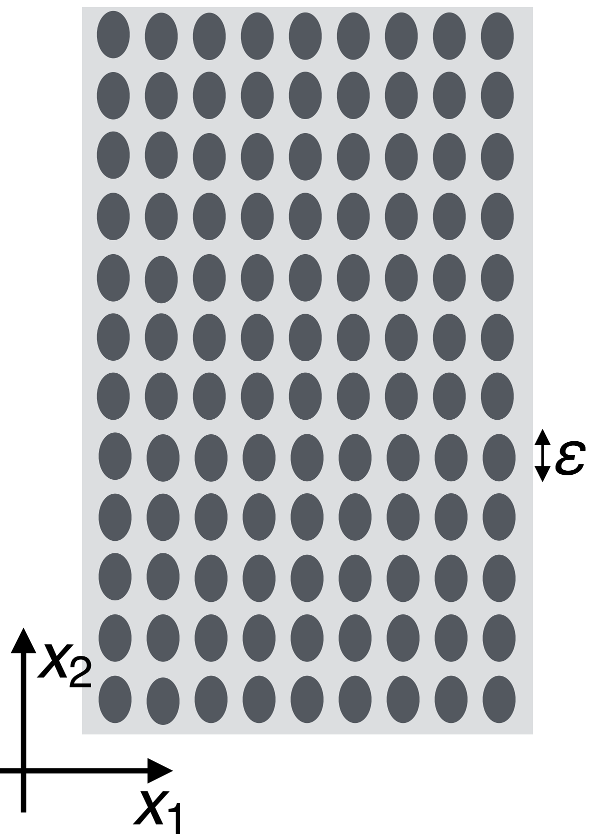
Reminder of the homogenization principles



$$-\nabla\cdot\left[a_p\left(\frac{\mathbf{x}}{\varepsilon}\right)\nabla\, u_\varepsilon(\mathbf{x})\right]-\omega^2\, u_\varepsilon(\mathbf{x})=f(\mathbf{x}),\qquad \mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$$

Ansatz for the solution

$$\mathbf{x}\in\mathbb{R}^2,\qquad u_\varepsilon(\mathbf{x})=\sum_{n\in\mathbb{N}}\varepsilon^n u_n\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\right)$$

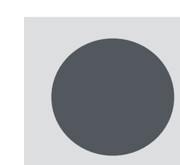
where $u_n(\mathbf{x},\mathbf{y})$ is 1-periodic with respect to $\mathbf{y}$.

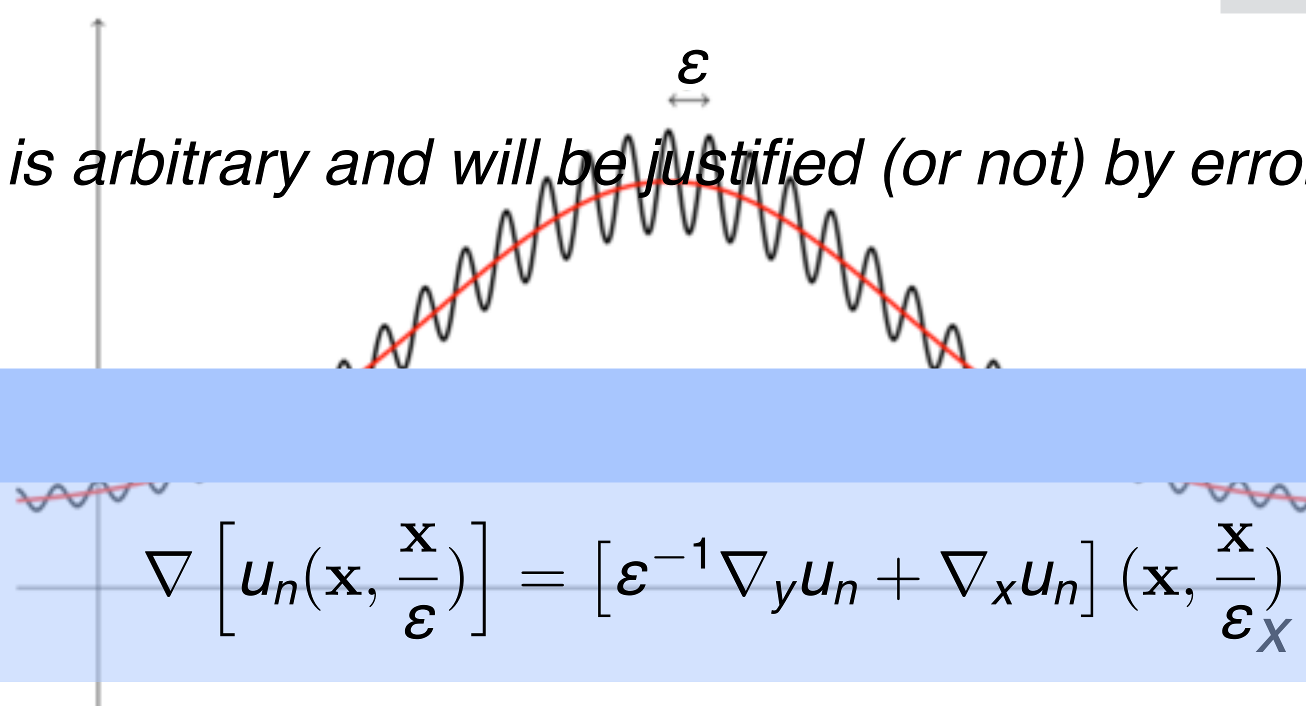
Slow (macroscopic) variable

$$\mathbf{x}\in\mathbb{R}^2$$

Fast (microscopic) variable

$$\mathbf{y}=\frac{\mathbf{x}}{\varepsilon}\in Y=(0,1)^2$$



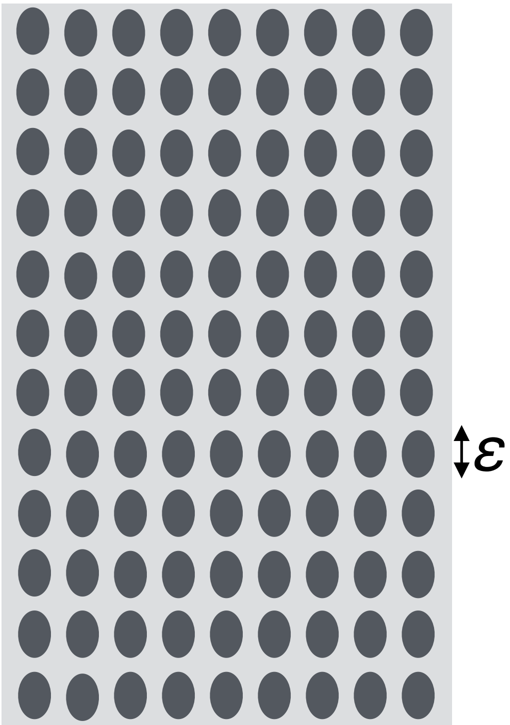


This choice is arbitrary and will be justified (or not) by error estimates.

Derivation rule

$$\nabla\left[u_n\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\right)\right]=\left[\varepsilon^{-1}\nabla_y u_n+\nabla_x u_n\right]\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\right)$$

Reminder of the homogenization principles



$$-\nabla \cdot \left[a_p(\frac{\mathbf{x}}{\varepsilon})\, \nabla\, u_\varepsilon(\mathbf{x}) \right] - \omega^2\, u_\varepsilon(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

Cascade of equations

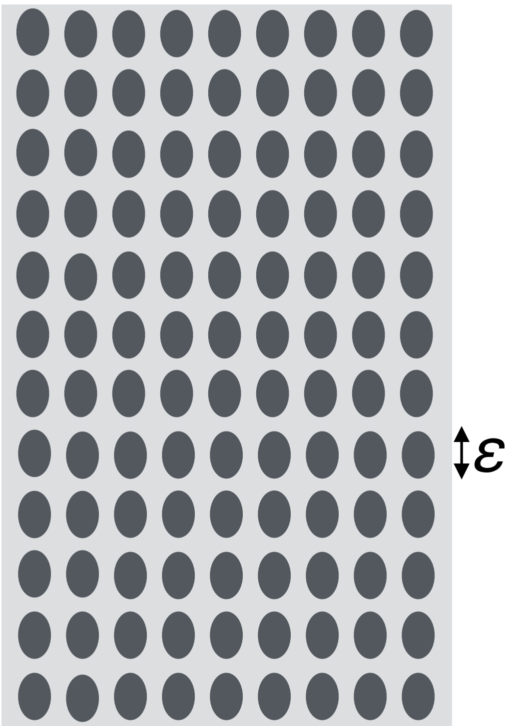
$$\varepsilon^{-2} \left[-\nabla_y \cdot a \nabla_y u_0 \right] (\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$$

$$+\varepsilon^{-1} \left[-\nabla_y \cdot a(\nabla_x u_0 + \nabla_y u_1) - \nabla_x \cdot a \nabla_y u_0 \right] (\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$$

$$+\varepsilon^{0} \left[-\nabla_x \cdot a(\nabla_x u_0 + \nabla_y u_1) - \nabla_y \cdot a(\nabla_x u_1 + \nabla_y u_2) - \omega^2\, u_0 \right] (\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$$

$$+\sum_{n=1}^{+\infty} \varepsilon^{n} \left[-\nabla_x \cdot a(\nabla_x u_n + \nabla_y u_{n+1}) - \nabla_y \cdot a(\nabla_x u_{n+1} + \nabla_y u_{n+2}) - \omega^2\, u_n \right] (\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \qquad = 0$$

Reminder of the homogenization principles



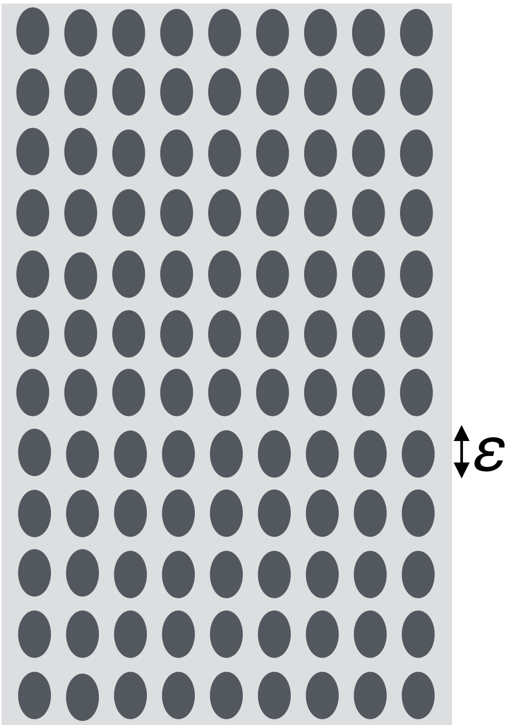
$$-\nabla \cdot \left[a_p(\frac{\mathbf{x}}{\varepsilon}) \, \nabla \, u_\varepsilon(\mathbf{x}) \right] - \omega^2 \, u_\varepsilon(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

Cascade of equations

$$\varepsilon^{-2} \left[-\nabla_y \cdot a \nabla_y u_0 \right] (\mathbf{x}, \mathbf{y}) = 0 \qquad \forall \mathbf{x} \in \mathbb{R}^2, \ \mathbf{y} \in Y$$

$$\Longrightarrow \qquad u_0(\mathbf{x}, \mathbf{y}) \equiv u_0(\mathbf{x})$$

Reminder of the homogenization principles



$$-\nabla \cdot \left[a_p(\frac{\mathbf{x}}{\varepsilon})\,\nabla\, u_\varepsilon(\mathbf{x})\right] - \omega^2\, u_\varepsilon(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

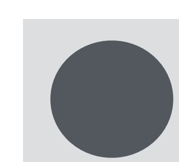
Cascade of equations

$$+\varepsilon^{-1}\left[-\nabla_{\mathbf{y}} \cdot \boldsymbol{a}(\nabla_x u_0 + \nabla_y u_1)\right](\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \;= 0 \quad \forall \mathbf{x} \in \mathbb{R}^2,\ \mathbf{y} \in Y$$

$$\Longrightarrow \qquad u_1(\mathbf{x}, \mathbf{y}) \equiv \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{y}) \\ w_2(\mathbf{y}) \end{bmatrix} + \hat{u}_1(\mathbf{x})$$

where for i=1,2, w_i is the unique solution in $H^1_{\text{per}}(Y)$

$$\left|\begin{array}{l} -\nabla_y \cdot a(\mathrm{e}_i + \nabla_y w_i) = 0, \quad y \in Y \\ \displaystyle\int_Y w_i = 0 \end{array}\right.$$



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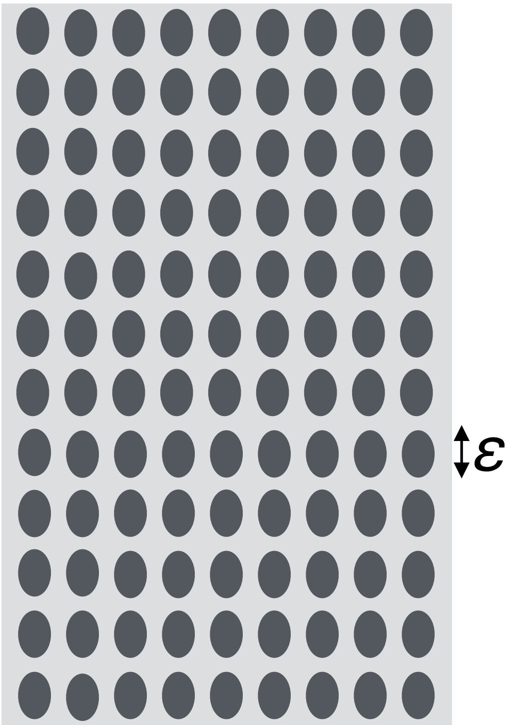


$$-\nabla \cdot \left[a_p(\frac{\mathbf{x}}{\varepsilon}) \, \nabla \, u_\varepsilon(\mathbf{x}) \right] - \omega^2 \, u_\varepsilon(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

Cascade of equations

$$\Longrightarrow$$

$$-\nabla_x \cdot [A^* \, \nabla_x \, u_0(\mathbf{x})] - \omega^2 \, u_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^2$$

$$\text{where } A^*_{ij} = \int_Y [(a(y) \nabla_y w_i) \cdot \mathrm{e}_j + \delta_{ij} a(y)] \; dy$$

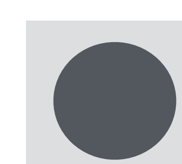
$$+\varepsilon^0 \left[-\nabla_x \cdot a(\nabla_x u_0 + \nabla_y u_1) - \nabla_y \cdot a(\nabla_x u_1 + \nabla_y u_2) - \omega^2 \, u_0 \right] (\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \qquad = 0$$

$$\Longrightarrow$$

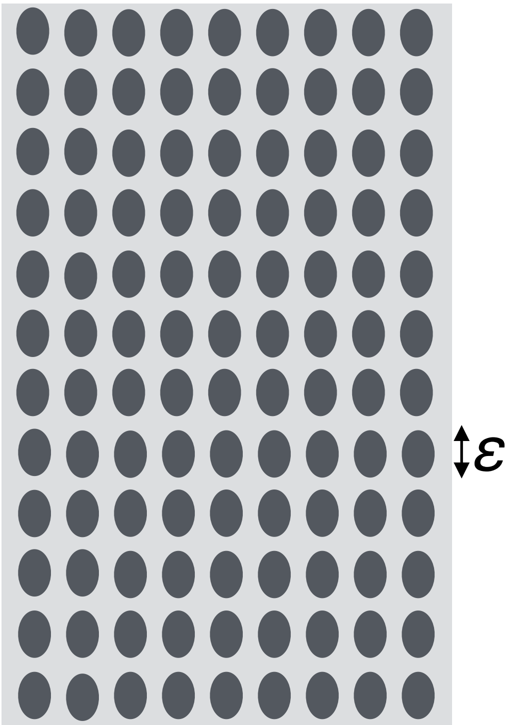
$$u_2(\mathbf{x}, \mathbf{y}) \equiv \nabla_x \cdot \begin{bmatrix} \theta_{11}(\mathbf{y}) & \theta_{12}(\mathbf{y}) \\ \theta_{21}(\mathbf{y}) & \theta_{22}(\mathbf{y}) \end{bmatrix} \nabla_x u_0(\mathbf{x}) + \nabla_x \hat{u}_1(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{y}) \\ w_2(\mathbf{y}) \end{bmatrix} + \hat{u}_2(\mathbf{x})$$

where for i,j=1,2, θ_{ij} is the unique solution in $H^1_{\text{per}}(Y)$

$$\left| \begin{array}{l} -\nabla_y \cdot a(\mathbf{y})(\nabla_y \theta_{ij}) = a(\mathbf{y}) \partial_{y_j} w_i + \partial_{y_j}(a \, w_i) + \delta_{ij} a - A^*_{ij}, \quad y \in Y \\ \displaystyle\int_Y \theta_{ij} = 0 \end{array} \right.$$



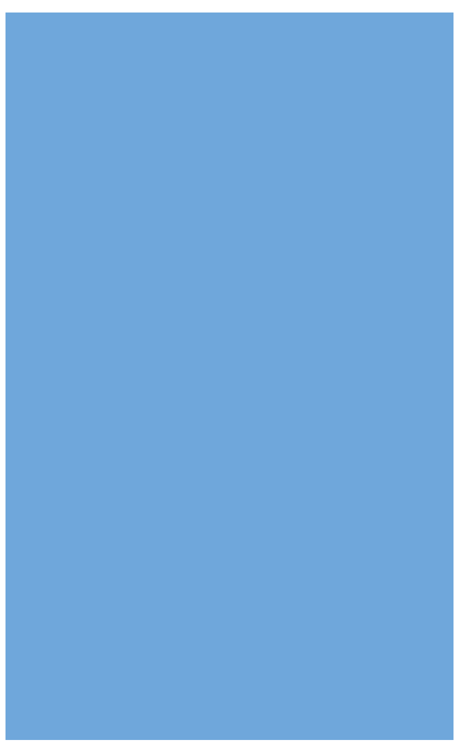
Reminder of the homogenization principles



$$-\nabla \cdot \left[a_p\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla\, u_\varepsilon(\mathbf{x}) \right] - \omega^2\, u_\varepsilon(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

$$u_\varepsilon(\mathbf{x}) = u_0(\mathbf{x}) + \varepsilon\, u_1\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2\, u_2\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) + \dots$$

where

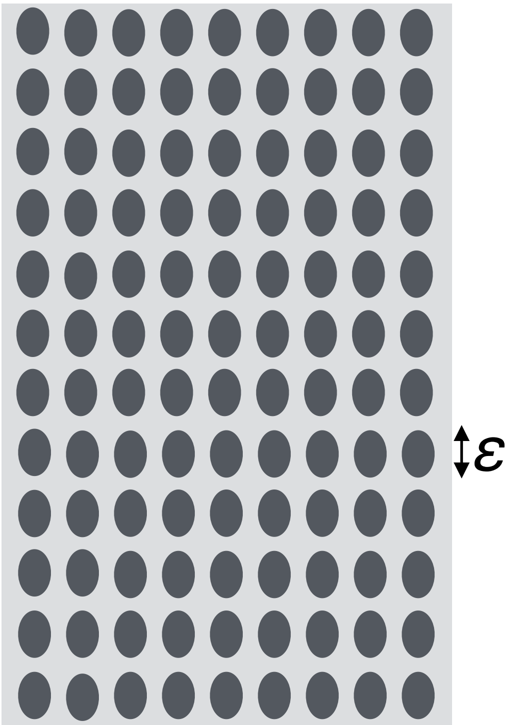


$$-\nabla_x \cdot [A^* \nabla_x u_0(\mathbf{x})] - \omega^2 u_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^2$$

$$\text{where } A^*_{ij} = \int_Y [(a(y)\nabla_y w_i) \cdot \mathrm{e}_j + \delta_{ij} a(y)]\, dy$$

- The homogenized tensor A^* is **symmetric** and **positive definite** (but not necessarily isotropic)
- It does not depend on ε.
- Cheap computation (cell problems and homogeneous media)

Reminder of the homogenization principles



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where

$$u_1(\mathbf{x}, \mathbf{y}) \equiv \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{y}) \\ w_2(\mathbf{y}) \end{bmatrix} + \hat{u}_1(\mathbf{x})$$

and $\hat{u}_1$ is the solution of the homogenized equation whose r.h.s depend on u_0 and the solutions of cell problems.

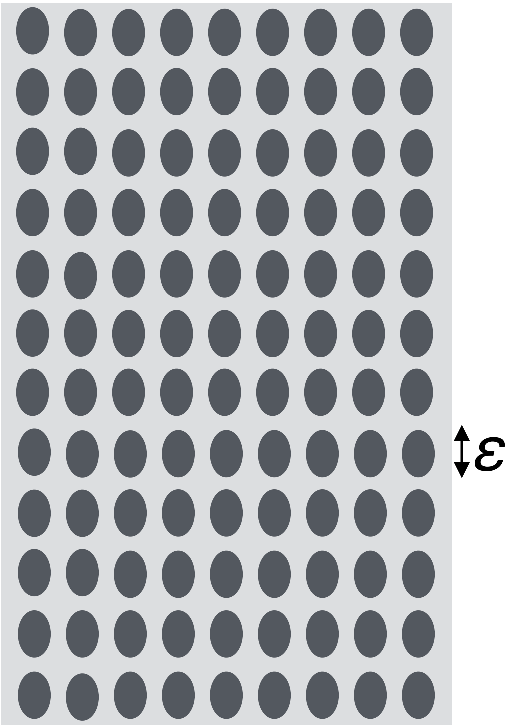
$$-\nabla_x \cdot [A^* \nabla_x \hat{u}_1(\mathbf{x})] - \omega^2 \hat{u}_1(\mathbf{x}) = \sum_{i,j,k=1}^{2} c_{ijk} \partial^3_{x_i x_j x_k} u_0, \qquad \mathbf{x} \in \mathbb{R}^2$$

0

$$\hat{u}_1 = 0$$

Santosa-Vogelius 1993, Moskow-Vogelius 1997

Reminder of the homogenization principles



$$-\nabla \cdot \left[a_p(\frac{\mathbf{x}}{\varepsilon}) \, \nabla \, u_\varepsilon(\mathbf{x}) \right] - \omega^2 \, u_\varepsilon(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

$$u_\varepsilon(\mathbf{x}) = u_0(\mathbf{x}) + \varepsilon \, u_1 \left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon} \right) + \varepsilon^2 \, u_2 \left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon} \right) + \dots$$

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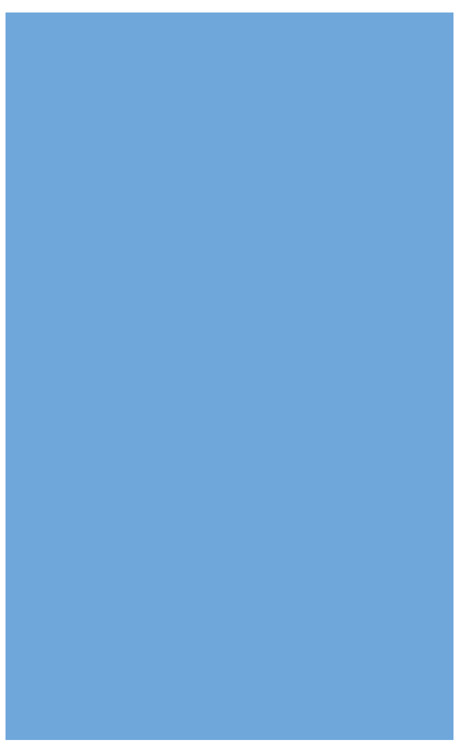
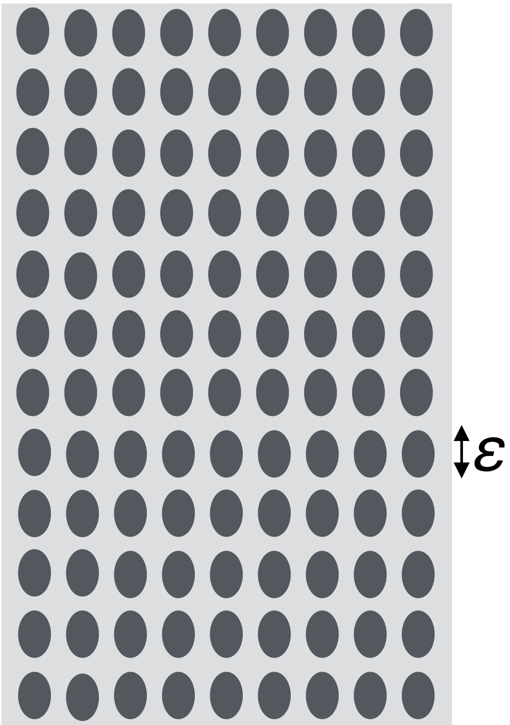


$$u_2(\mathbf{x}, \mathbf{y}) \equiv \nabla_x \cdot \begin{bmatrix} \theta_{11}(\mathbf{y}) & \theta_{12}(\mathbf{y}) \\ \theta_{21}(\mathbf{y}) & \theta_{22}(\mathbf{y}) \end{bmatrix} \nabla_x u_0(\mathbf{x}) + \nabla_x \hat{u}_1(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{y}) \\ w_2(\mathbf{y}) \end{bmatrix} + \hat{u}_2(\mathbf{x})$$

and $\hat{u}_2$ is the solution of the homogenized equation whose r.h.s depends on u_0 and the solutions of cell problems.

Reminder of the homogenization results



$$-\nabla\cdot\left[a_p\left(\frac{\mathbf{x}}{\varepsilon}\right)\nabla\, u_\varepsilon(\mathbf{x})\right]-\omega^2\, u_\varepsilon(\mathbf{x})=f(\mathbf{x}),\qquad \mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$$

$$u_\varepsilon(\mathbf{x})=u_0(\mathbf{x})+\varepsilon\, u_1\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\right)+\varepsilon^2\, u_2\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\right)+\ldots$$

$$\nabla u_\varepsilon(\mathbf{x})=\left[\nabla u_0(\mathbf{x})+\nabla_y\, u_1\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\right)\right]+\varepsilon\left[\nabla_x u_1\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\right)+\nabla_y\, u_2\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\right)\right]+\ldots$$

Theorem

Under suitable assumptions on the coefficients

$$\|u_\varepsilon-u_0\|_{L^2(\mathbb{R}^2)}=\mathcal{O}(\varepsilon)$$

$$\|u_\varepsilon-(u_0+\varepsilon u_1)\|_{H^1(\mathbb{R}^2)}=\mathcal{O}(\varepsilon)$$

Sánchez-Palencia 1980, Bakhalov&Panasenko 1990, Birman-Suslina 2001-2004-2006
Zhikov 2005-2006

$$-\nabla_x\cdot\left[A^*\,\nabla_x\, u_0(\mathbf{x})\right]-\omega^2 u_0(\mathbf{x})=0,\qquad \mathbf{x}\in\mathbb{R}^2$$

$$\text{where }\ A^*_{ij}=\int_Y\left[(a(y)\nabla_y w_i)\cdot \mathrm{e}_j+\delta_{ij}a(y)\right]\,dy$$

$$u_1(\mathbf{x},\mathbf{y})\equiv\nabla_x u_0(\mathbf{x})\cdot\begin{bmatrix} w_1(\mathbf{y})\\ w_2(\mathbf{y})\end{bmatrix}$$

Reminder of the homogenization results

Transmission problem between an homogeneous and a periodic media

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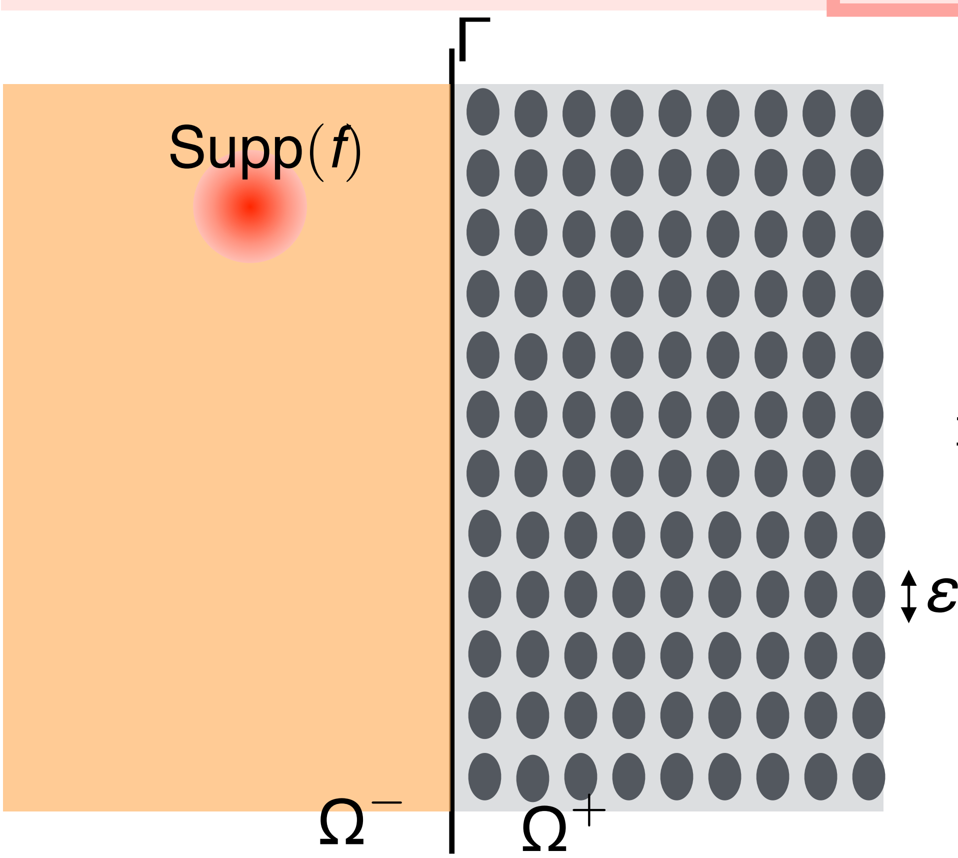
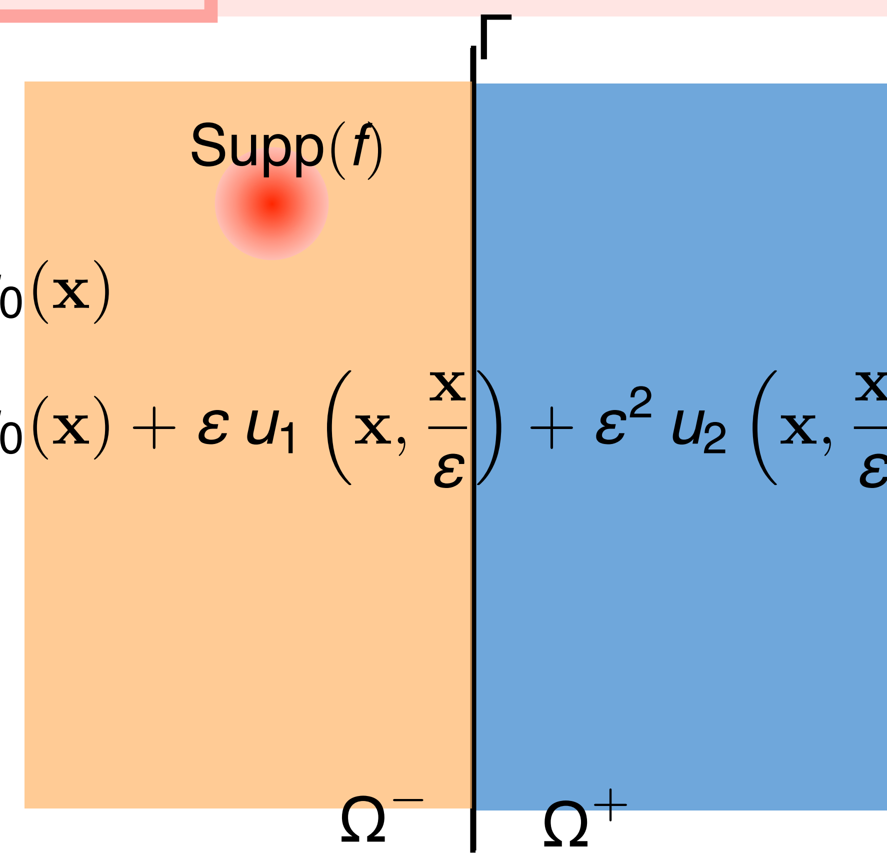
$$[u_\varepsilon]_\Gamma = 0 \quad \text{and} \quad \left[a \, \frac{\partial u_\varepsilon}{\partial x_1} \right]_\Gamma = 0$$



$$\xrightarrow{\varepsilon \to 0}$$

$$\mathbf{x} \in \Omega^-, \qquad u_\varepsilon(\mathbf{x}) = u_0(\mathbf{x})$$

$$\mathbf{x} \in \Omega^+, \qquad u_\varepsilon(\mathbf{x}) = u_0(\mathbf{x}) + \varepsilon \, u_1\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 \, u_2\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) + \ldots$$



Transmission problem between two homogeneous media

$$-\nabla_x \cdot [A^* \, \nabla_x \, u_0(\mathbf{x})] - \omega^2 u_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^2$$

$$[u_0]_\Gamma = 0 \quad \text{and} \quad [A_0^* \, \nabla u_0 \cdot \mathbf{e}_1]_\Gamma = 0 \qquad \text{where} \quad A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$$

Reminder of the homogenization results

Transmission problem between an homogeneous and a periodic media

$$-\nabla \cdot \left[a(\frac{\mathbf{x}}{\varepsilon})\,\nabla\, u_\varepsilon(\mathbf{x})\right] - \omega^2\, u_\varepsilon(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

$$[u_\varepsilon]_\Gamma = 0 \quad \text{and} \quad \left[a\,\frac{\partial u_\varepsilon}{\partial x_1}\right]_\Gamma = 0$$

Error estimates

Under suitable assumptions on the coefficients

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon)$$

$$\|u_\varepsilon - (u_0 + \varepsilon u_1)\|_{H^1(\Omega)} = \mathcal{O}(\sqrt{\varepsilon})$$

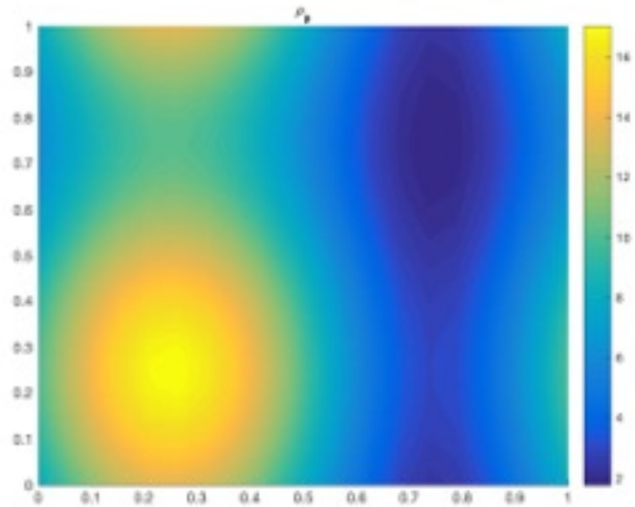
Sánchez-Palencia 1980, Bakhalov&Panasenko 1990, Moskow-Vogelius 1996, Allaire&Amar 1999
Birman-Suslina 2006, Zhikov-Pastukhova 2005, Griso 2004-2006

Transmission problem between two homogeneous media

$$-\nabla_x \cdot [A^*\,\nabla_x\, u_0(\mathbf{x})] - \omega^2 u_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^2$$

$$[u_0]_\Gamma = 0 \quad \text{and} \quad [A_0^*\,\nabla u_0 \cdot \mathbf{e}_1]_\Gamma = 0 \qquad \text{where } A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$$

Numerical results



Periodic coefficient in one cell

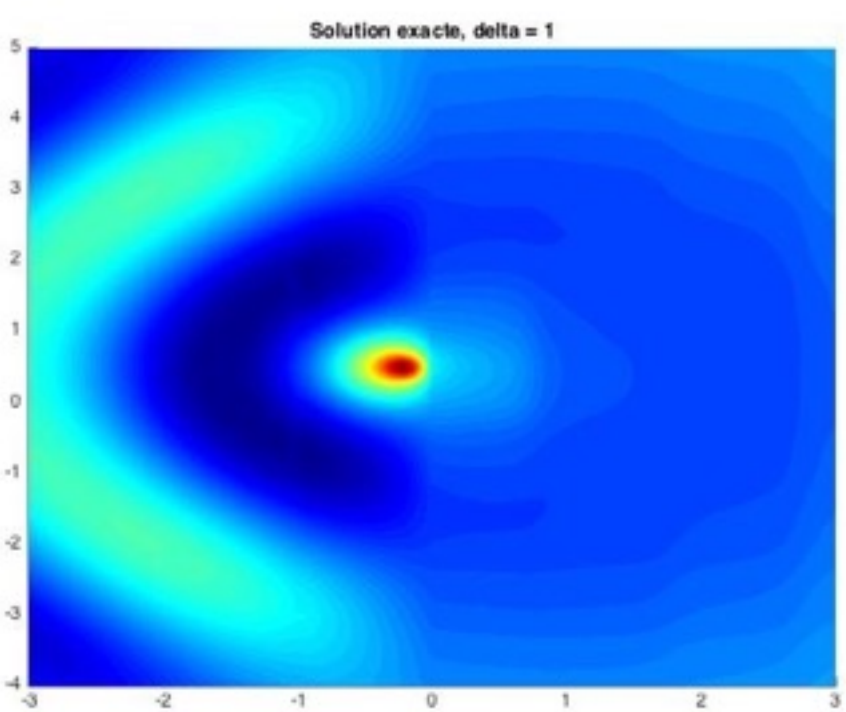
$a_0 = 1$

$\omega = 2 + 0.01\imath$

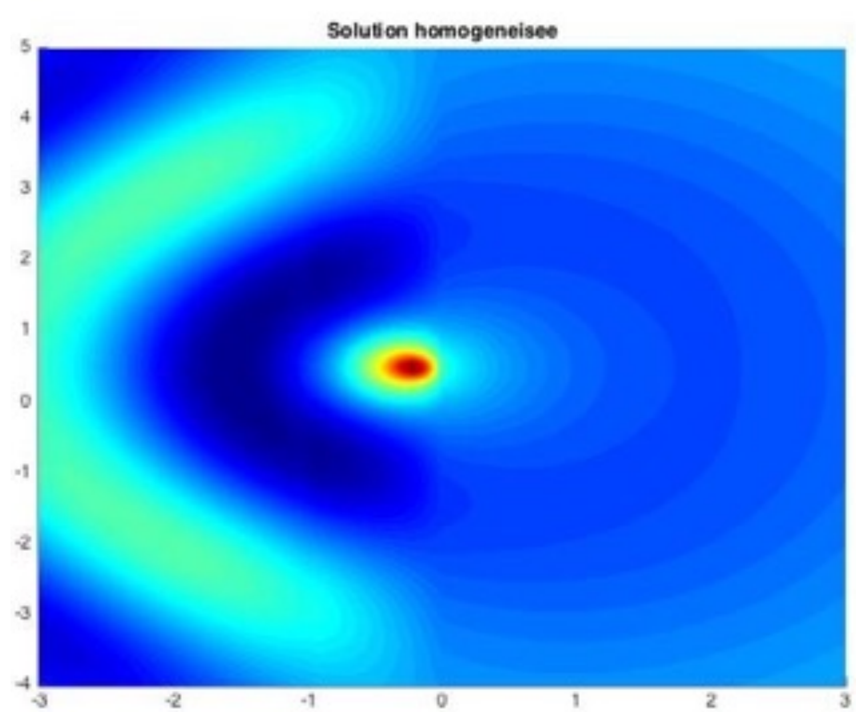
The source term in a gaussian localised near the interface.

We will describe latter the numerical method for the computation of the exact solution and the approximate solutions.

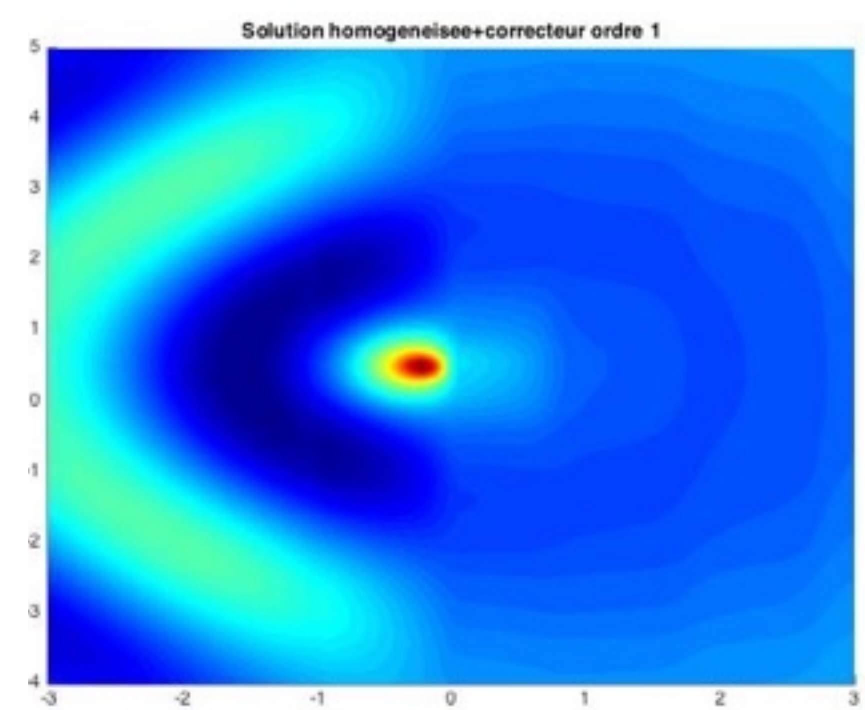
For $\varepsilon = 1$



u_ε

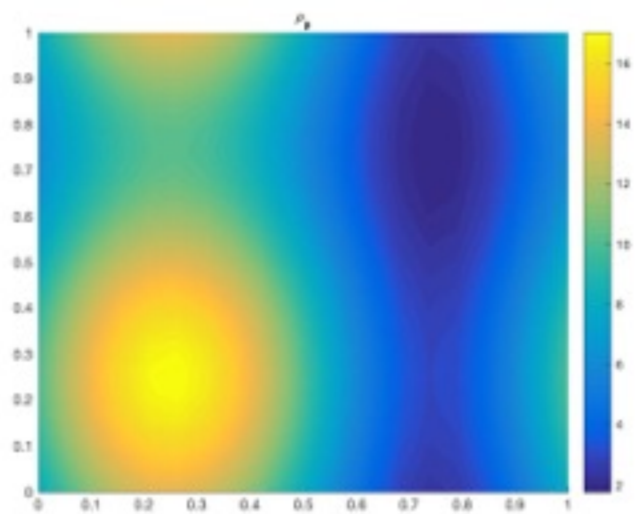


u_0



$u_0(\mathbf{x}) + \varepsilon \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+}$

Numerical results



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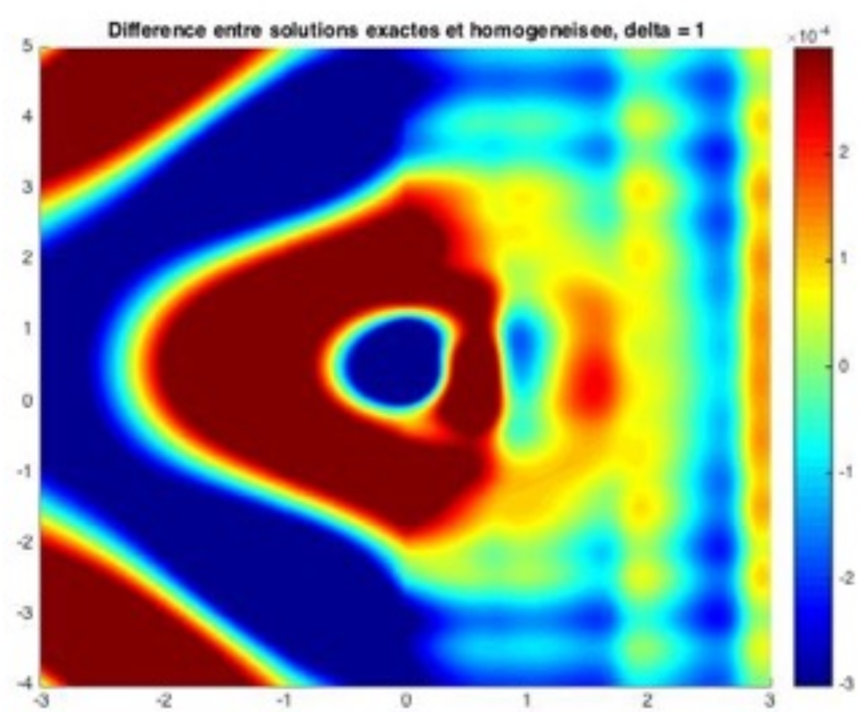
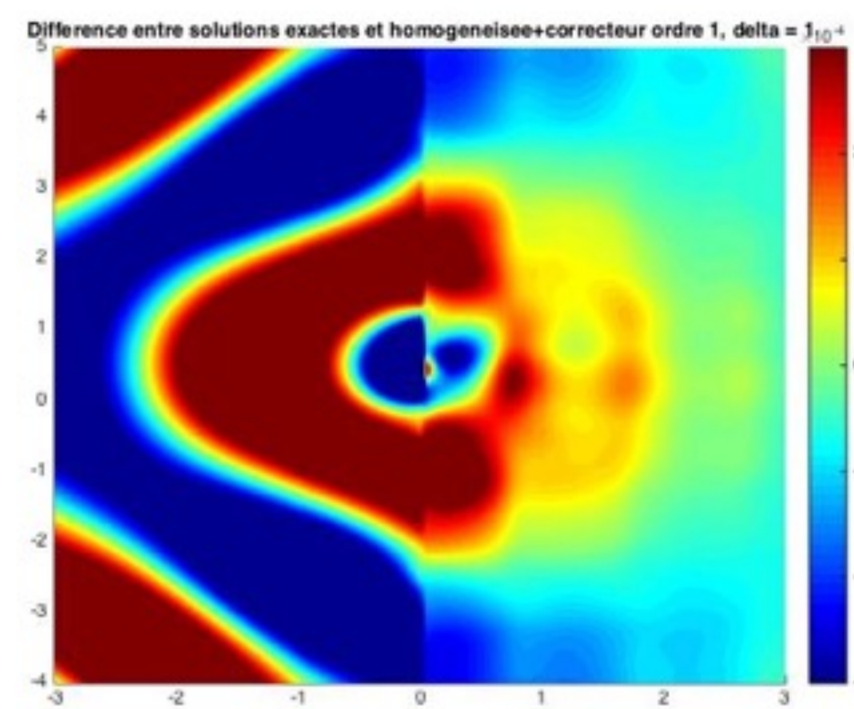
The source term in a gaussian localised near the interface.

For $\varepsilon = 1$

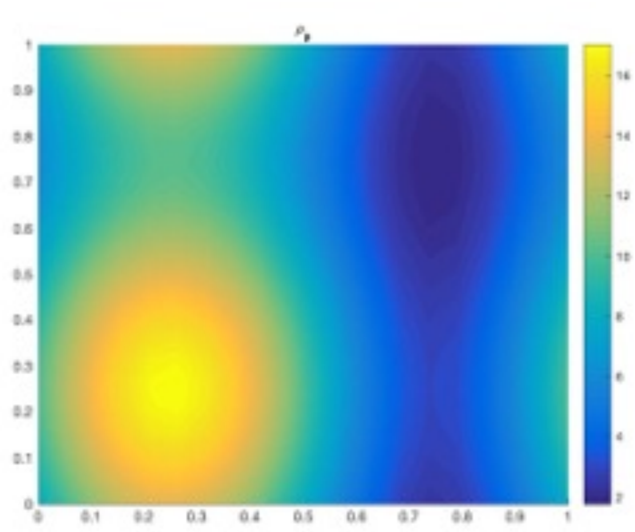


$u_\varepsilon - u_0$



$$u_\varepsilon - \left(u_0(\mathbf{x}) + \varepsilon \, \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$$

Numerical results



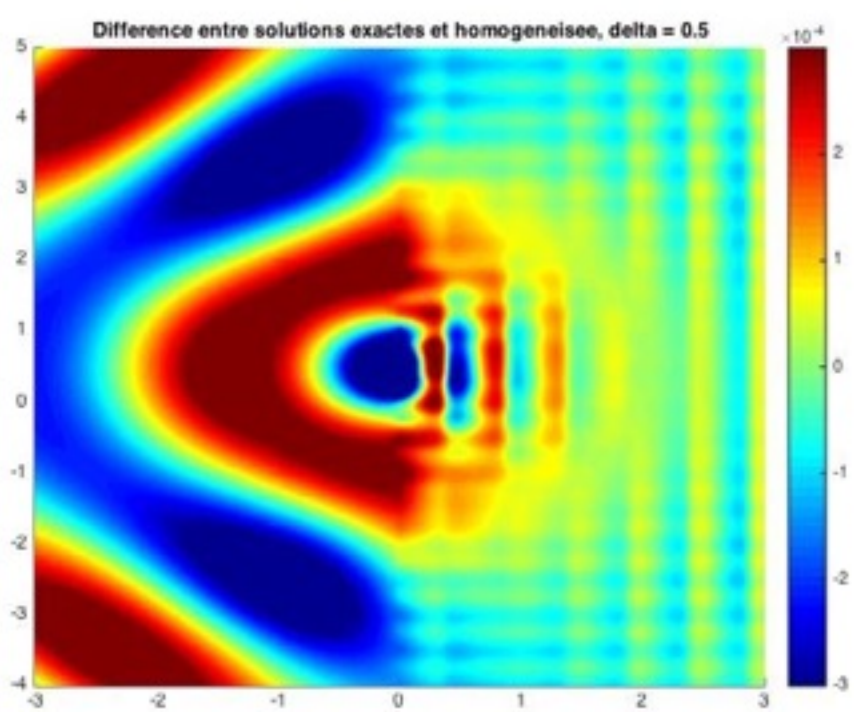
Periodic coefficient in one cell

$a_0 = 1$

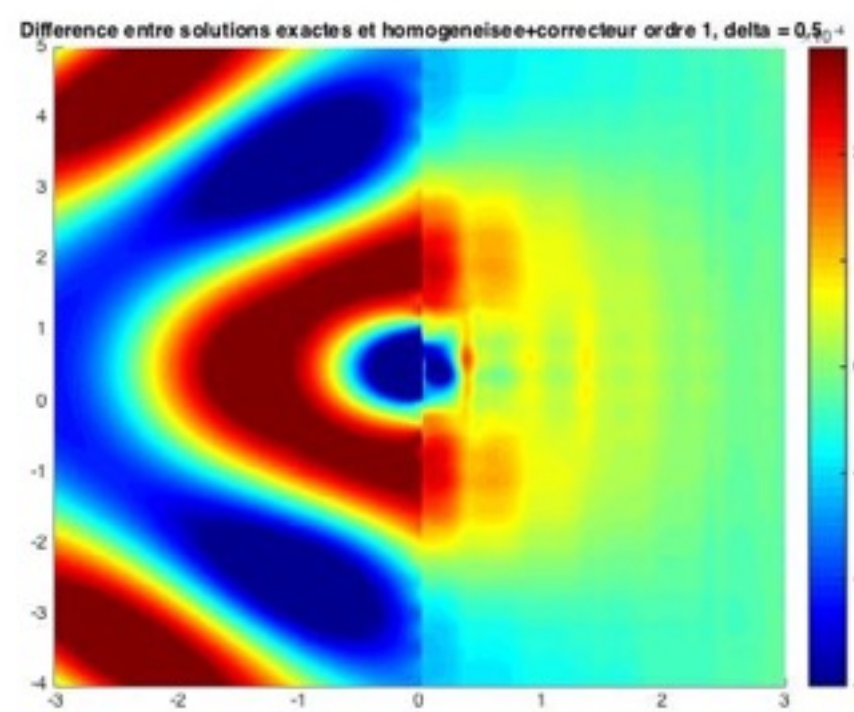
$\omega = 2 + 0.01\imath$

The source term in a gaussian localised near the interface.

For $\varepsilon = 0.5$

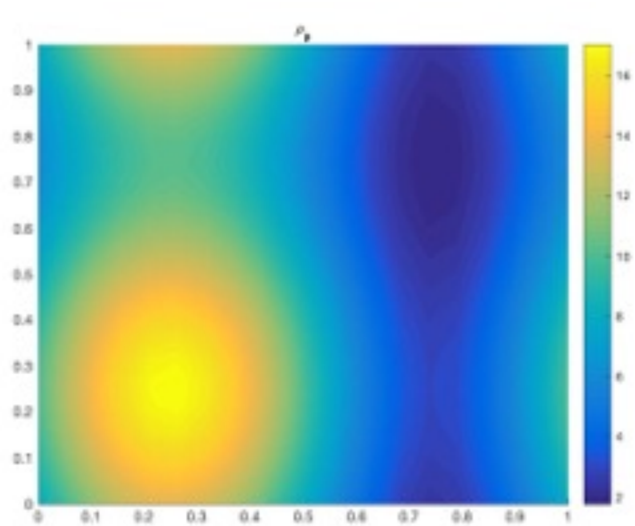


$u_\varepsilon - u_0$



$$u_\varepsilon - \left(u_0(\mathbf{x}) + \varepsilon \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$$

Numerical results



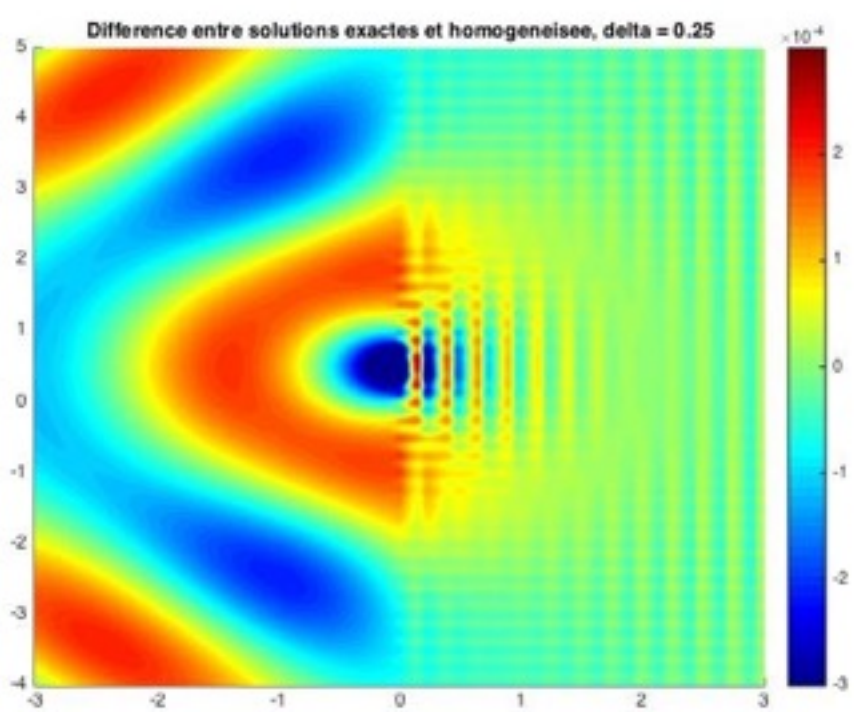
Periodic coefficient in one cell

$a_0 = 1$

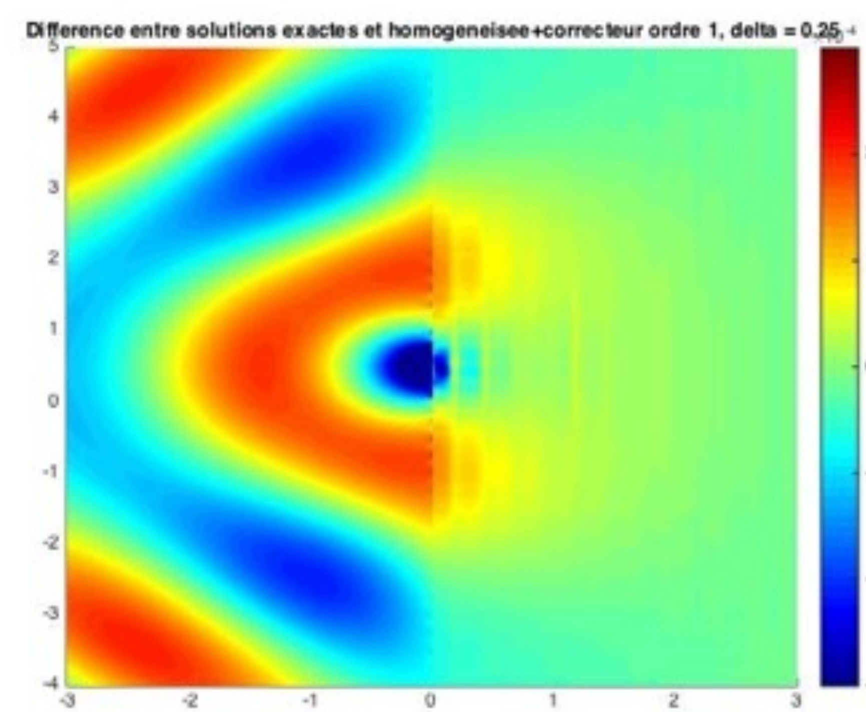
$\omega = 2 + 0.01\imath$

The source term in a gaussian localised near the interface.

For $\varepsilon = 0.25$

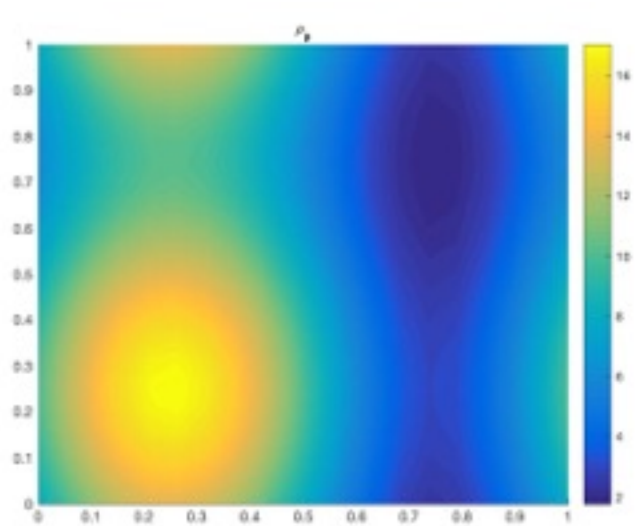


$u_\varepsilon - u_0$



$$u_\varepsilon - \left(u_0(\mathbf{x}) + \varepsilon \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$$

Numerical results



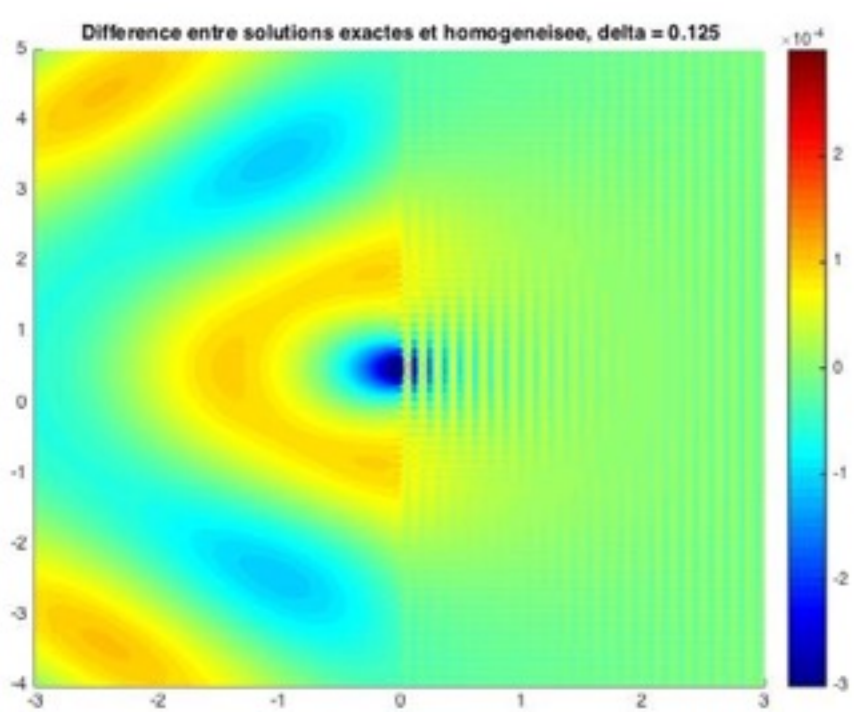
Periodic coefficient in one cell

$a_0 = 1$

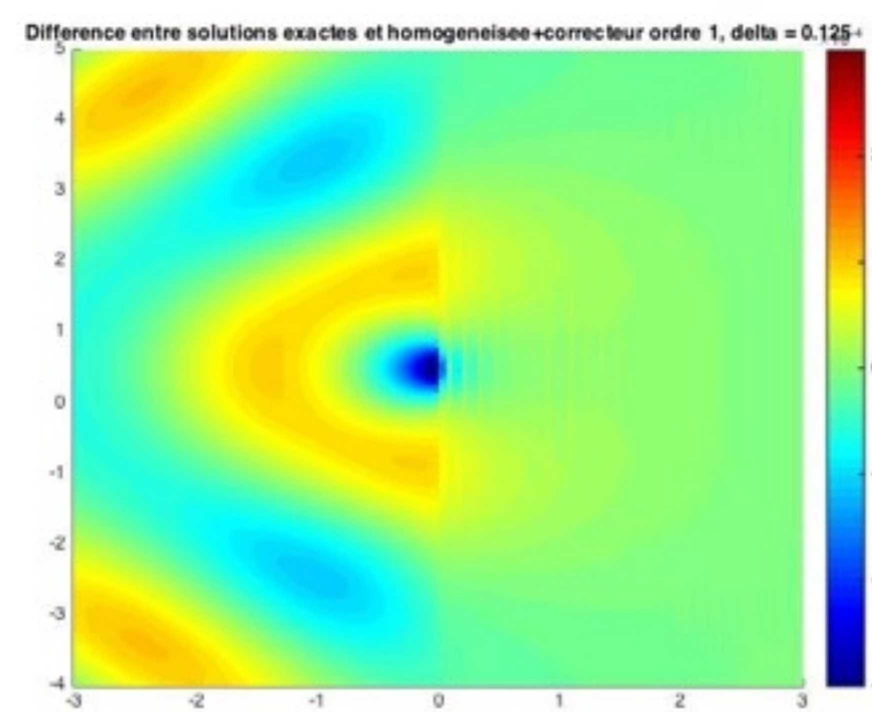
$\omega = 2 + 0.01\imath$

The source term in a gaussian localised near the interface.

For $\varepsilon = 0.125$

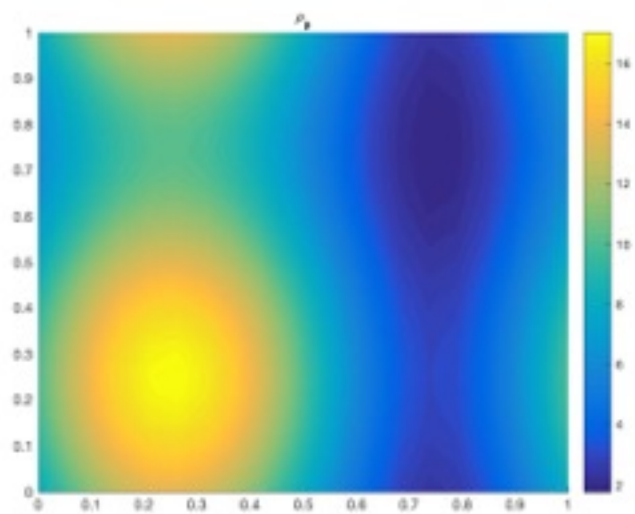


$u_\varepsilon - u_0$



$$u_\varepsilon - \left(u_0(\mathbf{x}) + \varepsilon \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$$

Numerical results

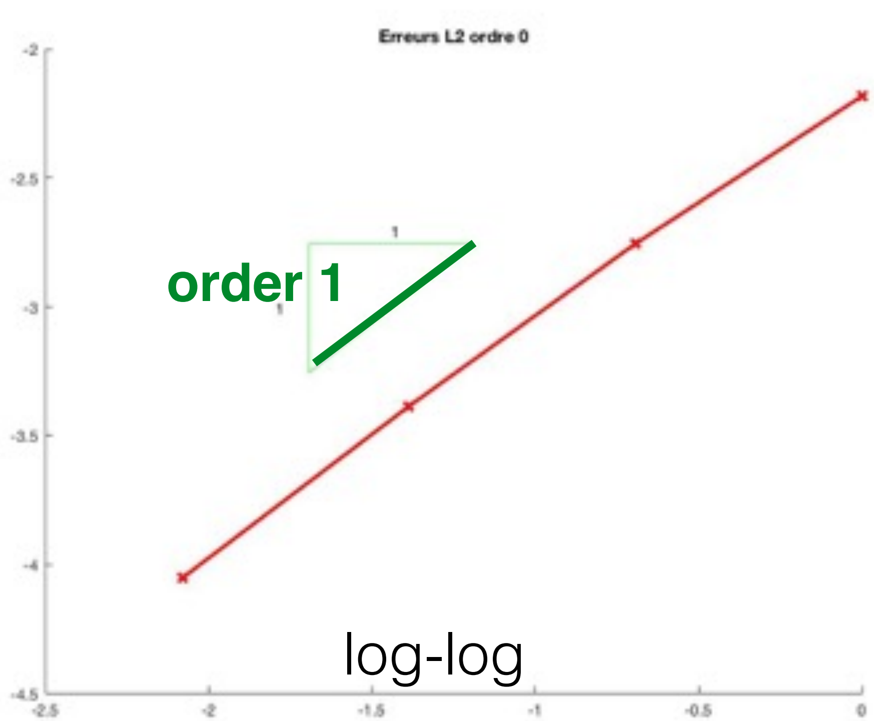


Periodic coefficient in one cell

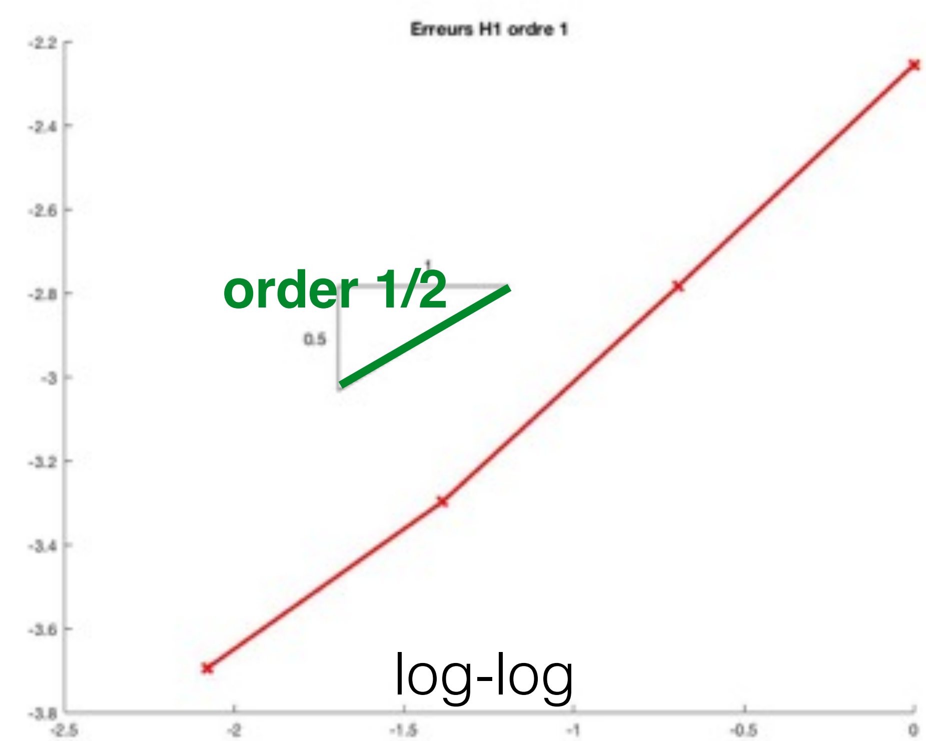
$a_0 = 1$

$\omega = 2 + 0.01\imath$

The source term in a gaussian localised near the interface.



$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon)$$



$$\|u_\varepsilon - (u_0 + \varepsilon u_1)\|_{H^1(\Omega)} = \mathcal{O}(\sqrt{\varepsilon})$$

Objectives of this work

✓ This problem is well known and linked to the **presence of boundary layers**.

✓ The **ansatz is adapted in infinite periodic media** but not in presence of boundaries or **interfaces**.

For Dirichlet or Neumann boundary conditions

Babuska 1977, Bensoussan-Lions-Papanicolaou 79, Brizzi-Chalot 1978, Sánchez-Palencia 1980, Bakhalov&Panasenko 1990, Moskow-Vogelius 1996, Allaire&Amar 1999 Birman-Suslina 2006, Zhikov-Pastukhova 2005, Griso 2004-2006 Gérard-Varet - Masmoudi 2006-2007

For transmission problems (very few works)

Cakoni-Guzina-Moskow (to appear)

Objectives of this work

- ✓ This problem is well known and linked to the **presence of boundary layers**.
- ✓ The **ansatz is adapted in infinite periodic media** but not in presence of boundaries or **interfaces**.
- ✓ One cannot expect a simple asymptotic expansion that would be valid uniformly in the whole space.
  We use the **matched asymptotic method** which allows to postulate different ansatz for the expansion of the solution.

  Van Dyke 1964, Il'in 1992, Maz'ya-Nazarov-Plamanevskii 2012

- ✓ We want to construct **higher order transmission conditions** at the interfaces.
  The equation in the bulk has to be as simple as the classical homogenized one.
- ✓ The **error analysis** allows us to justify rigorously the **approximate model**.

  Thin coatings : *Engquist-Nedelec, Bendali-Lemrabet, Artola-Cessenat, Haddar-Joly, Caloz-Costabel-Dauge-Vial, Leichleter, Jai-Peron-Poignard*

  Effective boundary conditions for periodic coatings : *Achdou-Pironneau, Abboud-Ammari, Sanchez-Palencia, Bendali-Poirier, Bonnet-Drissi, Valentin, Poignard, Delourme-Haddar-Joly, Claeys-Delourme*

The steps of our approach

1. **The asymptotic expansions of the solution**

    The formal steps of the matched asymptotic expansion

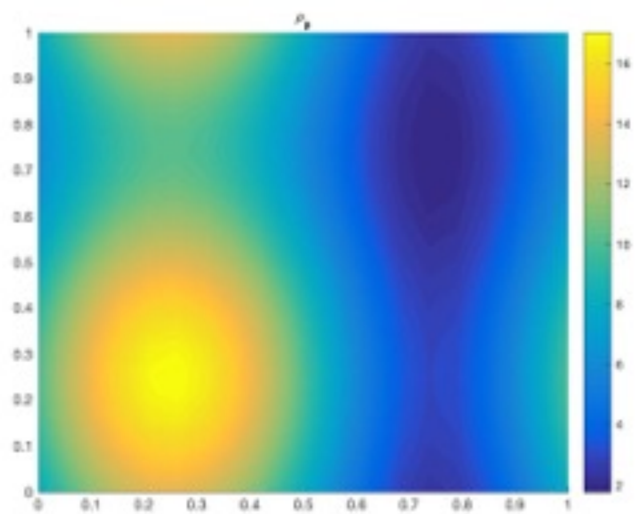
    Existence and uniqueness results

2. **Construction of the approximate conditions for the asymptotic expansion**

3. **Stability and error estimates for the approximate problem**

4. **Numerical implementation and validation**

Numerical results to motivate the rest of the talk



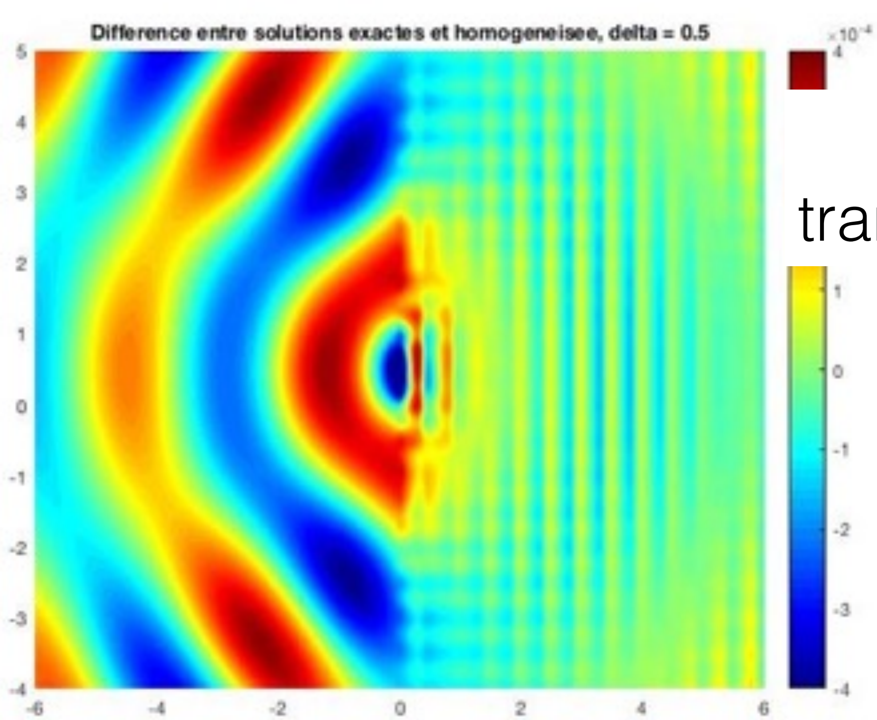
Periodic coefficient in one cell

$a_0 = 1$

$\omega = 5 + 0.01\imath$

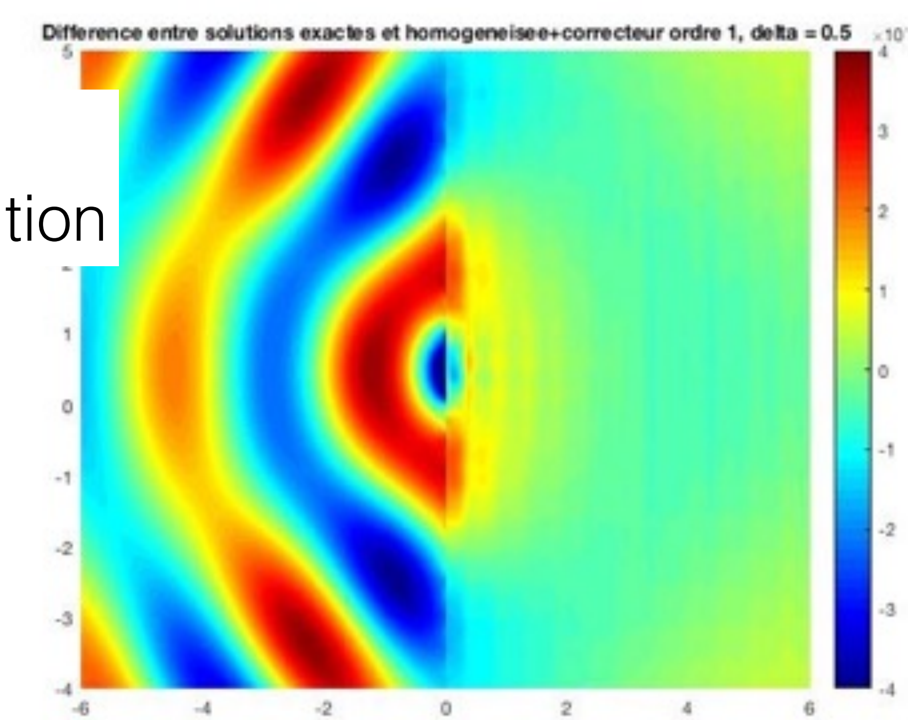
The source term in a gaussian localised near the interface.

For $\varepsilon = 0.5$



$u_\varepsilon - u_0$

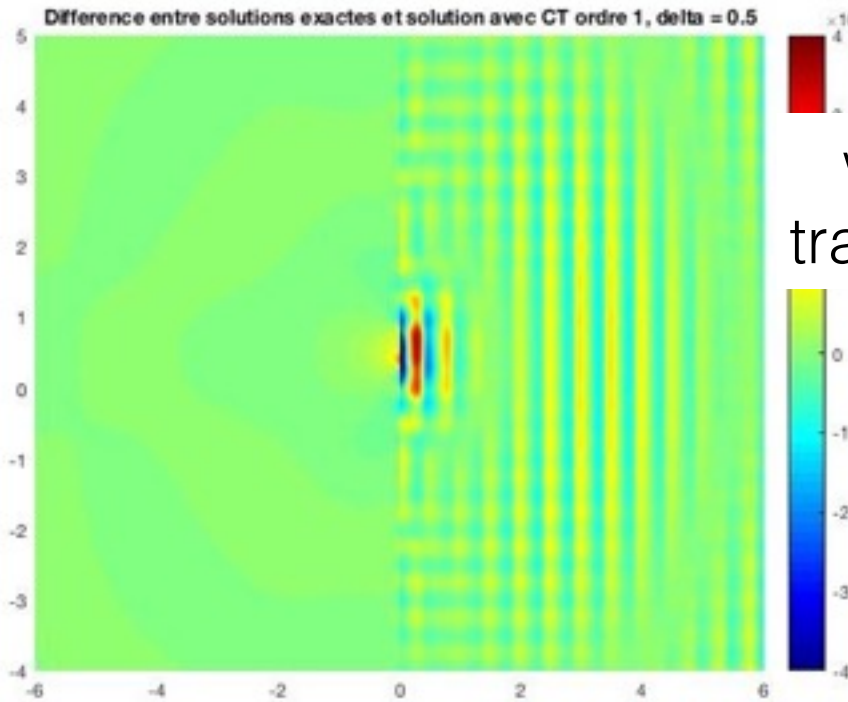
with classical transmission condition



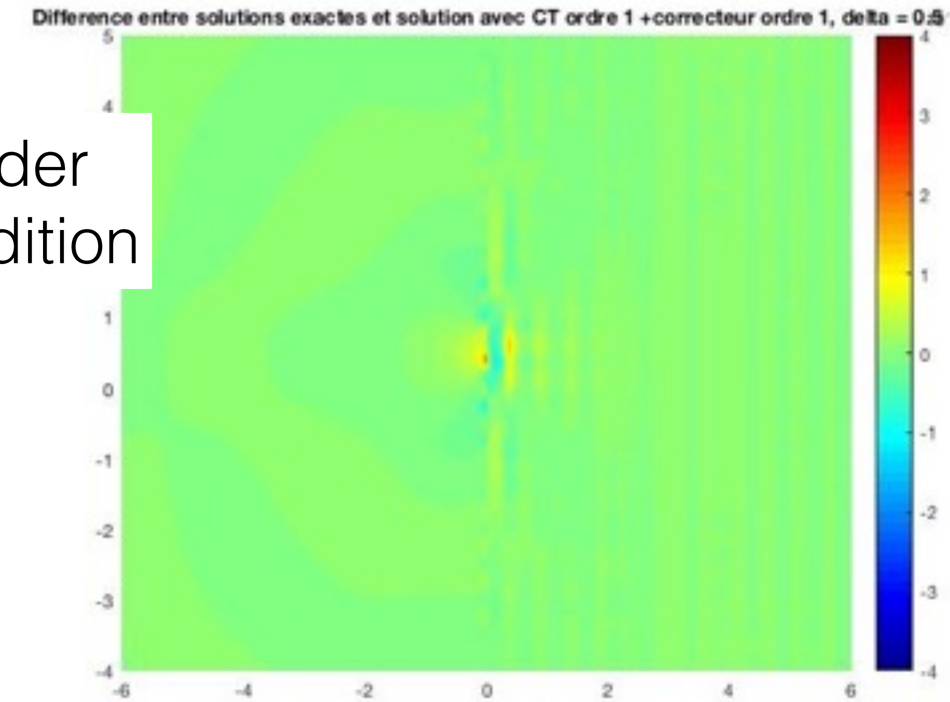
$$u_\varepsilon - \left(u_0(\mathbf{x}) + \varepsilon \, \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$$

Numerical results to motivate the rest of the talk

$u_\varepsilon - \tilde{v}_\varepsilon$

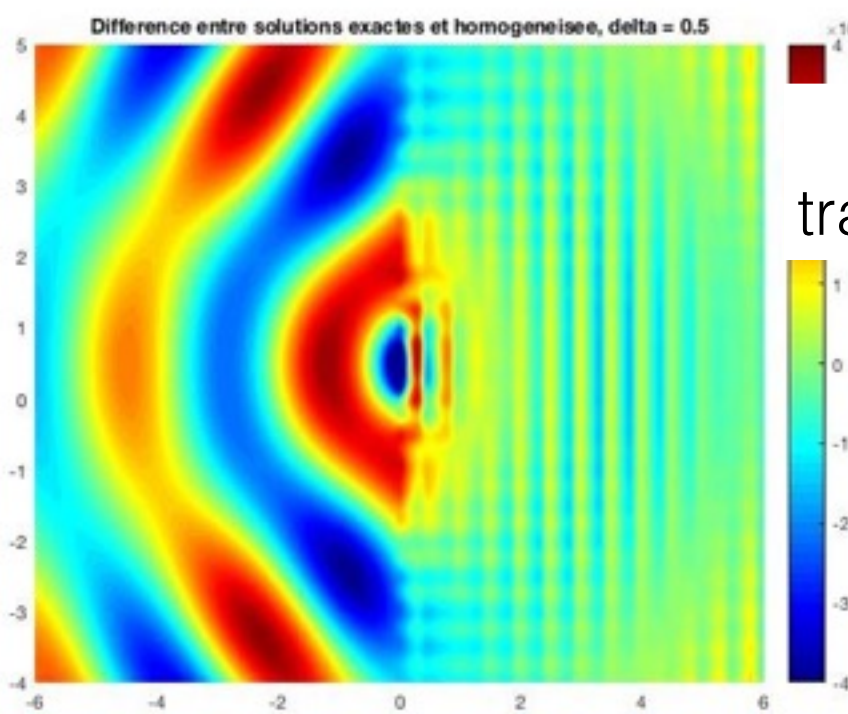


$u_\varepsilon - \left(\tilde{v}_\varepsilon(\mathbf{x}) + \varepsilon \nabla_x \tilde{v}_\varepsilon(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$

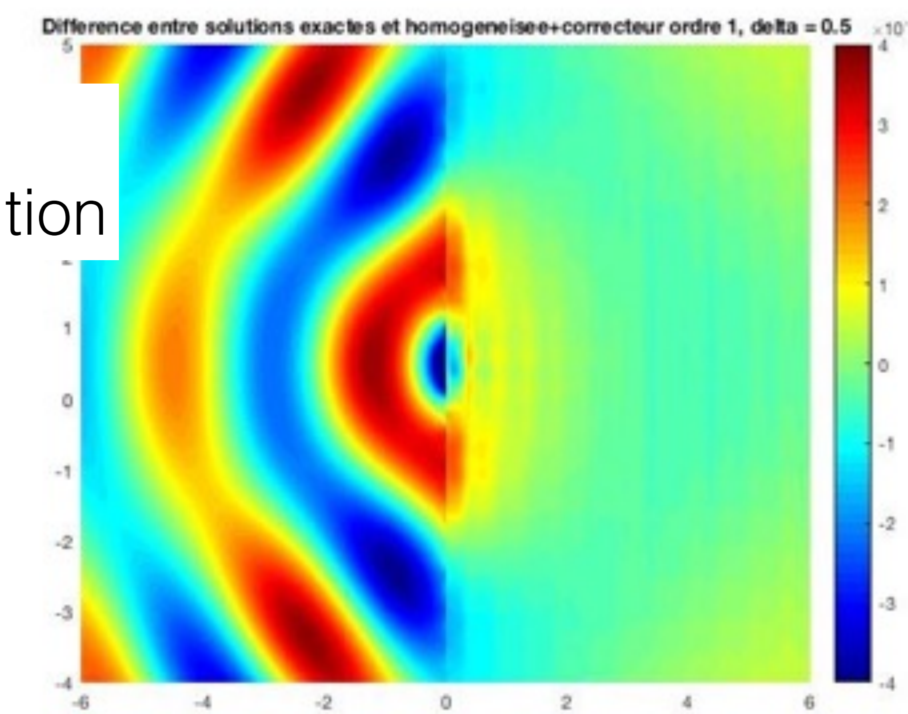


with our high order transmission condition

For $\varepsilon = 0.5$



$u_\varepsilon - u_0$



$u_\varepsilon - \left(u_0(\mathbf{x}) + \varepsilon \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$

with classical transmission condition

The steps of our approach

1. **The asymptotic expansions of the solution**

   The formal steps of the matched asymptotic expansion

   Existence and uniqueness results

2. **Construction of the approximate conditions for the asymptotic expansion**

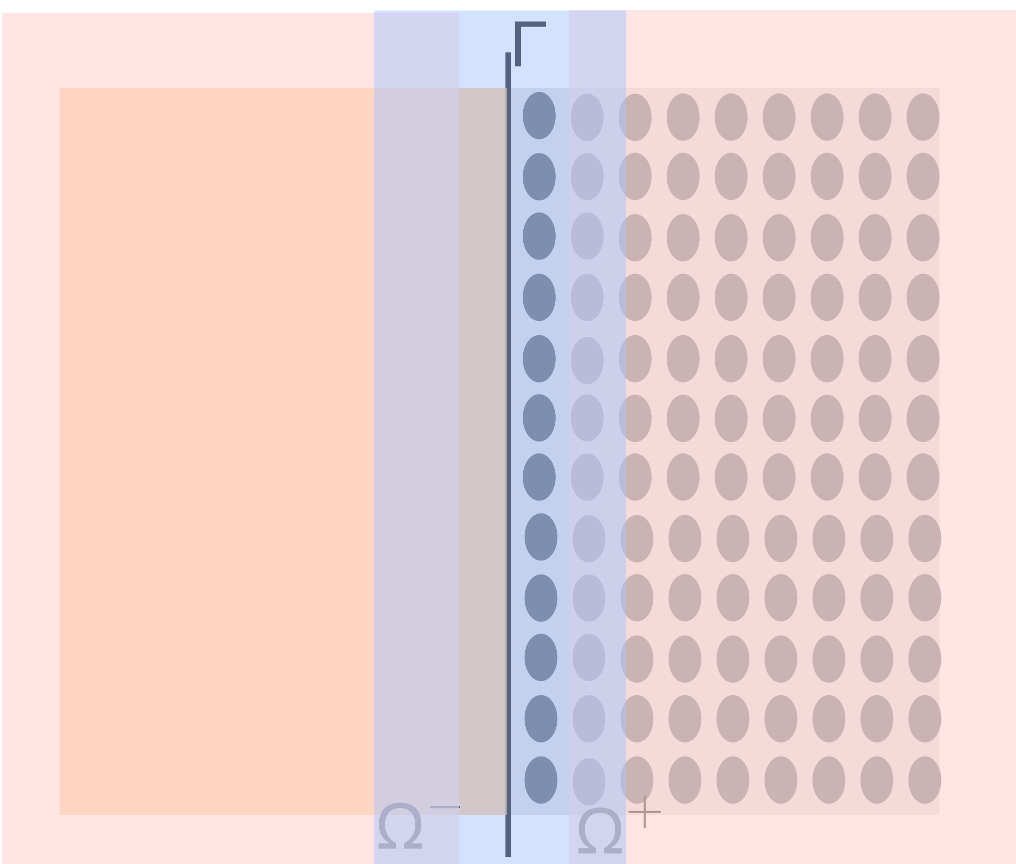
3. **Stability and error estimates for the approximate problem**

4. **Numerical implementation and validation**

The asymptotic expansions of the solution

We cannot expect a single asymptotic expansion far from the interface and in the neighbourhood of the interface.

We will distinguish different regions in which we postulate different ansatz.

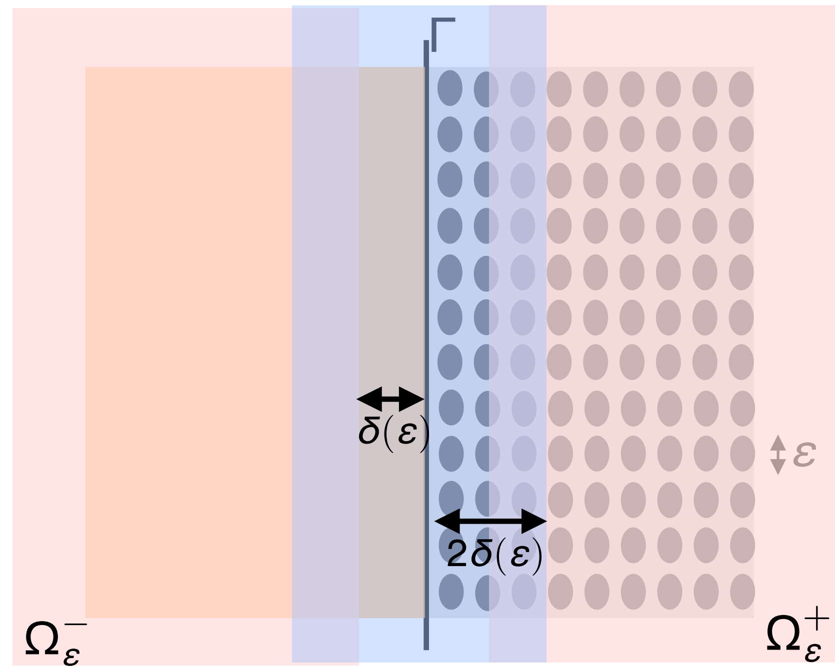


✓ ***Two Far Field zones*** : regions far from the interface

✓ ***One Near Field zone*** : region in the neighbourhood of the interface

The regions overlap and the different asymptotic expansions have to coincide in the transition zones (matching principle).

The asymptotic expansions of the solution



$$\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$$

$$\lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\varepsilon} = +\infty$$

$$\Omega_\varepsilon^- = \{(x_1, x_2), x_1 \leq -\delta(\varepsilon)\}$$

Ansatz in Ω_ε^-

$$u_\varepsilon(\mathbf{x}) = \sum_{n\in\mathbb{N}} \varepsilon^n u_n^-(\mathbf{x})$$

$$\Omega_\varepsilon^0 = \{(x_1, x_2), |x_1| \leq 2\delta(\varepsilon)\}$$

Ansatz in Ω_ε^0

$$u_\varepsilon(\mathbf{x}) = \sum_{n\in\mathbb{N}} \varepsilon^n U_n\left(x_2, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$$

where $U_n(x_2, y_1, y_2)$ is 1-periodic with respect to y_2 but not in y_1

$$\Omega_\varepsilon^+ = \{(x_1, x_2), x_1 \geq \delta(\varepsilon)\}$$

Ansatz in Ω_ε^+

$$u_\varepsilon(\mathbf{x}) = \sum_{n\in\mathbb{N}} \varepsilon^n u_n^+\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)$$

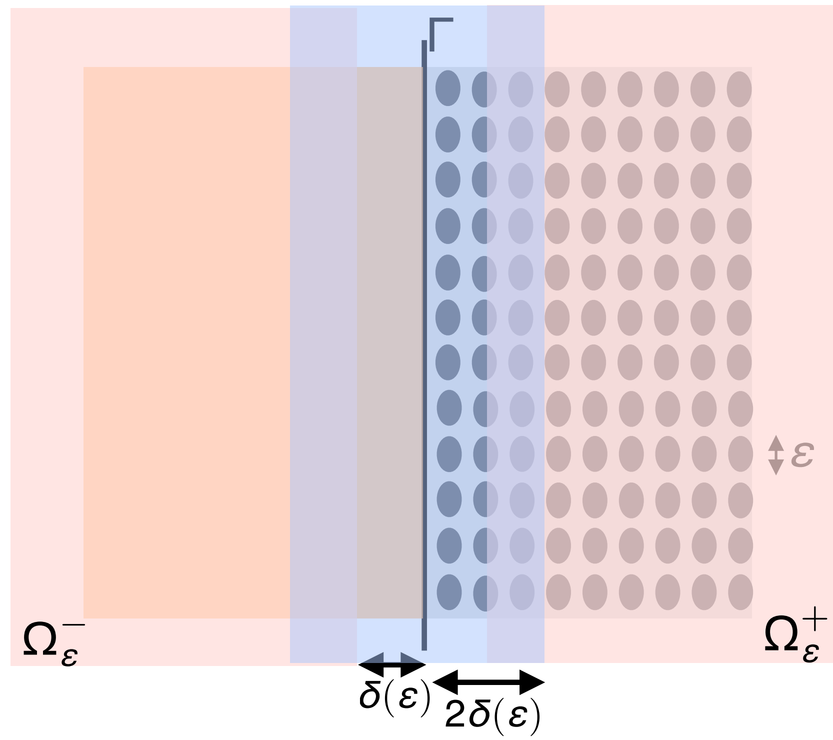
where $u_n^+(\mathbf{x}, \mathbf{y})$ is 1-periodic with respect to $\mathbf{y}$.

The different asymptotic expansions have to coincide in the overlapping zone.

The matching conditions link the behaviour u_n^+'s at Γ to the behaviour of U_n's at $+\infty$.

the behaviour u_n^-'s at Γ to the behaviour of U_n's at $-\infty$.

The asymptotic expansions of the solution



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Ansatz in Ω_ε^0

$$u_\varepsilon(\mathbf{x}) = \sum_{n\in\mathbb{N}} \varepsilon^n U_n\left(x_2, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$$

where $U_n(x_2, y_1, y_2)$ is 1-periodic with respect to y_2 but not in y_1

$\Omega_\varepsilon^+ = \{(x_1, x_2), x_1 \ge \delta(\varepsilon)\}$

Ansatz in Ω_ε^+

$$u_\varepsilon(\mathbf{x}) = \sum_{n\in\mathbb{N}} \varepsilon^n u_n^+\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)$$

where $u_n^+(\mathbf{x}, \mathbf{y})$ is 1-periodic with respect to $\mathbf{y}$.

The different asymptotic expansions have to coincide in the overlapping zone.

The matching step relies on Taylor expansion of the far field terms $u_n^\pm$ and the behaviour at $\pm\infty$ of the near field terms U_n .

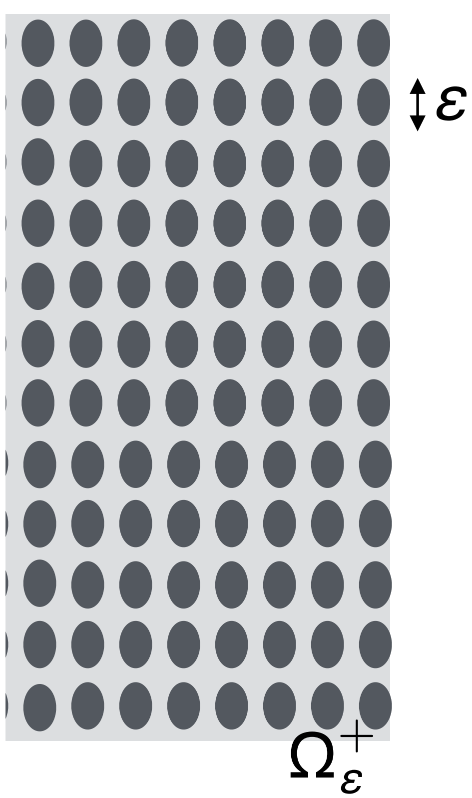
Equations for the far fields

Ansatz : $u_\varepsilon(\mathbf{x}) = \sum_{n\in\mathbb{N}} \varepsilon^n u_n^-(\mathbf{x})$

$$-a_0 \triangle_x u_n^- - \omega^2\, u_n^- = \begin{cases} f & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

To determine uniquely the far field, we need additional conditions :

there will be provided by the matching conditions



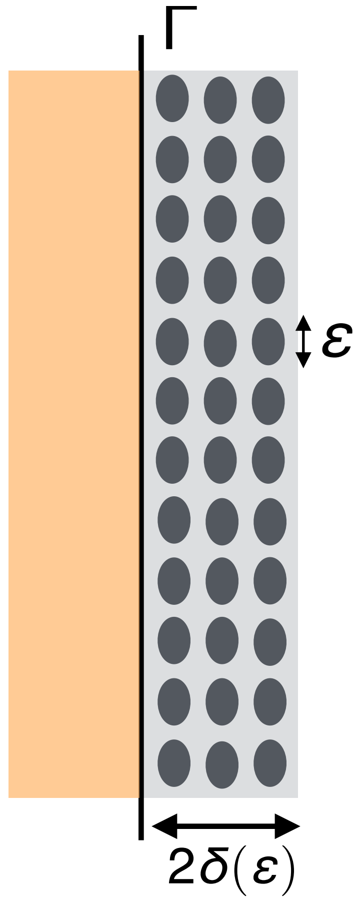
Ansatz : $u_\varepsilon(\mathbf{x}) = \sum_{n\in\mathbb{N}} \varepsilon^n u_n^+\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)$ where $u_n^+(\mathbf{x},\mathbf{y})$ is 1-periodic with respect to $\mathbf{y}$.

$$-\nabla_x \cdot \left[A^*\, \nabla_x\, u_0^+\right] - \omega^2\, u_0^+ = 0$$

$$u_1^+(\mathbf{x},\mathbf{y}) \equiv \nabla_x u_0^+(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{y}) \\ w_2(\mathbf{y}) \end{bmatrix} + \hat{u}_1^+(\mathbf{x})$$

$$u_2^+(\mathbf{x},\mathbf{y}) \equiv \nabla_x \cdot \begin{bmatrix} \theta_{11}(\mathbf{y}) & \theta_{12}(\mathbf{y}) \\ \theta_{21}(\mathbf{y}) & \theta_{22}(\mathbf{y}) \end{bmatrix} \nabla_x u_0^+(\mathbf{x}) + \nabla_x \hat{u}_1^+(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{y}) \\ w_2(\mathbf{y}) \end{bmatrix} + \hat{u}_2^+(\mathbf{x})$$

The asymptotic expansion in the near field zone



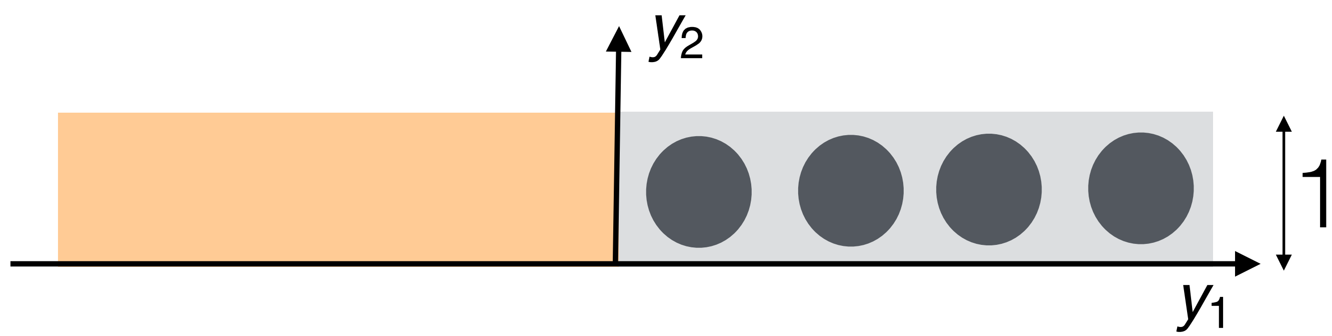
Ansatz : $u_\varepsilon(\mathbf{x}) = \sum_{n\in\mathbb{N}} \varepsilon^n U_n\left(x_2, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$

macroscopic — boundary layer — homogenization

$U_n(x_2, y_1, y_2) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ \rightarrow \ \mathbb{C}$

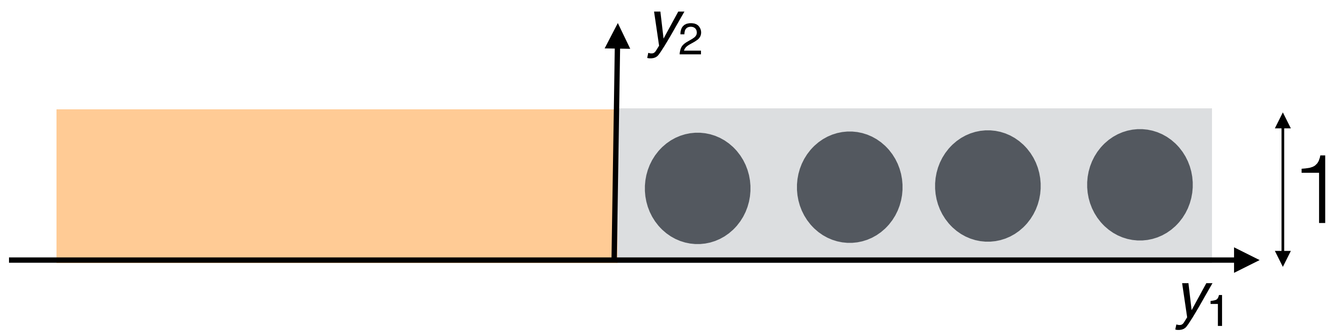
$U_n(x_2, y_1, y_2)$ is 1-periodic with respect to y_2.

By periodicity $U_n(x_2, \cdot; \cdot)$ can be studied in the infinite strip $\forall x_2,$



$S = \mathbb{R} \times (0, 1)$

The equations for the near field terms



$$-\nabla_y \cdot (a(\mathbf{y})\nabla_y U_0) = 0$$

$$-\nabla_y \cdot (a(\mathbf{y})\nabla_y U_1) = \partial_{y_2}(a(\mathbf{y})\partial_{x_2} U_0) + \partial_{x_2}(a(\mathbf{y})\partial_{y_2} U_0)$$

$$-\nabla_y \cdot (a(\mathbf{y})\nabla_y U_2) = \partial_{y_2}(a(\mathbf{y})\partial_{x_2} U_1) + \partial_{x_2}(a(\mathbf{y})\partial_{y_2} U_1) + a(\mathbf{y})\partial^2_{x_2} U_0 - \omega^2 U_0$$

$$\ldots$$

The U_n are solutions of **Laplace equations** for (y_1, y_2) in the infinite strip $S = \mathbb{R} \times (0,1)$, and in which x_2 plays a role of a **parameter** through the right hand side.

We impose that U_n does not increase exponentially at infinity but it may increase **polynomially**.

V^1_+ : functions locally H^1, periodic w.r.t. y_2,increasing at least polynomially at infinity

These equations determine by induction the U_n up to an element of the **kernel** of

$$L_0 = -\nabla_y(a(\mathbf{y})\nabla_y \cdot)$$

for each value of x_2 .

The equations for the near field terms

$$L_0 = -\nabla_y(a(\mathbf{y})\nabla_y\cdot)$$

$$V^1_\pm = \{U \in H^1_{\mathrm{loc}}(S), U(\cdot, y_2+1) = U(\cdot, y_2), \sum_{\alpha+\beta\leq 1}\int_B |\partial^\alpha\partial^\beta U|^2 e^{-(\pm\gamma|y_1|)}\, d\mathbf{y} < +\infty\}$$

V^1_+ / V^1_- : functions exponentially increasing/decreasing at infinity

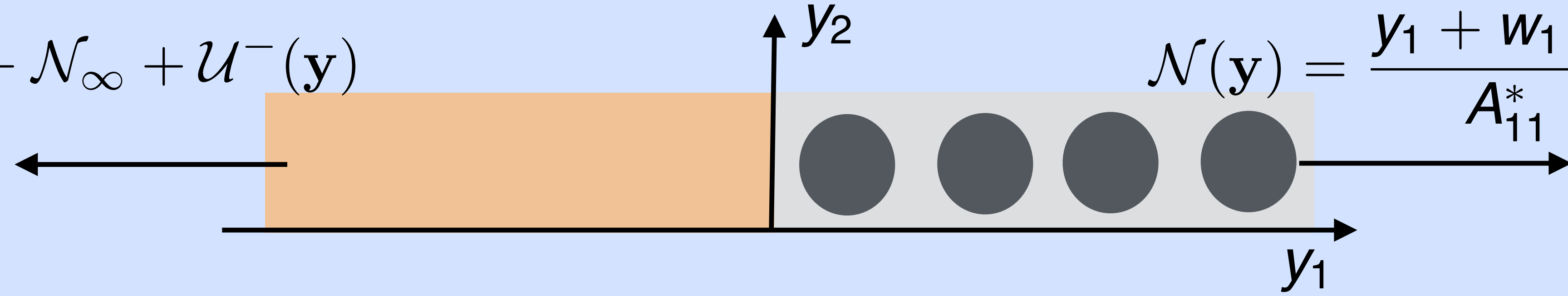
Theorem

In V^1_+, the kernel of L_0 is of dimension 2 and more precisely

$$\mathrm{Ker}(L_0) = \{1, \mathcal{N}\}$$

where $\mathcal{N}$, called a profile function, is defined by

$$\mathcal{N}(\mathbf{y}) = \frac{y_1}{a_0} - \mathcal{N}_\infty + \mathcal{U}^-(\mathbf{y}) \qquad \mathcal{N}(\mathbf{y}) = \frac{y_1 + w_1(\mathbf{y})}{A^*_{11}} + \mathcal{N}_\infty + \mathcal{U}^+(\mathbf{y})$$



and $\mathcal{U}^\pm$ exponentially decreasing at $\pm\infty$

Tools for the proof : Floquet-Bloch Transformation + Kondratiev Theory

Kondratiev, Kozlov-Maz'ja-Rossmann, Kuchment, Nazarov

$$L_0\, 1 = 0$$

If $a(\mathbf{y}) = a_0$, $L_0\, y_1 = 0$ If $a(\mathbf{y}) = a_p(\mathbf{y})$, $L_0\,(y_1 + w_1(\mathbf{y})) = 0$

The equations for the near field terms

$$L_0 = -\nabla_y (a(\mathbf{y}) \nabla_y \cdot)$$

$$V^1_{\pm} = \{ U \in H^1_{loc}(S), U(\cdot, y_2 + 1) = U(\cdot, y_2), \sum_{\alpha+\beta\leq 1} \int_B |\partial^\alpha \partial^\beta U|^2 e^{-(\pm\gamma|y_1|)} \, d\mathbf{y} < +\infty \}$$

V^1_+ / V^1_- : functions exponentially increasing/decreasing at infinity

$(V^1_+)'$: functions exponentially decreasing at infinity

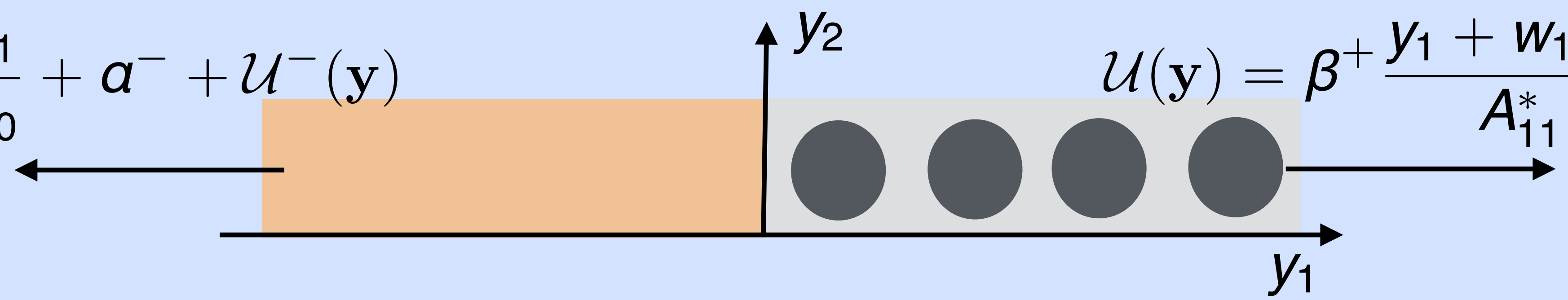
Proposition

For all $g \in (V^1_+)'$, the solution $\mathcal{U}$ in V^1_+

$$L_0 \mathcal{U} = g$$

have the following behaviour

$$\mathcal{U}(\mathbf{y}) = \beta^- \frac{y_1}{a_0} + \alpha^- + \mathcal{U}^-(\mathbf{y}) \qquad \mathcal{U}(\mathbf{y}) = \beta^+ \frac{y_1 + w_1(\mathbf{y})}{A^*_{11}} + \alpha^+ + \mathcal{U}^+(\mathbf{y})$$



and $\mathcal{U}^{\pm} \in V^1_-$.

Tools for the proof : Floquet-Bloch Transformation + Kondratiev Theory

The equations for the near field terms

$$L_0 = -\nabla_y (a(\mathbf{y}) \nabla_y \cdot)$$

$$V^1_\pm = \{U \in H^1_{loc}(S), U(\cdot, y_2+1) = U(\cdot, y_2), \sum_{\alpha+\beta \le 1} \int_B |\partial^\alpha \partial^\beta U|^2 e^{-(\pm\gamma|y_1|)}\, d\mathbf{y} < +\infty\}$$

V^1_+ / V^1_- : functions exponentially increasing/decreasing at infinity

$(V^1_+)'$: functions exponentially decreasing at infinity

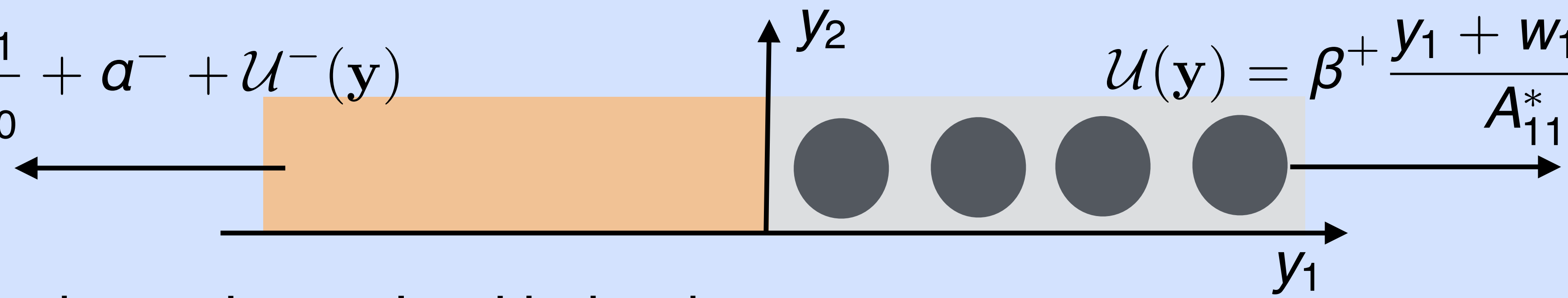
Existence and uniqueness result

For all $g \in (V^1_+)'$, $\alpha,\ \beta \in \mathbb{C}$, **it exists an unique solution** solution $\mathcal{U}$ in V^1_+ of

$$\left| \begin{array}{c} L_0 \mathcal{U} = g \\ <\alpha^\pm> = \alpha \quad \text{and} \quad <\beta^\pm> = \beta \end{array} \right.$$

where

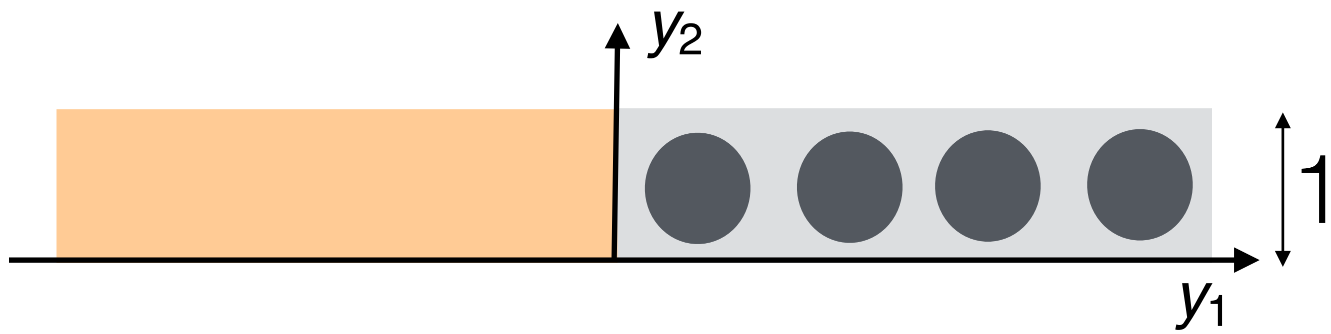
$$\mathcal{U}(\mathbf{y}) = \beta^- \frac{y_1}{a_0} + \alpha^- + \mathcal{U}^-(\mathbf{y}) \qquad \mathcal{U}(\mathbf{y}) = \beta^+ \frac{y_1 + w_1(\mathbf{y})}{A^*_{11}} + \alpha^+ + \mathcal{U}^+(\mathbf{y})$$



and $\mathcal{U}$ depends continuously with the datas.

Definition : $\quad <\alpha^\pm> \equiv \dfrac{\alpha^+ + \alpha^-}{2} \qquad <\beta^\pm> \equiv \dfrac{\beta^+ + \beta^-}{2}$

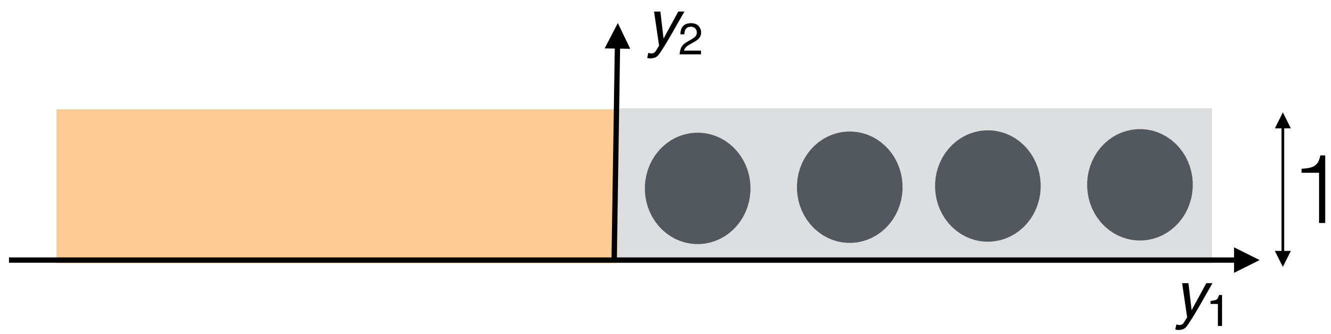
The equations for the near field terms



- $-\nabla_y \cdot (a(\mathbf{y})\nabla_y U_0) = 0$

$$U_0(x_2, y_1, y_2) = \alpha_0(x_2) + \beta_0(x_2)\,\mathcal{N}(y_1, y_2)$$

The equations for the near field terms



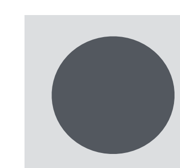
- $$-\nabla_y \cdot (a(\mathbf{y})\nabla_y U_1) = \underbrace{\partial_{y_2} a(\mathbf{y})\partial_{x_2} U_0}_{\notin (V^1_+)'}$$

$$U_1(x_2, y_1, y_2) = a_1(x_2) + \beta_1(x_2)\mathcal{N}(y_1, y_2) + U_0'(x_2)\left[\chi(y_1) w_2(y_1, y_2) + \mathcal{Z}_1(y_1, y_2)\right]$$

For $y_1 \in \mathbb{R}^-$ $\partial_{y_2} a(\mathbf{y}) = 0$ and for $y_1 \in \mathbb{R}^+$ $\partial_{y_2} a(\mathbf{y}) = \partial_{y_2} a_p(\mathbf{y})$

where w_2 is the solution of the cell problem

$$-\nabla_y \cdot (a(\mathbf{y})\nabla_y w_2) = \partial_{y_2} a(\mathbf{y}), \quad \text{in } Y$$

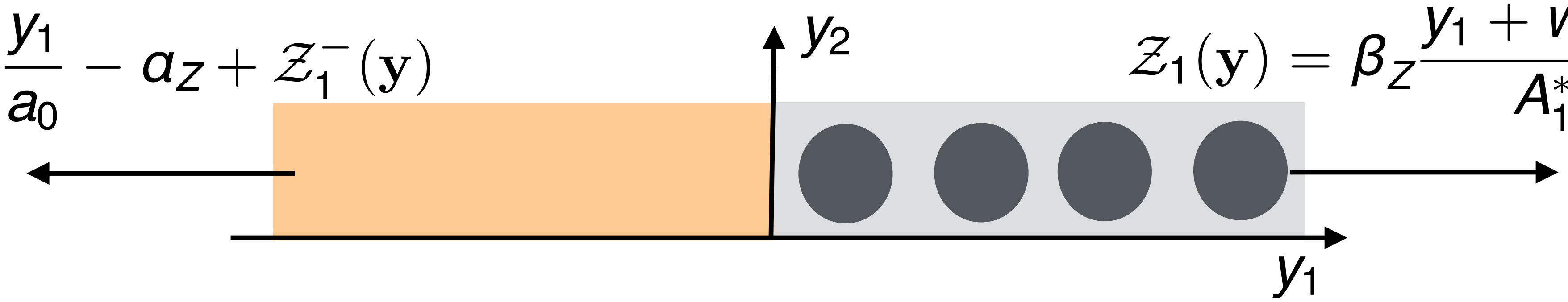


$Y = (0,1)^2$

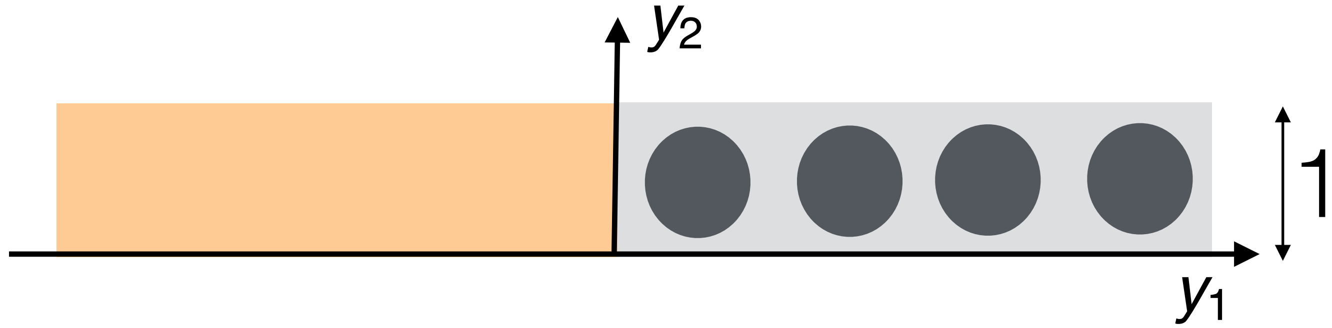
$\mathcal{Z}_1$ is a profile function, solution in the infinite strip of

$$-\nabla_y \cdot (a(\mathbf{y})\nabla_y \mathcal{Z}_1) = \partial_{y_2} a(\mathbf{y}) - \nabla_y \cdot (a(\mathbf{y})\nabla_y \chi w_2)$$

$$\mathcal{Z}_1(\mathbf{y}) = -\beta_Z \frac{y_1}{a_0} - \alpha_Z + \mathcal{Z}_1^-(\mathbf{y}) \qquad \mathcal{Z}_1(\mathbf{y}) = \beta_Z \frac{y_1 + w_1(\mathbf{y})}{A^*_{11}} + \alpha_Z + \mathcal{Z}_1^+(\mathbf{y})$$



The equations for the near field terms



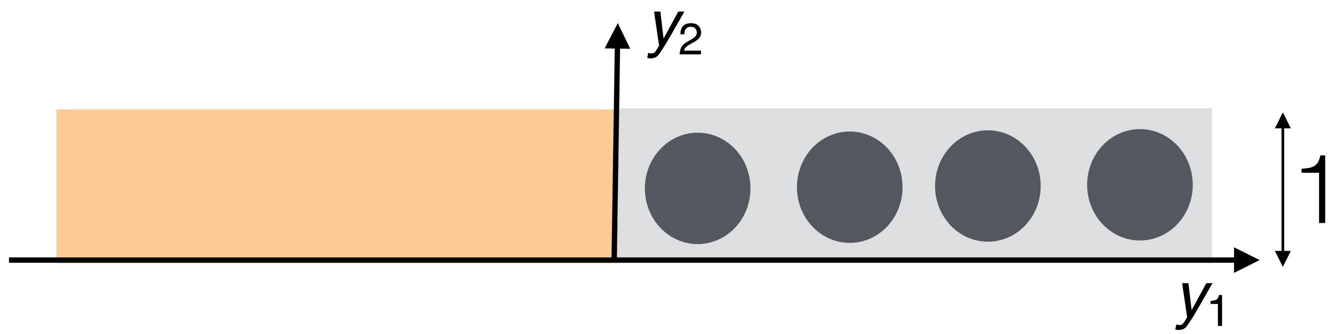
- $-\nabla_y \cdot (a(\mathbf{y})\nabla_y U_2) = \partial_{y_2}(a(\mathbf{y})\partial_{x_2} U_1) + \partial_{x_2}(a(\mathbf{y})\partial_{y_2} U_1) + a(\mathbf{y})\partial^2_{x_2} U_0 - \omega^2 U_0$

$$
\begin{aligned}
U_2(x_2, y_1, y_2) = {} & a_2(x_2) + \beta_2(x_2)\mathcal{N} + a_1'(x_2)\left[\chi w_2 + \mathcal{Z}_1\right] \\
& -\omega^2 a_0(x_2)\left[-(1-\chi)\frac{y_1^2}{2a_0} - \chi\,\frac{y_1^2/2 + y_1 w_1 + \theta_{11}}{A^*_{11}} + \mathcal{Z}_2^{(1)}\right] \\
& +\beta_1'(x_2)\left[\chi\left(\mathcal{N}_\infty w_2 + \frac{1}{A^*_{11}}[2\theta_{21} + y_1 w_2 + 2A^*_{12}(y_1^2/2 + y_1 w_1 + \theta_{11})]\right) + \mathcal{Z}_2^{(2)}\right] \\
& +a_0''(x_2)\left[\chi\left(a_Z w_2 + \theta_{22} + \frac{\beta_Z}{A^*_{11}}[2\theta_{21} + y_1 w_2] + \frac{2A^*_{12}\beta_Z + A^*_{22}}{A^*_{11}}(y_1^2/2 + y_1 w_1 + \theta_{11})]\right)\right. \\
& \left. + (1-\chi)y_1^2/2 + \mathcal{Z}_2^{(3)}\right]
\end{aligned}
$$

solutions of cell problems

profile functions, solution of band problems

The equations for the near field terms



- $U_0(x_2, y_1, y_2) = \alpha_0(x_2) + \beta_0(x_2)\mathcal{N}(y_1, y_2)$

- $U_1(x_2, y_1, y_2) = \alpha_1(x_2) + \beta_1(x_2)\mathcal{N}(y_1, y_2) + U_0'(x_2)\left[\chi(y_1)w_2(y_1, y_2) + \mathcal{Z}_1(y_1, y_2)\right]$

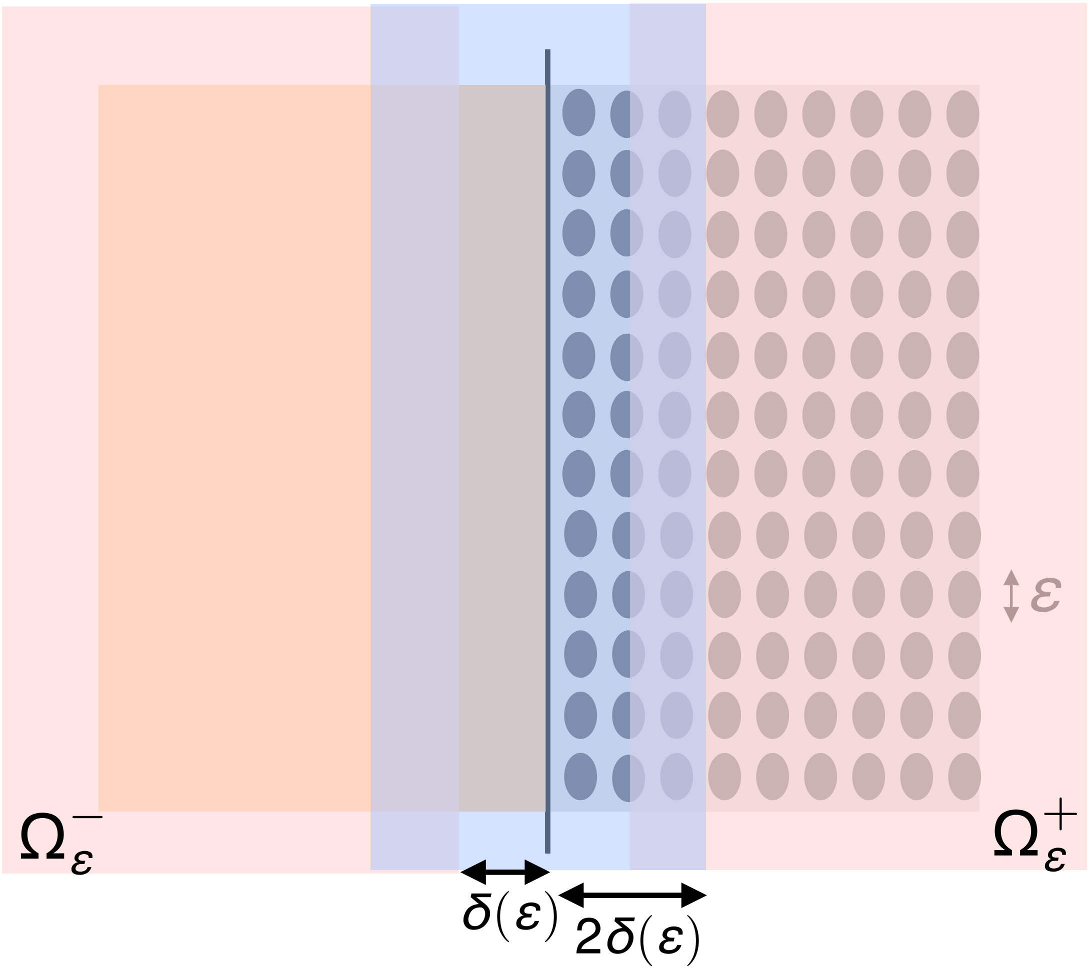
- $U_2(x_2, y_1, y_2) = \alpha_2(x_2) + \beta_2(x_2)\mathcal{N} + \alpha_1'(x_2)\left[\chi w_2 + \mathcal{Z}_1\right]$ + …

To determine uniquely the near field, we need two additional conditions :

Matching conditions

The matching conditions

The missing information will be provided by the matching conditions that are obtained by expressing that the expansions have to coincide in the overlapping zone.



$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$$

$$\lim_{\varepsilon \to 0} \frac{\delta(\varepsilon)}{\varepsilon} = +\infty$$

The matching conditions link the u_n^+ at Γ to the behaviour of U_n at $+\infty$.

the u_n^- at Γ to the behaviour of U_n at $-\infty$.

> The matching step relies on Taylor expansion of the far field terms $u_n^\pm$ and the behaviour at $\pm\infty$ of the near field terms U_n .

One gets **transmission conditions** for (u_n^-, u_n^+) by eliminating the U_n .

The matching conditions

At order 0:

Ω_ε^- | Ω_ε^0 | Ω_ε^+

$$u_\varepsilon = u_0^- + \mathcal{O}(\varepsilon)$$

$$u_\varepsilon = u_0^+ + \mathcal{O}(\varepsilon)$$

$$u_\varepsilon = a_0(x_2) + \beta_0(x_2)\mathcal{N}\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + \mathcal{O}(\varepsilon)$$

$$u_0^-\Big|_\Gamma = a_0(x_2) + \beta_0(x_2)\left(-\mathcal{N}_\infty + \frac{x_1/\varepsilon}{a_0}\right)$$

0

$$u_0^+\Big|_\Gamma = a_0(x_2) + \beta_0(x_2)\left(\mathcal{N}_\infty + \frac{x_1/\varepsilon + w_1(\mathrm{x}/\varepsilon)}{A_{11}^*}\right)$$

0

Dirichlet conditions for the order 0

$$\boxed{u_0^-\Big|_\Gamma = u_0^+\Big|_\Gamma} \equiv a_0(x_2)$$

The matching conditions

At order 1:

Ω_ε^- Ω_ε^0 Ω_ε^+

$$u_\varepsilon(\mathbf{x}) = u_0^-(\mathbf{x}) + \varepsilon u_1^-(\mathbf{x}) + \mathcal{O}(\varepsilon^2)$$

$$u_\varepsilon(\mathbf{x}) = u_0^+(\mathbf{x}) + \varepsilon\, u_1^+(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \mathcal{O}(\varepsilon^2)$$

$$u_\varepsilon = a_0(x_2) + \varepsilon\left(a_1(x_2) + \beta_1(x_2)\mathcal{N}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) + a_0'(x_2)[\chi(\frac{x_1}{\varepsilon})w_2(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) + \mathcal{Z}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})]\right) + \mathcal{O}(\varepsilon^2)$$

$$u_0^-\Big|_\Gamma + x_1 \frac{\partial u_0^-}{\partial x_1}\Big|_\Gamma + \varepsilon u_1^-\Big|_\Gamma$$
$$= a_0(x_2) + \varepsilon\left(a_1(x_2) + \beta_1(x_2)\left[-\mathcal{N}_\infty + \frac{x_1/\varepsilon}{a_0}\right] + a_0'(x_2)\left[-a_Z - \beta_Z \frac{x_1/\varepsilon}{a_0}\right]\right)$$

The matching conditions

At order 1:

Ω_ε^- | Ω_ε^0 | Ω_ε^+

$$u_\varepsilon(\mathbf{x}) = u_0^-(\mathbf{x}) + \varepsilon u_1^-(\mathbf{x}) + \mathcal{O}(\varepsilon^2)$$

$$u_\varepsilon(\mathbf{x}) = u_0^+(\mathbf{x}) + \varepsilon\, u_1^+(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \mathcal{O}(\varepsilon^2)$$

$$u_\varepsilon = a_0(x_2) + \varepsilon\left(a_1(x_2) + \beta_1(x_2)\mathcal{N}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) + a_0'(x_2)[\chi(\frac{x_1}{\varepsilon})w_2(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) + \mathcal{Z}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})]\right) + \mathcal{O}(\varepsilon^2)$$

$$x_1 \frac{\partial u_0^-}{\partial x_1}\Big|_\Gamma + \varepsilon u_1^-\Big|_\Gamma = x_1\left(\frac{\beta_1(x_2)}{a_0} - \frac{a_0'(x_2)\beta_Z}{a_0}\right) + \varepsilon\left(a_1(x_2) - \beta_1(x_2)\mathcal{N}_\infty - a_0'(x_2)a_Z\right)$$

$$u_0^+\Big|_\Gamma + x_1\frac{\partial u_0^+}{\partial x_1}\Big|_\Gamma + \varepsilon\left(\frac{\partial u_0^+}{\partial x_1}w_1 + \frac{\partial u_0^+}{\partial x_2}w_2 + \hat{u}_1^+\Big|_\Gamma\right)$$

$$= a_0(x_2) + \varepsilon\left(a_1(x_2) + \beta_1(x_2)\left[\mathcal{N}_\infty + \frac{x_1/\varepsilon + w_1}{A_{11}^*}\right] + a_0'(x_2)\left[w_2 + a_Z + \beta_Z\frac{x_1/\varepsilon + w_1}{A_{11}^*}\right]\right)$$

The matching conditions

At order 1:

Ω_ε^- | Ω_ε^0 | Ω_ε^+

$$u_\varepsilon(\mathbf{x}) = u_0^-(\mathbf{x}) + \varepsilon u_1^-(\mathbf{x}) + \mathcal{O}(\varepsilon^2)$$

$$u_\varepsilon(\mathbf{x}) = u_0^+(\mathbf{x}) + \varepsilon\, u_1^+(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \mathcal{O}(\varepsilon^2)$$

$$u_\varepsilon = a_0(x_2) + \varepsilon \left(a_1(x_2) + \beta_1(x_2)\mathcal{N}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) + a_0'(x_2)[\chi(\frac{x_1}{\varepsilon}) w_2(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) + \mathcal{Z}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})] \right) + \mathcal{O}(\varepsilon^2)$$

Neumann conditions for the order 0

$$\boxed{a_0 \frac{\partial u_0^-}{\partial x_1}\Big|_\Gamma = A^* \nabla u_0^+ \cdot \mathbf{e}_1}$$

Dirichlet conditions for the order 1

$$\boxed{\hat{u}_1^+\big|_\Gamma - u_1^-\big|_\Gamma = 2\mathcal{N}_\infty \frac{a_0 \partial_{x_1} u_0^- |_\Gamma + A^* \nabla u_0^+ \cdot \mathbf{e}_1}{2} + (2a_Z - \mathcal{N}_\infty)\partial_{x_2} u_0\big|_\Gamma}$$

The matching conditions

The matching conditions at order 2 give

$\dots$

Neumann conditions for the order 1

$$A^* \nabla \hat{u}_1^+ \cdot \mathrm{e}_1 \big|_\Gamma - a_0 \frac{\partial u_1^-}{\partial x_1} \big|_\Gamma = Y_1 \partial^2_{x_1 x_2} u_0^- \big|_\Gamma + Y_2 \partial^2_{x_2} u_0^- \big|_\Gamma + Y_3\, \omega^2\, u_0^- \big|_\Gamma$$

where Y_1, Y_2 and Y_3 depend on the **profile functions**.

Dirichlet conditions for the order 2

$\dots$

The first far field problems

$$-a_0 \triangle u_0^- - \omega^2 u_0^- = f, \quad \text{in } \Omega^-$$

$$-\nabla \cdot \left[A^* \nabla u_0^+\right] - \omega^2 u_0^+ = 0, \quad \text{in } \Omega^+$$

$$[u_0]_\Gamma = 0 \ \text{ and } \quad [A_0^* \nabla u_0 \cdot \mathrm{e}_1]_\Gamma = 0$$

with $A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$

$$-a_0 \triangle u_1^- - \omega^2 u_1^- = 0, \quad \text{in } \Omega^-$$

$$-\nabla_x \cdot \left[A^* \nabla_x \hat{u}_1^+\right] - \omega^2 \hat{u}_1^+ = 0, \quad \text{in } \Omega^+$$

$$[u_1]_\Gamma = C_1^{(1)} < A_0^* \nabla u_0 \cdot \mathrm{e}_1 >_\Gamma + C_2^{(1)} < \partial_{x_2} u_0 >_\Gamma$$

$$\left[A_0^* \nabla \hat{u}_1^+ \cdot \mathrm{e}_1\right]_\Gamma = C_1^{(2)} < \partial_{x_2}(A_0^* \nabla u_0 \cdot \mathrm{e}_1) >_\Gamma + C_2^{(2)} < \partial_{x_2}^2 u_0 >_\Gamma + C_3^{(2)} \omega^2 < u_0 >_\Gamma$$

with $< A_0^* \nabla u_0 \cdot \mathrm{e}_1 >_\Gamma = \dfrac{A_0^* \nabla u_0^+ \cdot \mathrm{e}_1 |_\Gamma + a_0 \partial u_0^- |_\Gamma}{2}$

$$< \partial_{x_2} u_0 >_\Gamma = \frac{\partial_{x_2} u_0^+ |_\Gamma + \partial_{x_2} u_0^- |_\Gamma}{2} \quad \ldots$$

where all the constants are defined thanks to **cell solutions** and **profile functions**.

The first far field problems

$$-a_0 \triangle u_0^- - \omega^2 u_0^- = f, \quad \text{in } \Omega^-$$

$$-\nabla \cdot \left[A^* \nabla u_0^+ \right] - \omega^2 u_0^+ = 0, \quad \text{in } \Omega^+$$

$$[u_0]_\Gamma = 0 \;\; \text{and} \qquad [A_0^* \nabla u_0 \cdot \mathbf{e}_1]_\Gamma = 0$$

with $A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$

$$-a_0 \triangle u_1^- - \omega^2 u_1^- = 0, \quad \text{in } \Omega^-$$

$$-\nabla_x \cdot \left[A^* \nabla_x \hat{u}_1^+ \right] - \omega^2 \hat{u}_1^+ = 0, \quad \text{in } \Omega^+$$

$$[u_1]_\Gamma = C_1^{(1)} < A_0^* \nabla u_0 \cdot \mathbf{e}_1 >_\Gamma + C_2^{(1)} < \partial_{x_2} u_0 >_\Gamma$$

$$\left[A_0^* \nabla \hat{u}_1^+ \cdot \mathbf{e}_1 \right]_\Gamma = C_1^{(2)} < \partial_{x_2} (A_0^* \nabla u_0 \cdot \mathbf{e}_1) >_\Gamma + C_2^{(2)} < \partial_{x_2}^2 u_0 >_\Gamma + C_3^{(2)} \omega^2 < u_0 >_\Gamma$$

The near fields U_0 and U_1 can then be determined thanks to the far fields.

The steps of our approach

1. **The asymptotic expansions of the solution**

    The formal steps of the matched asymptotic expansion

    Existence and uniqueness results

    Error estimates

2. **Construction of the approximate conditions for the asymptotic expansion**

3. **Stability and error estimate for the approximate problem**

4. **Numerical implementation and validation**

The approximate problem

A natural candidate to provide a better approximation of the problem is

$$u_{\varepsilon,1}(\mathbf{x}) = \begin{cases} u_0^-(\mathbf{x}) + \varepsilon u_1^-(\mathbf{x}) & \mathbf{x} \in \ \Omega^- \\ u_0^+(\mathbf{x}) + \varepsilon \left(\nabla u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} + \hat{u}_1^+(\mathbf{x}) \right) & \mathbf{x} \in \ \Omega^+ \end{cases}$$

Let us introduce

$$\tilde{u}_{\varepsilon,1}(\mathbf{x}) = \begin{cases} u_0^-(\mathbf{x}) + \varepsilon u_1^-(\mathbf{x}) & \mathbf{x} \in \ \Omega^- \\ u_0^+(\mathbf{x}) + \varepsilon \hat{u}_1^+(\mathbf{x}) & \mathbf{x} \in \ \Omega^+ \end{cases}$$

$$-\nabla_x \cdot [A_0^* \nabla_x \tilde{u}_{\varepsilon,1}] - \omega^2 \tilde{u}_{\varepsilon,1} = f, \quad \text{in } \Omega^+ \cup \Omega^-$$

$$[\tilde{u}_{\varepsilon,1}]_\Gamma = \varepsilon C_1^{(1)} < A_0^* \nabla \tilde{u}_{\varepsilon,1} \cdot \mathbf{e}_1 >_\Gamma + \varepsilon C_2^{(1)} < \partial_{x_2} \tilde{u}_{\varepsilon,1} >_\Gamma + \mathcal{O}(\varepsilon^2)$$

$$[A_0^* \nabla \tilde{u}_{\varepsilon,1} \cdot \mathbf{e}_1]_\Gamma = \varepsilon C_1^{(2)} < \partial_{x_2}(A_0^* \nabla \tilde{u}_{\varepsilon,1} \cdot \mathbf{e}_1) >_\Gamma + \varepsilon C_2^{(2)} < \partial_{x_2}^2 \tilde{u}_{\varepsilon,1} >_\Gamma$$

$$+ \varepsilon C_3^{(2)} \omega^2 < \tilde{u}_{\varepsilon,1} >_\Gamma + \mathcal{O}(\varepsilon^2)$$

with $A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$

The higher order transmission problem

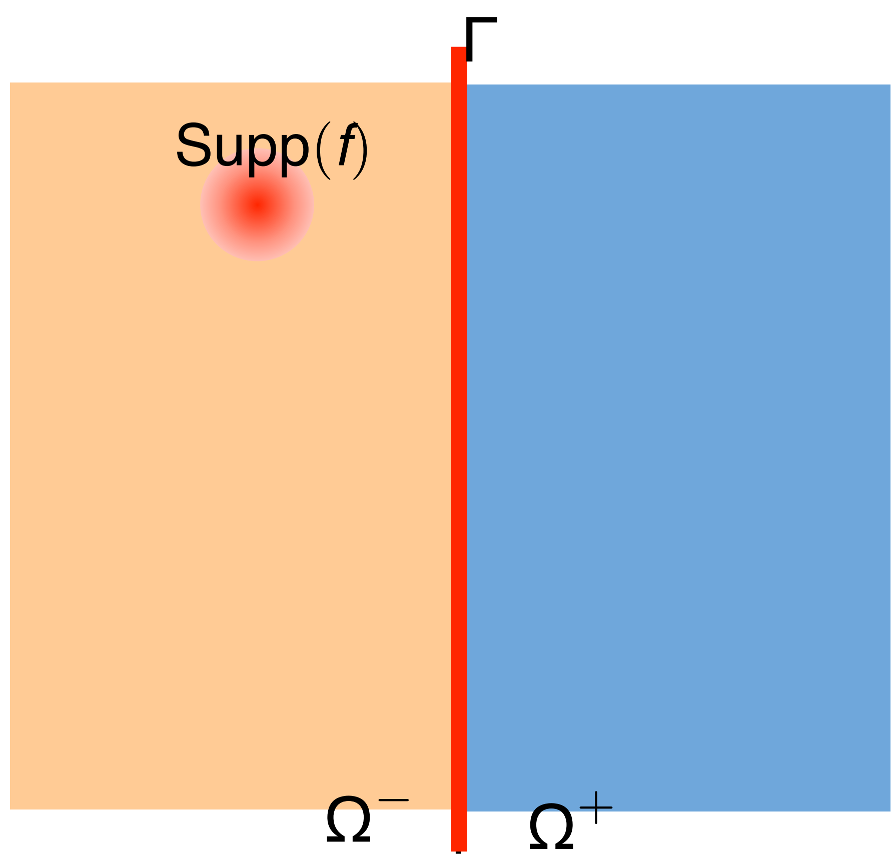
$$-\nabla_x \cdot [A_0^* \nabla_x \tilde{v}_\varepsilon] - \omega^2 \tilde{v}_\varepsilon = f \qquad \text{in } \Omega^- \cup \Omega^+$$

$$[\tilde{v}_\varepsilon]_\Gamma = \varepsilon C_1^{(1)} < A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1 >_\Gamma + \varepsilon C_2^{(1)} < \partial_{x_2} \tilde{v}_\varepsilon >_\Gamma$$

$$[A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1]_\Gamma = \varepsilon C_1^{(2)} < \partial_{x_2} (A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1) >_\Gamma + \varepsilon C_2^{(2)} < \partial_{x_2}^2 \tilde{v}_\varepsilon >_\Gamma + \varepsilon C_3^{(2)} \omega^2 < \tilde{v}_\varepsilon >_\Gamma$$

with $A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$

✓ The volumic equation is as **simple** as the original homogenized problem.



✓ The problem depends simply on ε **without introducing a microscopic scale**.

The higher order transmission problem

$$-\nabla_x \cdot [A_0^* \nabla_x \tilde{v}_\varepsilon] - \omega^2 \tilde{v}_\varepsilon = f \qquad \text{in } \Omega^- \cup \Omega^+$$

$$[\tilde{v}_\varepsilon]_\Gamma = \varepsilon C_1^{(1)} < A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1 >_\Gamma + \varepsilon C_2^{(1)} < \partial_{x_2} \tilde{v}_\varepsilon >_\Gamma$$

$$[A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1]_\Gamma = \varepsilon C_1^{(2)} < \partial_{x_2}(A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1) >_\Gamma + \varepsilon C_2^{(2)} < \partial_{x_2}^2 \tilde{v}_\varepsilon >_\Gamma + \varepsilon C_3^{(2)} \omega^2 < \tilde{v}_\varepsilon >_\Gamma$$

with $A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$

where $[\tilde{v}_\varepsilon]_\Gamma = \tilde{v}_\varepsilon\big|_{\Gamma+} - \tilde{v}_\varepsilon\big|_{\Gamma-} \neq 0$ $\quad [A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1]_\Gamma = A^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1\big|_{\Gamma+} - a_0 \partial_{x_1} \tilde{v}_\varepsilon\big|_{\Gamma-} \neq 0$

$$< \tilde{v}_\varepsilon >_\Gamma = \frac{\tilde{v}_\varepsilon\big|_{\Gamma+} + \tilde{v}_\varepsilon\big|_{\Gamma-}}{2} \qquad < A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1 >_\Gamma = \frac{A^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1\big|_{\Gamma+} + a_0 \partial_{x_1} \tilde{v}_\varepsilon\big|_{\Gamma-}}{2}$$

Differential operators of order 2 on the interface.

The higher order transmission problem

$$-\nabla_x \cdot [A_0^* \, \nabla_x \, \tilde{v}_\varepsilon] - \omega^2 \tilde{v}_\varepsilon = f \qquad \text{in } \Omega^- \cup \Omega^+$$

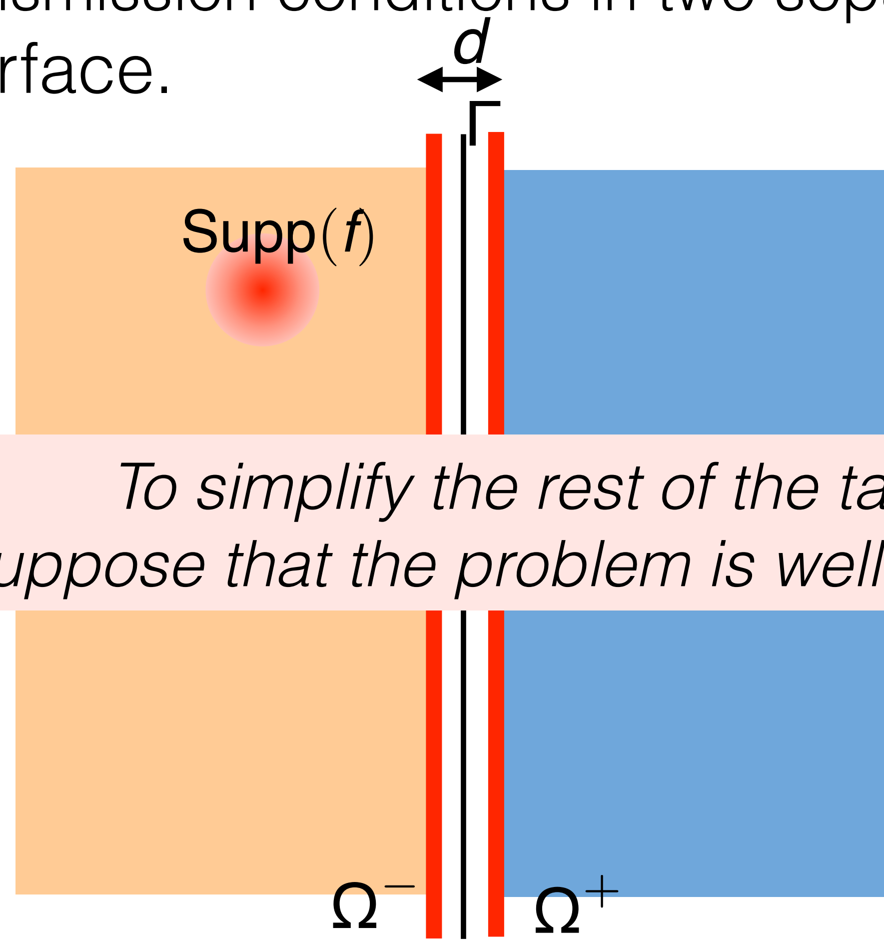
$$[\tilde{v}_\varepsilon]_\Gamma = \varepsilon C_1^{(1)} < A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1 >_\Gamma + \varepsilon C_2^{(1)} < \partial_{x_2} \tilde{v}_\varepsilon >_\Gamma$$

$$[A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1]_\Gamma = \varepsilon C_1^{(2)} < \partial_{x_2} (A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1) >_\Gamma + \varepsilon C_2^{(2)} < \partial_{x_2}^2 \tilde{v}_\varepsilon >_\Gamma + \varepsilon C_3^{(2)} \omega^2 < \tilde{v}_\varepsilon >_\Gamma$$

with $A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$

✓ This problem is not necessarily well posed.

Remedy : write the transmission conditions in two separated boundaries on both sides of the interface.



To simplify the rest of the talk,
I suppose that the problem is well posed…

Delourme-Haddar-Joly

The higher order transmission problem

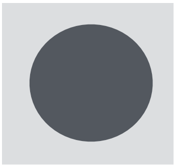
$$-\nabla_x \cdot [A_0^* \nabla_x \tilde{v}_\varepsilon] - \omega^2 \tilde{v}_\varepsilon = f \qquad \text{in } \Omega^- \cup \Omega^+$$

$$[\tilde{v}_\varepsilon]_\Gamma = \varepsilon C_1^{(1)} < A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1 >_\Gamma + \varepsilon C_2^{(1)} < \partial_{x_2} \tilde{v}_\varepsilon >_\Gamma$$

$$[A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1]_\Gamma = \varepsilon C_1^{(2)} < \partial_{x_2} (A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathrm{e}_1) >_\Gamma + \varepsilon C_2^{(2)} < \partial_{x_2}^2 \tilde{v}_\varepsilon >_\Gamma + \varepsilon C_3^{(2)} \omega^2 < \tilde{v}_\varepsilon >_\Gamma$$

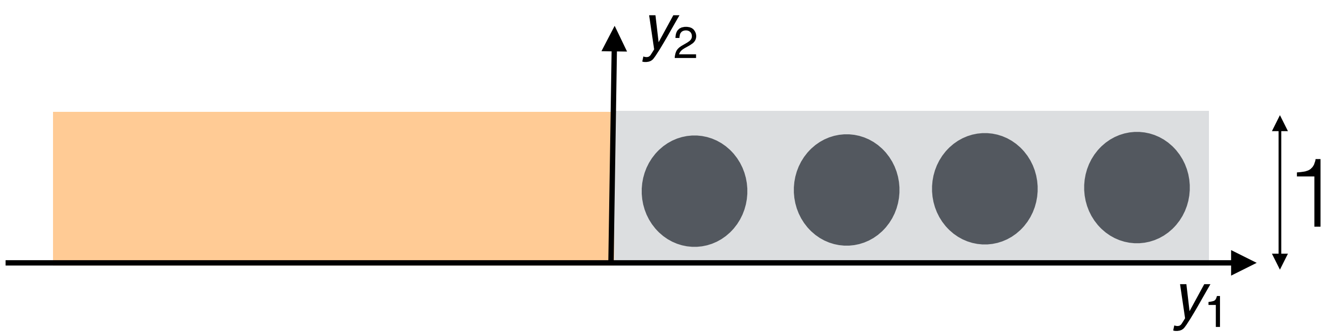
with $A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$

- All the constants appearing in the transmission conditions are determined via the solutions of **cell problems**



$Y = (0,1)^2$

and solutions of **Laplace equation in the band**



The higher order transmission problem

$$-\nabla_x \cdot [A_0^* \nabla_x \tilde{v}_\varepsilon] - \omega^2 \tilde{v}_\varepsilon = f \qquad \text{in } \Omega^- \cup \Omega^+$$

$$[\tilde{v}_\varepsilon]_\Gamma = \varepsilon C_1^{(1)} < A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathbf{e}_1 >_\Gamma + \varepsilon C_2^{(1)} < \partial_{x_2} \tilde{v}_\varepsilon >_\Gamma$$

$$[A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathbf{e}_1]_\Gamma = \varepsilon C_1^{(2)} < \partial_{x_2} (A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathbf{e}_1) >_\Gamma + \varepsilon C_2^{(2)} < \partial_{x_2}^2 \tilde{v}_\varepsilon >_\Gamma + \varepsilon C_3^{(2)} \omega^2 < \tilde{v}_\varepsilon >_\Gamma$$

with $A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$

We expect that

$$v_\varepsilon(\mathbf{x}) = \begin{cases} \tilde{v}_\varepsilon(\mathbf{x}) & \mathbf{x} \in \Omega^- \\ \tilde{v}_\varepsilon(\mathbf{x}) + \varepsilon \nabla \tilde{v}_\varepsilon(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} & \mathbf{x} \in \Omega^+ \end{cases}$$

is a **better approximation** of the original solution u_ε .

Error estimates

For any open set $\mathcal{O} \subset \Omega^- \cup \Omega^+$

$$\|u_\varepsilon - v_\varepsilon\|_{H^1(\mathcal{O})} \leq C\varepsilon$$

$$\|u_\varepsilon - v_\varepsilon\|_{L^2(\mathcal{O})} \leq C\varepsilon^2$$

The higher order transmission problem

$$-\nabla_x \cdot [A_0^* \nabla_x \tilde{v}_\varepsilon] - \omega^2 \tilde{v}_\varepsilon = f \qquad \text{in } \Omega^- \cup \Omega^+$$

$$[\tilde{v}_\varepsilon]_\Gamma = \varepsilon C_1^{(1)} < A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathbf{e}_1 >_\Gamma + \varepsilon C_2^{(1)} < \partial_{x_2} \tilde{v}_\varepsilon >_\Gamma$$

$$[A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathbf{e}_1]_\Gamma = \varepsilon C_1^{(2)} < \partial_{x_2} (A_0^* \nabla \tilde{v}_\varepsilon \cdot \mathbf{e}_1) >_\Gamma + \varepsilon C_2^{(2)} < \partial_{x_2}^2 \tilde{v}_\varepsilon >_\Gamma + \varepsilon C_3^{(2)} \omega^2 < \tilde{v}_\varepsilon >_\Gamma$$

with $A_0^* = \begin{cases} a_0 & \text{in } \Omega^- \\ A^* & \text{in } \Omega^+ \end{cases}$

We expect that

$$v_\varepsilon(\mathbf{x}) = \begin{cases} \tilde{v}_\varepsilon(\mathbf{x}) & \mathbf{x} \in \Omega^- \\ \tilde{v}_\varepsilon(\mathbf{x}) + \varepsilon \nabla \tilde{v}_\varepsilon(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} + \varepsilon^2 \nabla_x \cdot \begin{bmatrix} \theta_{11}(\mathbf{y}) & \theta_{12}(\mathbf{y}) \\ \theta_{21}(\mathbf{y}) & \theta_{22}(\mathbf{y}) \end{bmatrix} \nabla_x \tilde{v}_\varepsilon(\mathbf{x}) & \mathbf{x} \in \Omega^+ \end{cases}$$

is a **better approximation** of the original solution u_ε .

Error estimates

For any open set $\mathcal{O} \subset \Omega^- \cup \Omega^+$

$$\| u_\varepsilon - \tilde{v}_\varepsilon \|_{H^1(\mathcal{O})} \leq C \varepsilon^{3/2}$$

If we perform the asymptotic expansion at order 3, we could show

$$\| u_\varepsilon - \tilde{v}_\varepsilon \|_{H^1(\mathcal{O})} \leq C \varepsilon^2$$

The steps of our approach

1. **The asymptotic expansions of the solution**

   The formal steps of the matched asymptotic expansion

   Existence and uniqueness results

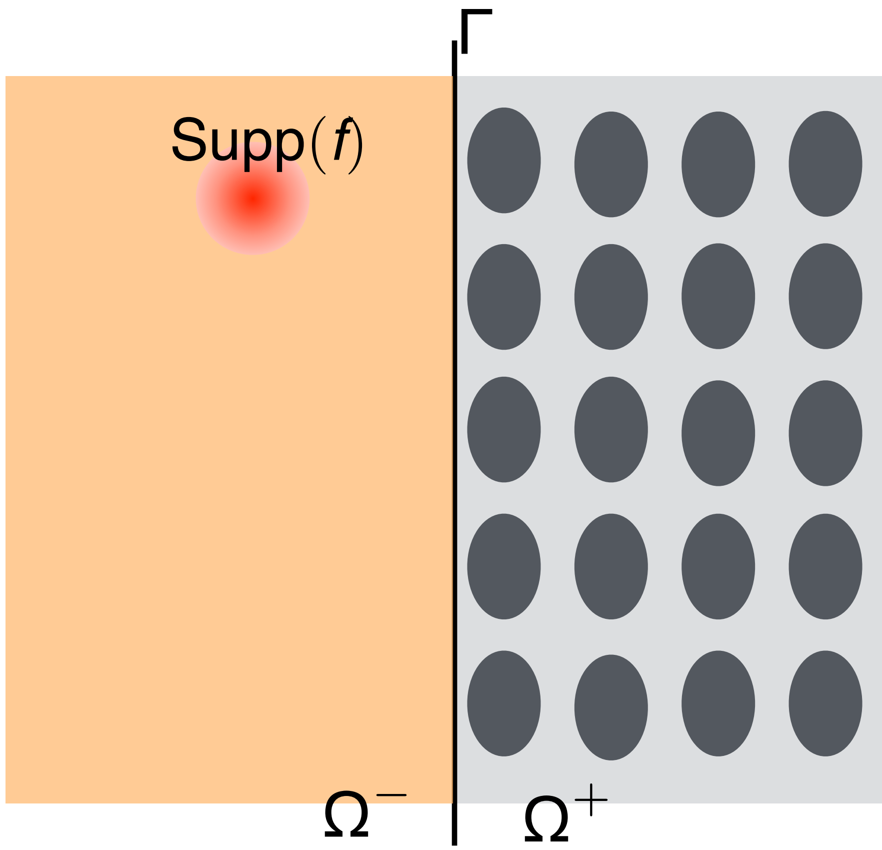
   Error estimates

2. **Construction of the approximate conditions for the asymptotic expansion**

3. **Stability and error analysis for the approximate problem**

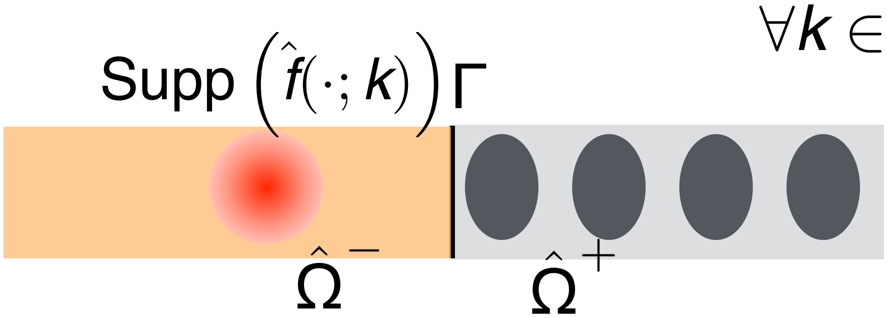
4. **Numerical implementation and validation**

Numerical method for the exact solution



$$-\nabla \cdot \left[a(\frac{\mathbf{x}}{\varepsilon}) \nabla u_\varepsilon(\mathbf{x}) \right] - \omega^2 u_\varepsilon(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

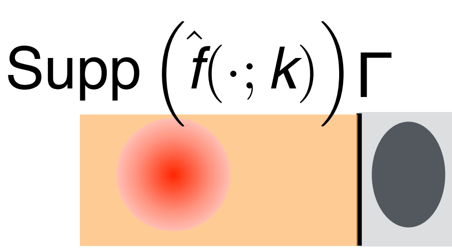
Floquet Bloch Transform — *Kuchment 1993*



$$\forall k \in (-\pi/\varepsilon, \pi/\varepsilon) \qquad \mathbf{x} \in \mathbb{R} \times (0, \varepsilon)$$

$$\left| \begin{array}{l} -(\nabla - \imath k) \cdot \left[a(\frac{\mathbf{x}}{\varepsilon}) (\nabla - \imath k)\, \hat{u}_\varepsilon(\mathbf{x}, k) \right] - \omega^2 \hat{u}_\varepsilon(\mathbf{x}, k) = \hat{f}(\mathbf{x}, k), \\ \hat{u}_\varepsilon(\cdot; k) \text{ per. in the } x_2 - \text{direction} \end{array} \right.$$

Construction of DtN operators — *Fliss-Joly-Li, Fliss-Joly*

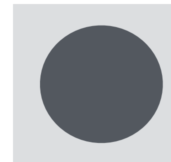


$$\forall k \in (-\pi/\varepsilon, \pi/\varepsilon) \qquad \mathbf{x} \in (-L, L) \times (0, \varepsilon)$$

$$\left| \begin{array}{l} -(\nabla - \imath k) \cdot \left[a(\frac{\mathbf{x}}{\varepsilon}) (\nabla - \imath k)\, \hat{u}_\varepsilon(\mathbf{x}, k) \right] - \omega^2 \hat{u}_\varepsilon(\mathbf{x}, k) = \hat{f}(\mathbf{x}, k), \\ \hat{u}_\varepsilon(\cdot; k) \text{ per. in the } x_2 - \text{direction} \\ \pm \frac{\partial}{\partial x_1} \hat{u}_\varepsilon(\cdot; k) + \Lambda^\pm(k) \hat{u}_\varepsilon(\cdot; k) = 0 \quad \text{on } x_1 = \pm L \end{array} \right.$$

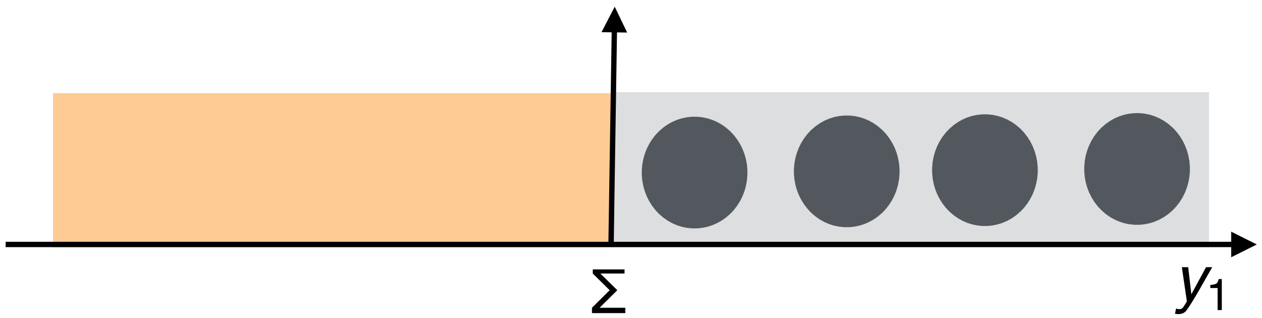
Numerical method for the approximate solution

✓ Resolution of the cell problems, computation of w_i and θ_{ij}, for $i, j \in \{1, 2\}$



$Y = (0, 1)^2$

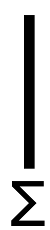
✓ Resolution of the band problems, computation of profile functions



$$-\nabla a(\mathbf{y})\nabla \mathcal{U} = g$$
$$< a^{\pm} >= a \quad \text{and} \quad < \beta^{\pm} >= \beta$$

Construction of DtN operators

Fliss-Joly-Li, Fliss-Joly

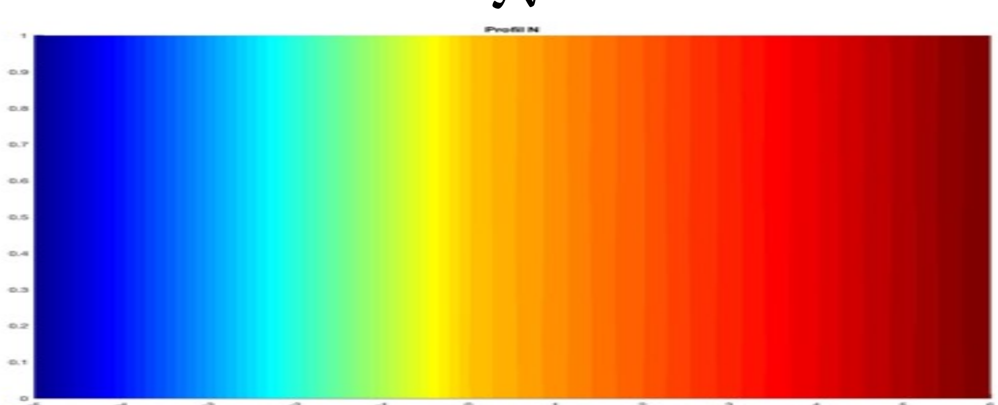


$\mathcal{U}$ is the solution up to *a of*

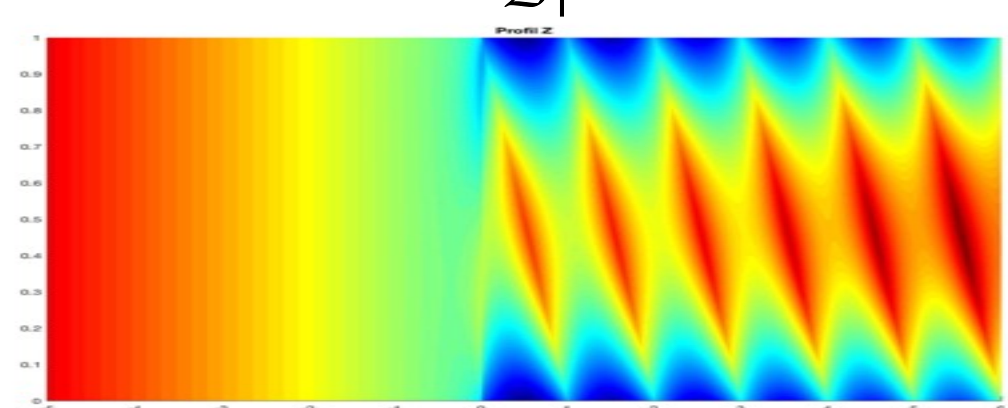
$$(\Lambda^{-} - \Lambda^{+})\mathcal{U} = \mathcal{F}(g, \beta) \quad \text{on } \Sigma$$

$\mathcal{U}$ can be reconstructed in the whole band

$\mathcal{N}$

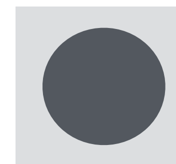


$\mathcal{Z}_1$



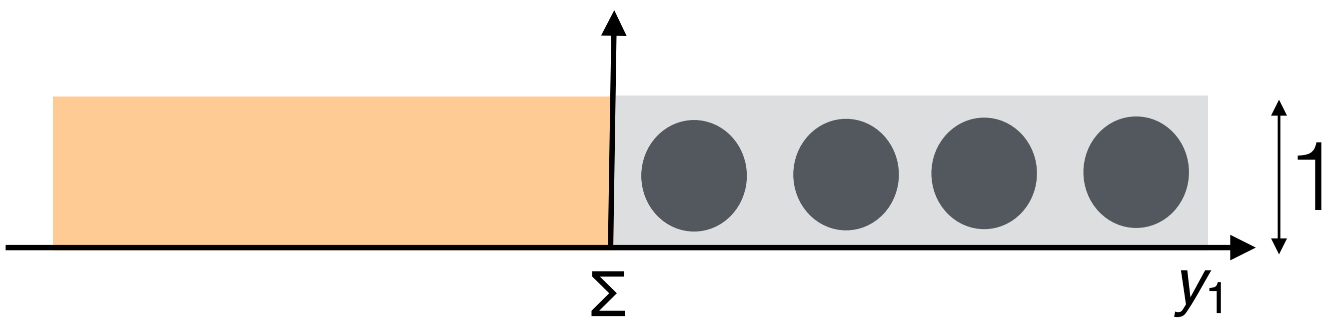
Numerical method for the approximate solution

✓ Resolution of the cell problems, computation of w_i and θ_{ij}, for $i, j \in \{1, 2\}$



$Y = (0, 1)^2$

✓ Resolution of the band problems, computation of profile functions



$$-\nabla a(\mathbf{y}) \nabla \mathcal{U} = g$$

$$< a^{\pm} >= a \quad \text{and} \quad < \beta^{\pm} >= \beta$$

Construction of DtN operators

Fliss-Joly-Li, Fliss-Joly

$\mathcal{U}$ is the solution up to *a of*

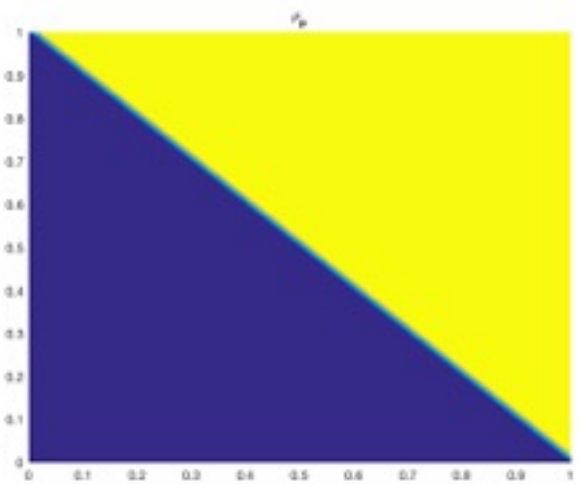
$$(\Lambda^- - \Lambda^+)\mathcal{U} = \mathcal{F}(g, \beta) \quad \text{on } \Sigma$$

$\mathcal{U}$ can be reconstructed in the whole band

✓ Computation of all the constants

✓ Solve the approximate problem

Numerical results



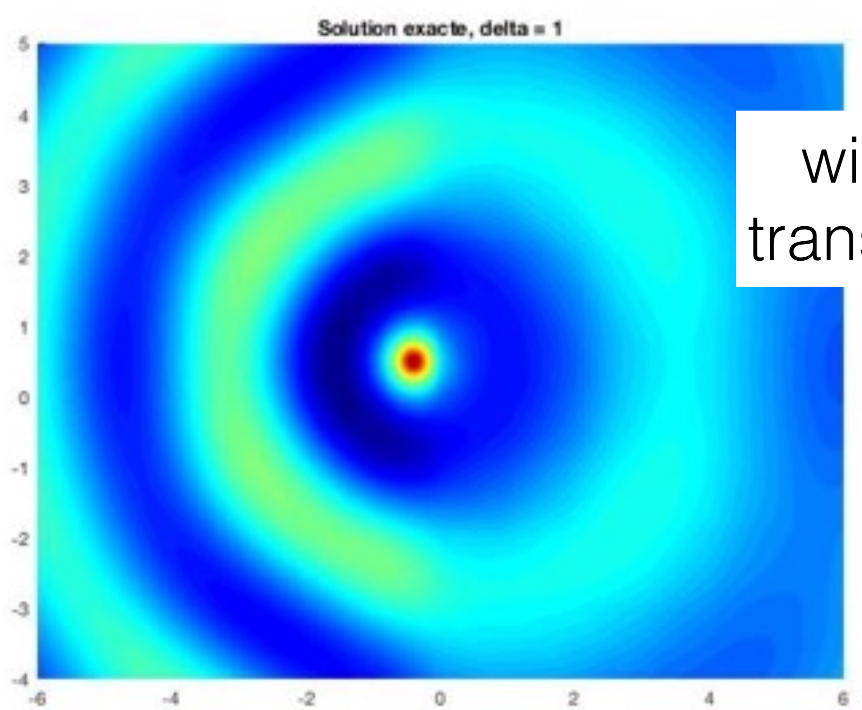
Periodic coefficient in one cell

$a_0 = 1$

$\omega = 2 + 0.01\imath$

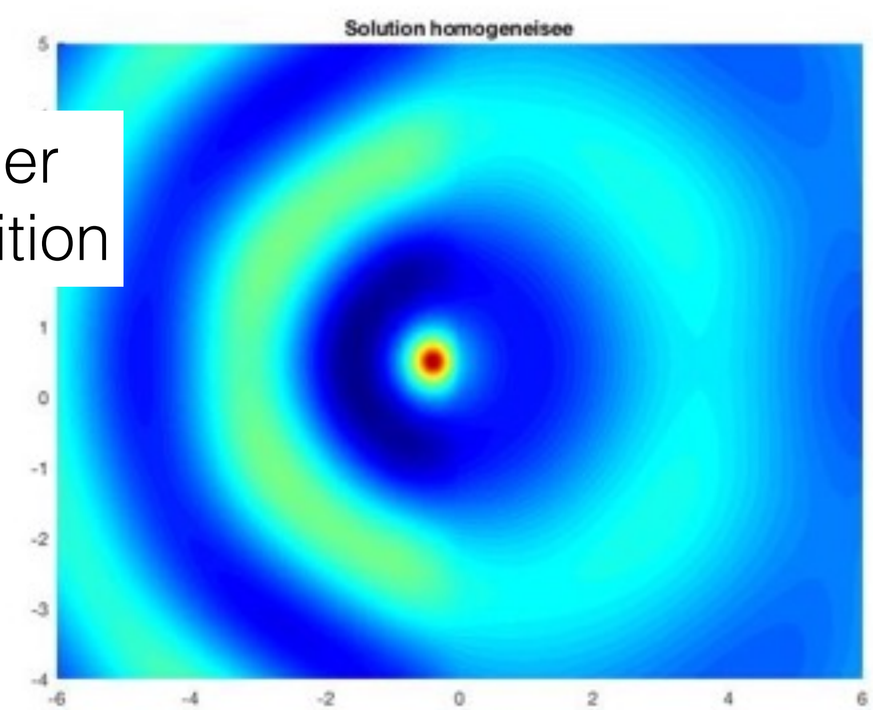
The source term in a gaussian localised near the interface.

For $\varepsilon = 1$



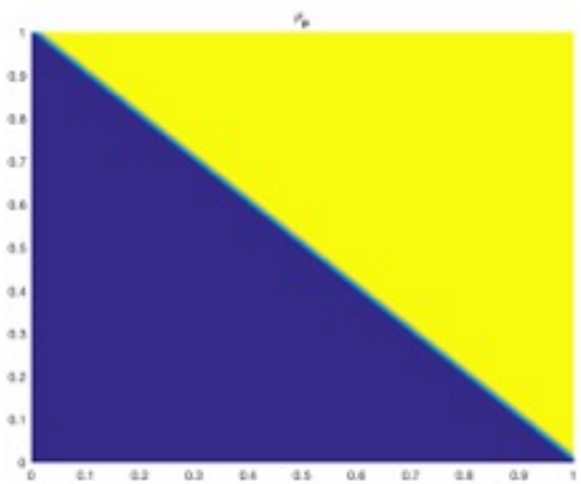
u_ε

with our high order transmission condition



$\tilde{v}_\varepsilon$

Numerical results



Periodic coefficient in one cell

$a_0 = 1$

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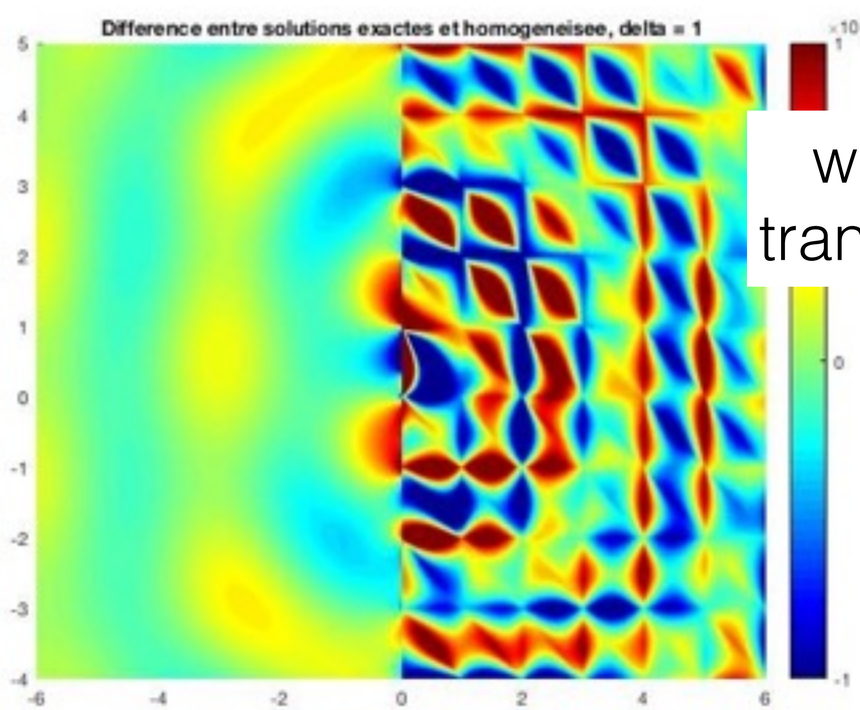
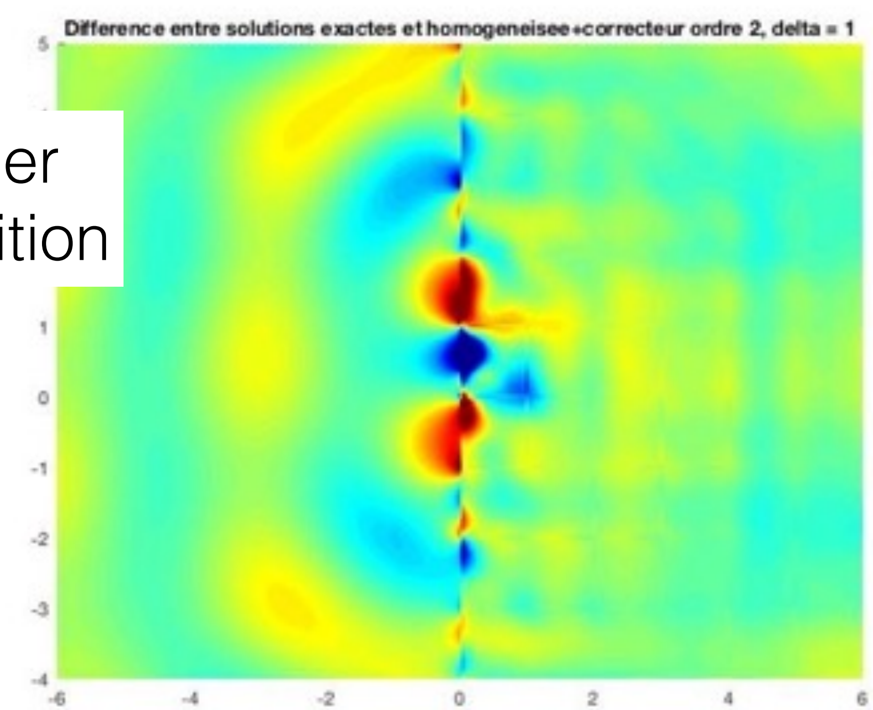
The source term in a gaussian localised near the interface.

For $\varepsilon = 1$



$u_\varepsilon - \tilde{v}_\varepsilon$

with our high order transmission condition

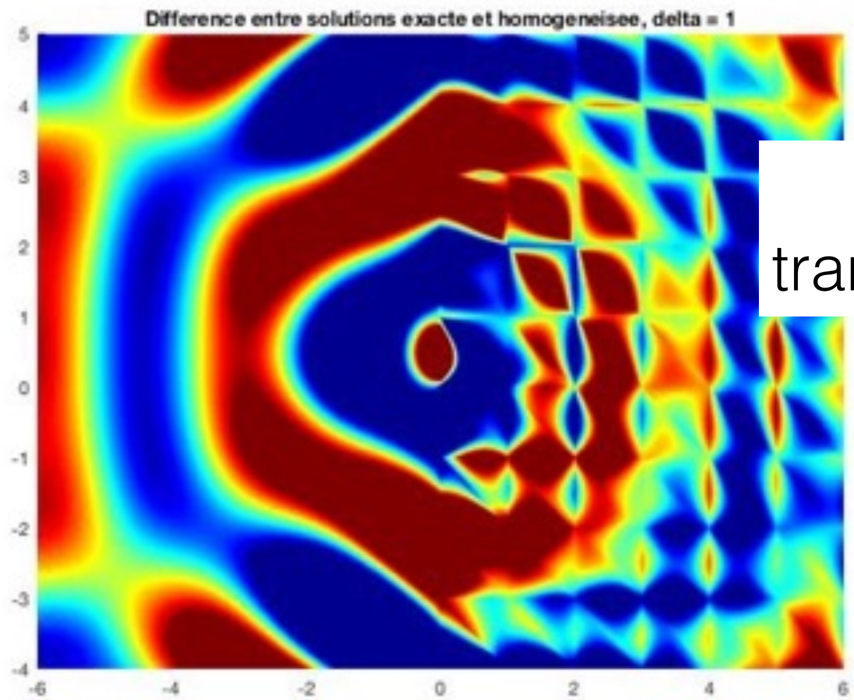


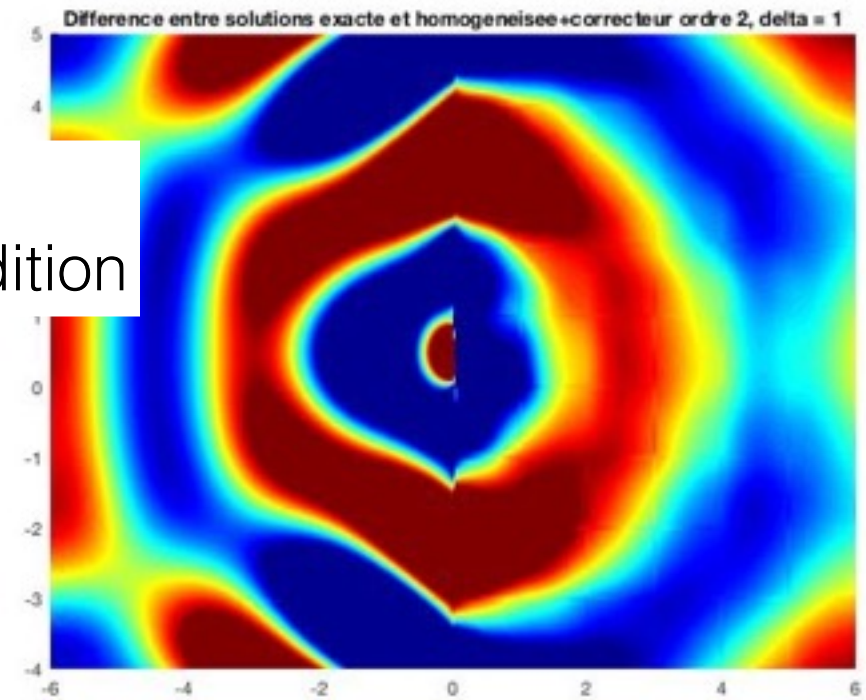
$u_\varepsilon - v_\varepsilon$

Numerical results

$u_\varepsilon - u_0$

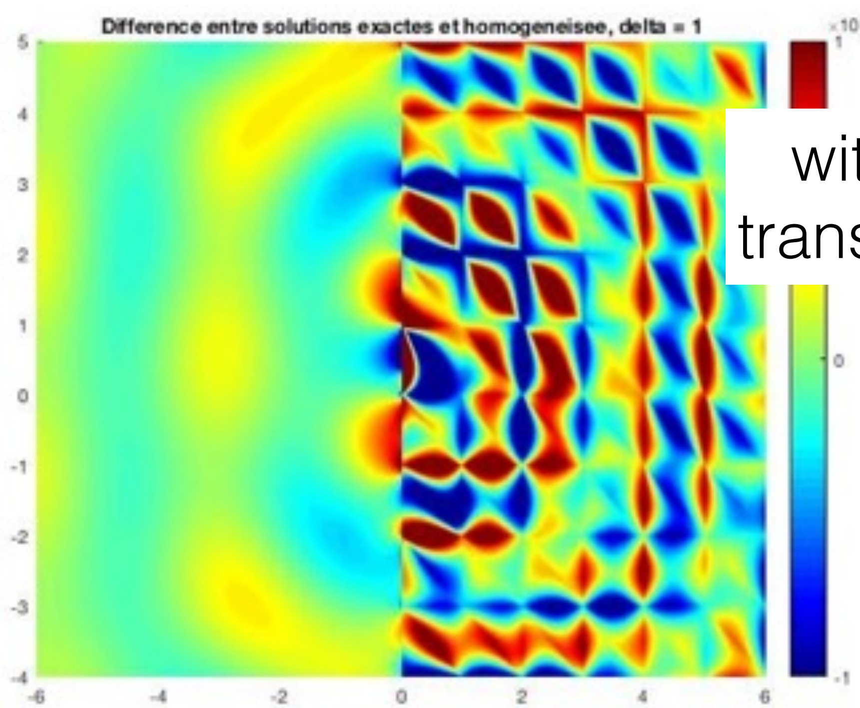
$u_\varepsilon - \left(u_0(\mathbf{x}) + \varepsilon \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$



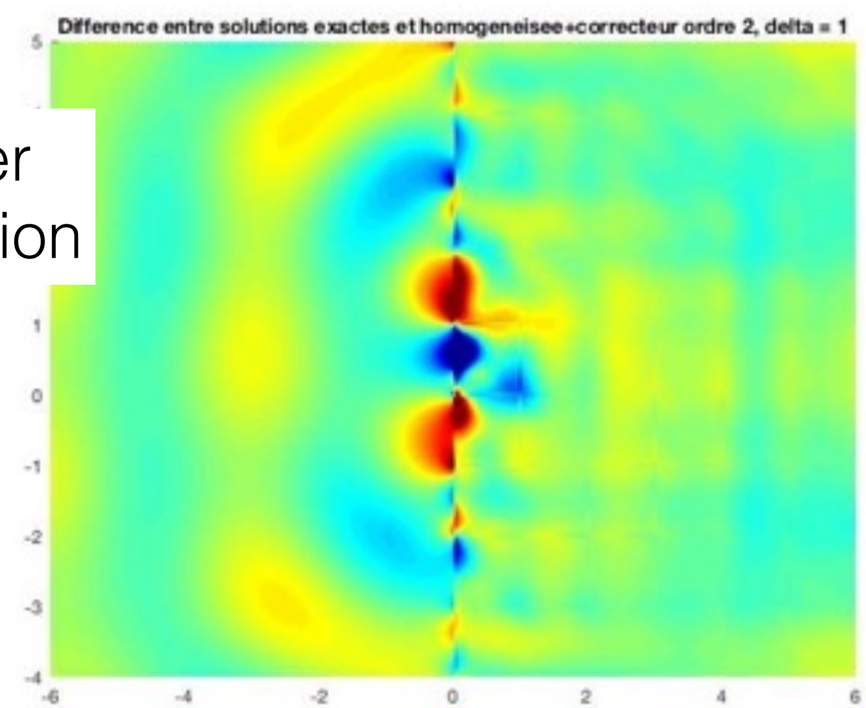


with classical
transmission condition

For $\varepsilon = 1$



with our high order
transmission condition

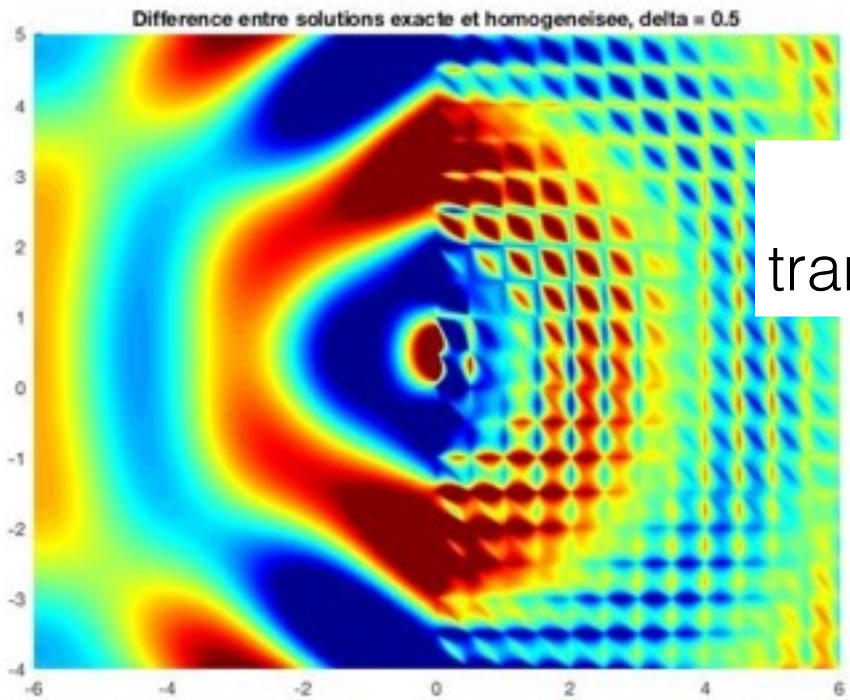


$u_\varepsilon - \tilde{v}_\varepsilon$

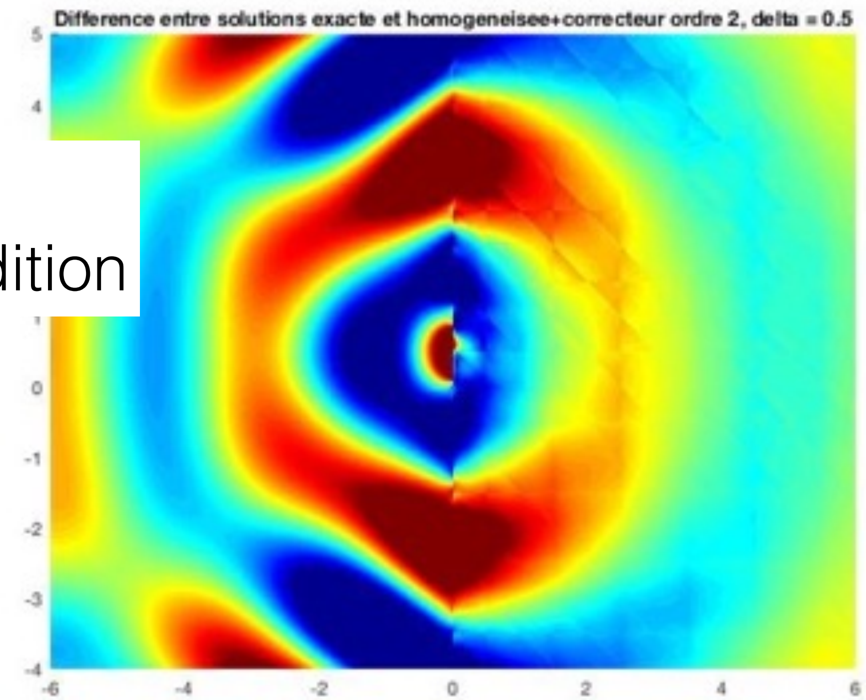
$u_\varepsilon - v_\varepsilon$

Numerical results

$u_\varepsilon - u_0$

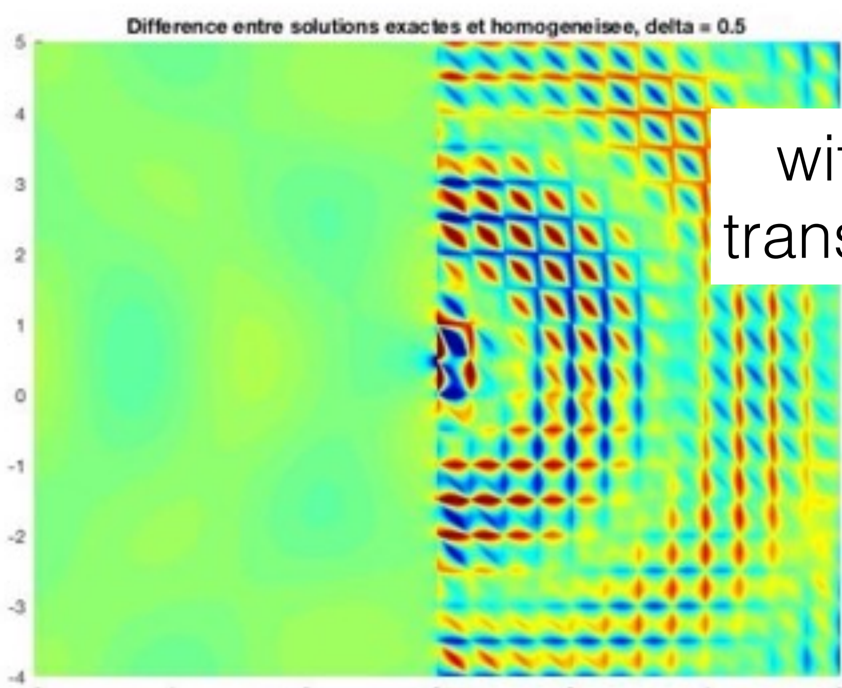


$u_\varepsilon - \left(u_0(\mathbf{x}) + \varepsilon \, \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$

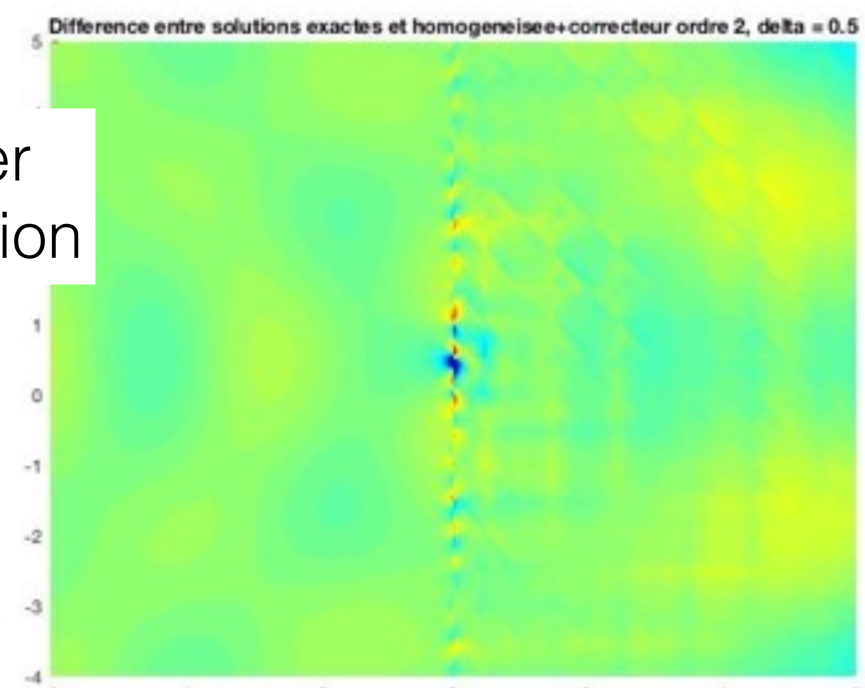


with classical transmission condition

For $\varepsilon = 0.5$



with our high order transmission condition

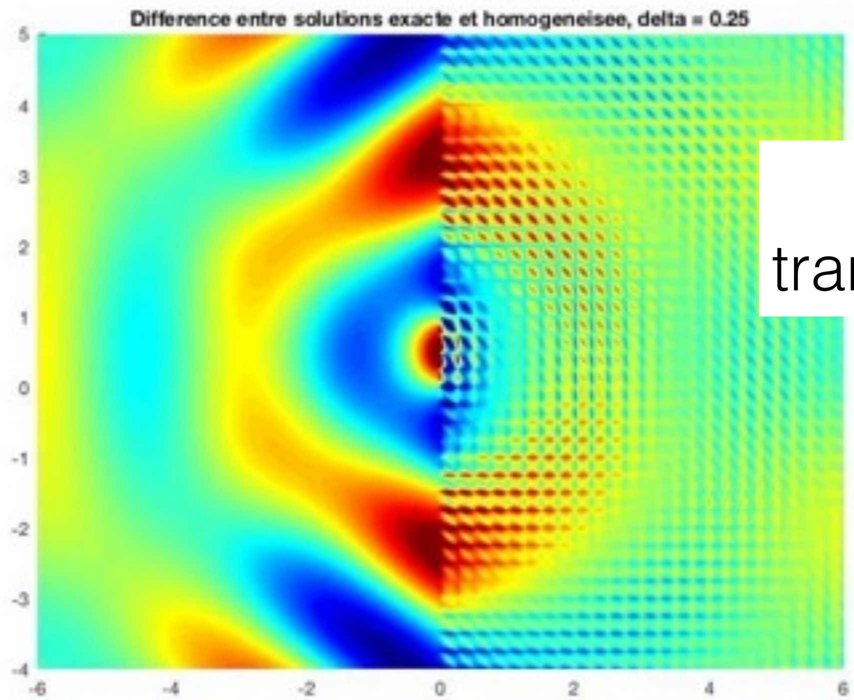


$u_\varepsilon - \tilde{v}_\varepsilon$

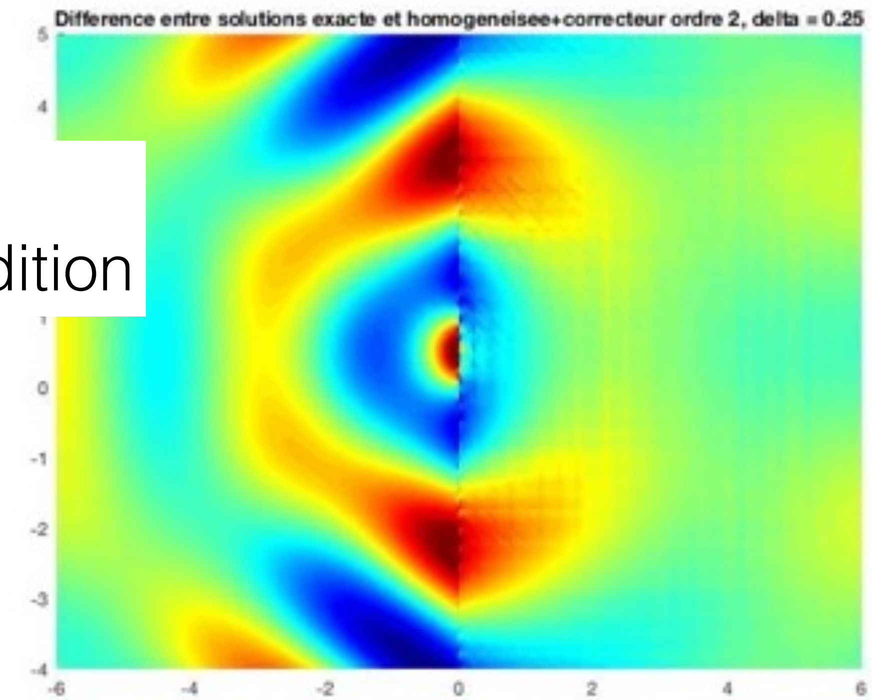
$u_\varepsilon - v_\varepsilon$

Numerical results

$u_\varepsilon - u_0$

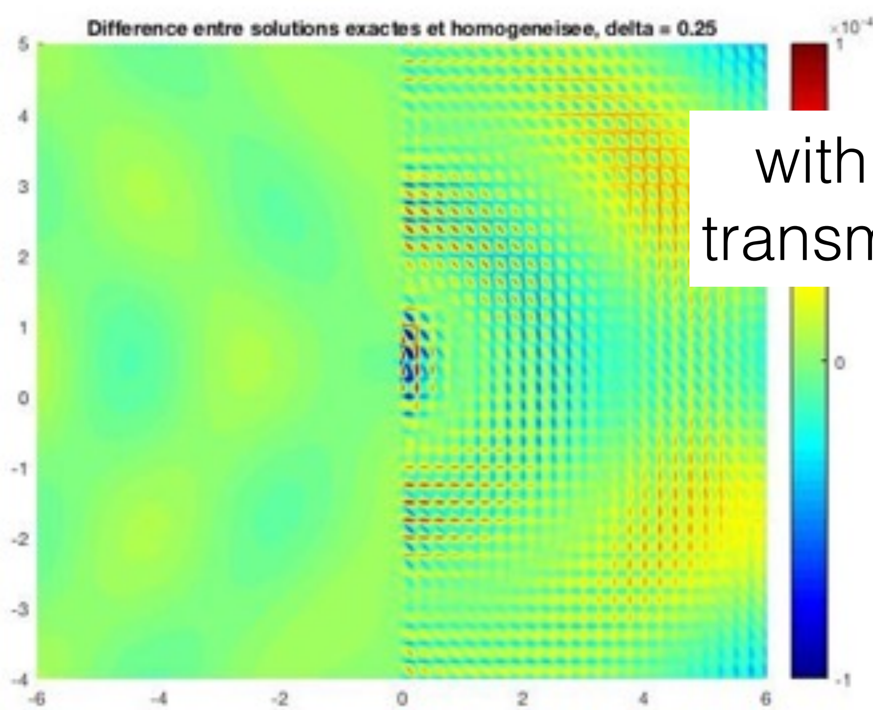


$$u_\varepsilon - \left(u_0(\mathbf{x}) + \varepsilon \, \nabla_x u_0(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} \chi_{\Omega^+} \right)$$



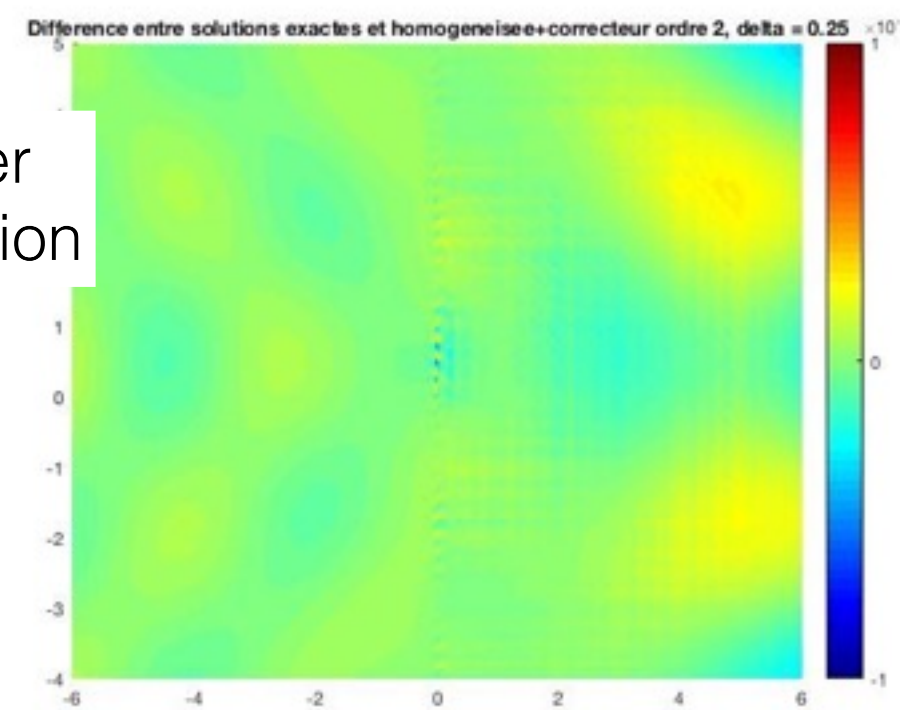
with classical transmission condition

For $\varepsilon = 0.25$



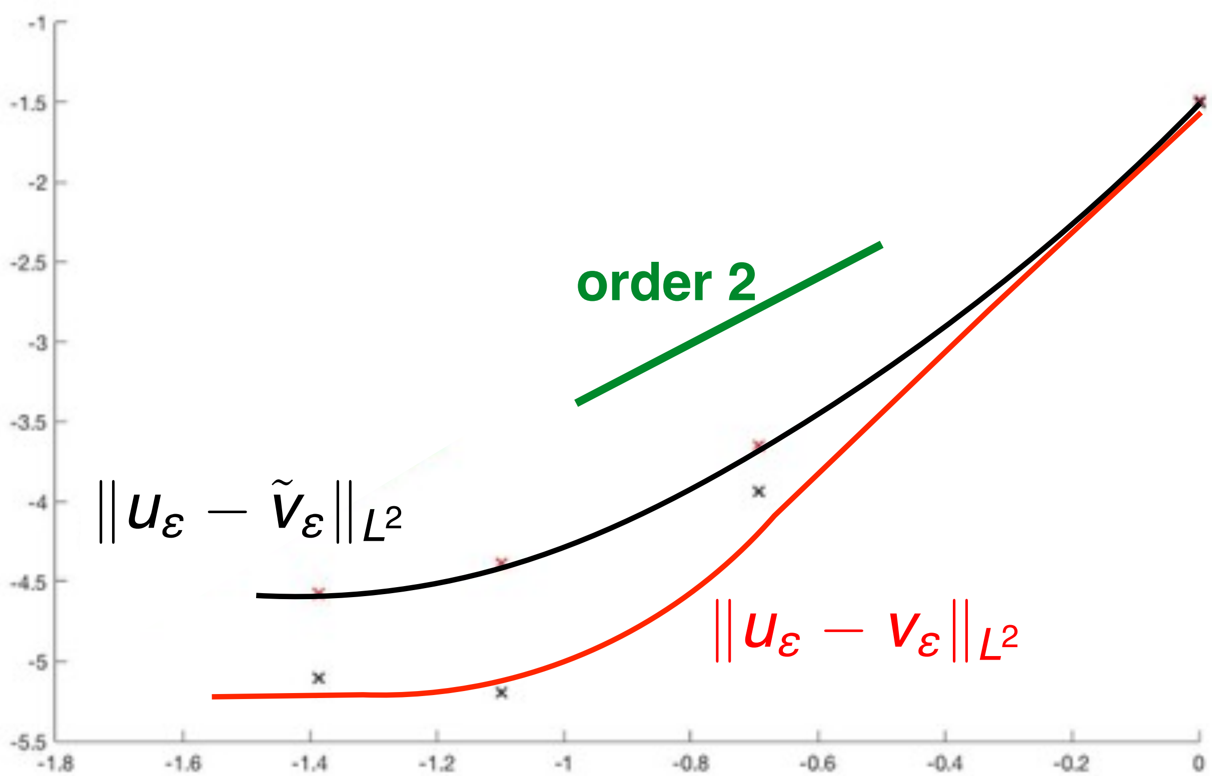
$u_\varepsilon - \tilde{v}_\varepsilon$

with our high order transmission condition

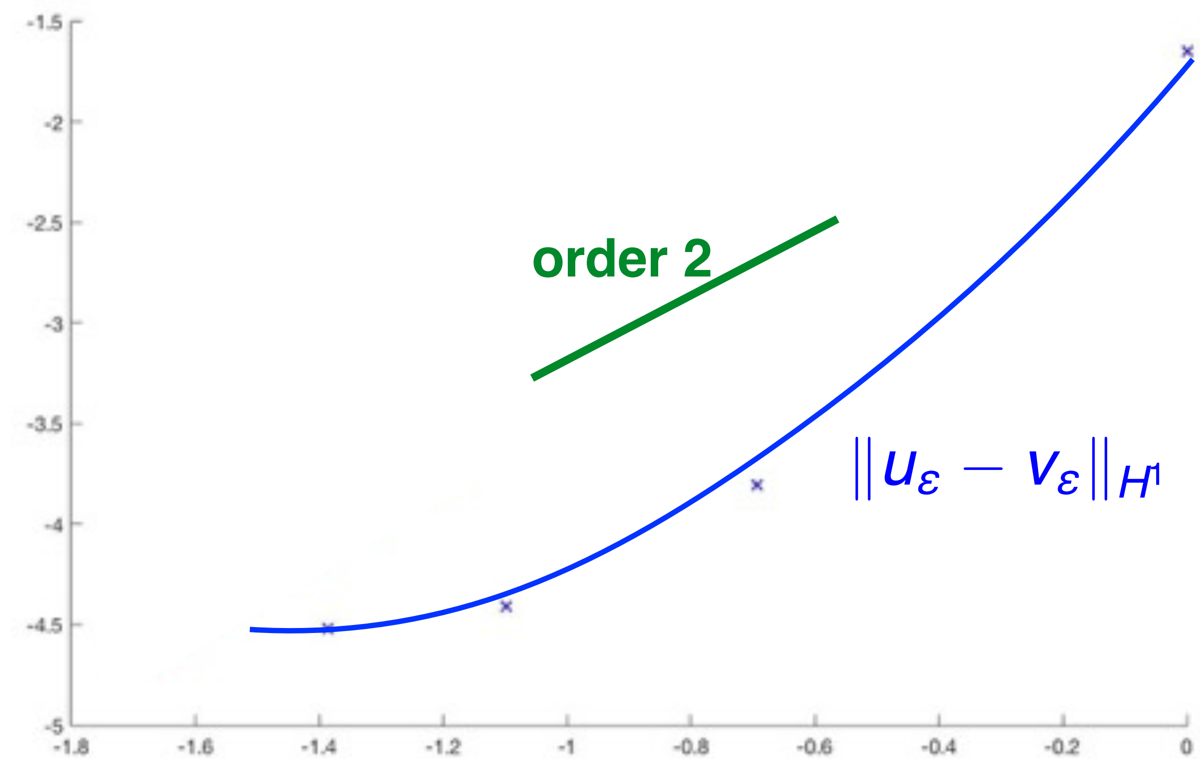


$u_\varepsilon - v_\varepsilon$

Numerical results



log-log



log-log

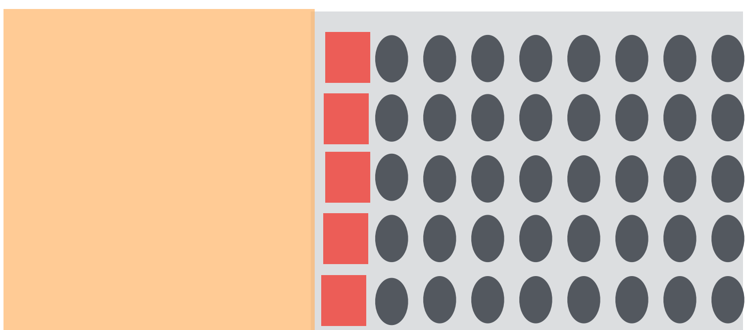
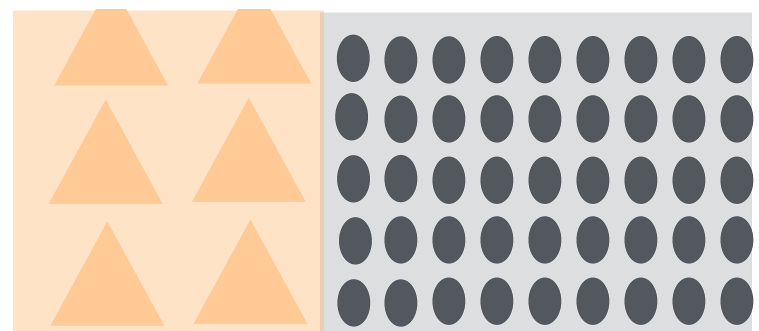
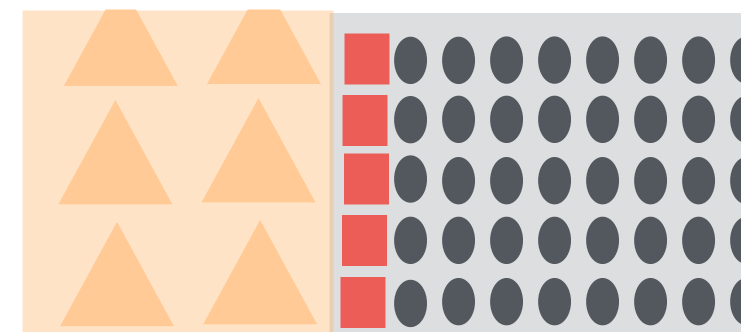
$$v_\varepsilon(\mathbf{x}) = \begin{cases} \tilde{v}_\varepsilon(\mathbf{x}) & \mathbf{x} \in \Omega^- \\ \tilde{v}_\varepsilon(\mathbf{x}) + \varepsilon \nabla \tilde{v}_\varepsilon(\mathbf{x}) \cdot \begin{bmatrix} w_1(\mathbf{x}/\varepsilon) \\ w_2(\mathbf{x}/\varepsilon) \end{bmatrix} + \varepsilon^2 \nabla_x \cdot \begin{bmatrix} \theta_{11}(\mathbf{y}) & \theta_{12}(\mathbf{y}) \\ \theta_{21}(\mathbf{y}) & \theta_{22}(\mathbf{y}) \end{bmatrix} \nabla_x \tilde{v}_\varepsilon(\mathbf{x}) & \mathbf{x} \in \Omega^+ \end{cases}$$

Ongoing works

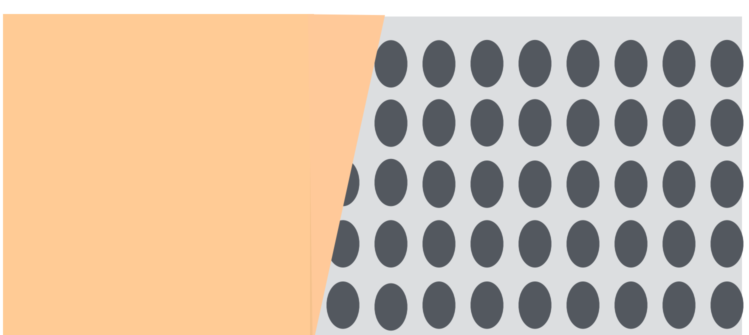
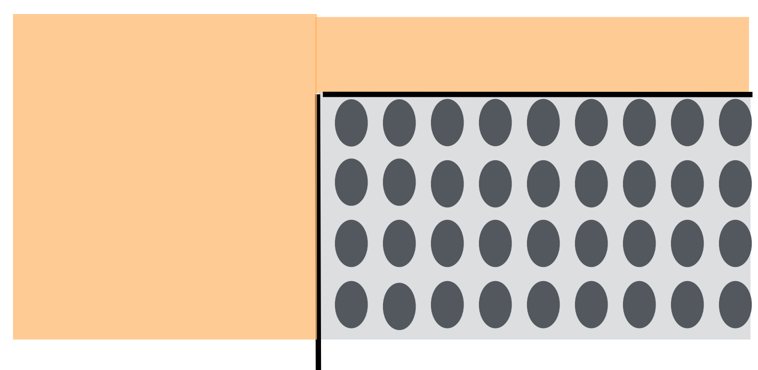
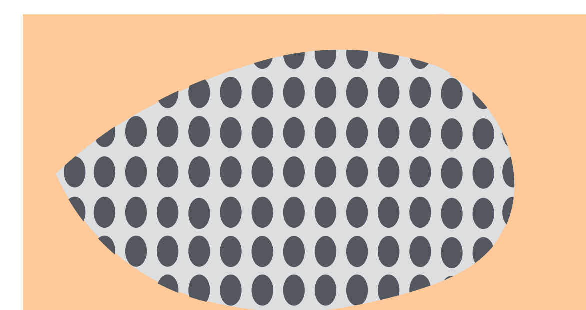
- ✓ Higher order approximate problems

- ✓ Extension of the method to the following situations ?

Direct extensions

Extensions raising challenging questions

Gérard-Varet - Masmoudi 2006-2007

- ✓ Extension of the case without dissipation (the transmission problem is not well-posed. What radiation condition in infinite periodic media?)

- ✓ Extension of the method to 3D and Maxwell equations

Ongoing works

- Why have we spent so much effort for so little?

Some high contrast materials behave as effective negative materials at some ranges of frequencies.

Joly’s, Cherednichenko's Talks, Open Question Session

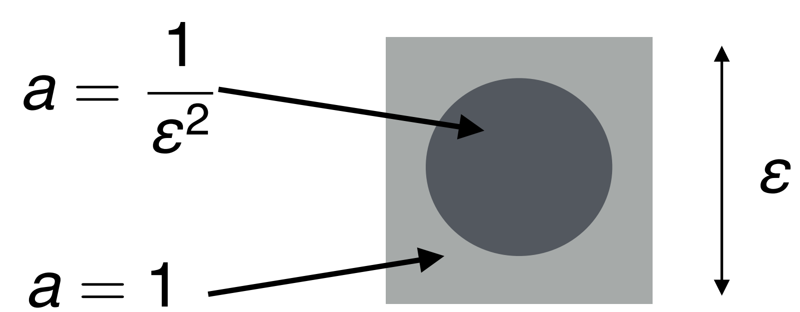
For frequencies for which the contrasts of permittivity or/and of permeability is/are equal to -1, the transmission problem at order 0 can be ***ill-posed.***

Nguyen's Talk

We think that higher order transmission condition will settle the problem.

Extension to **high contrast homogenization**

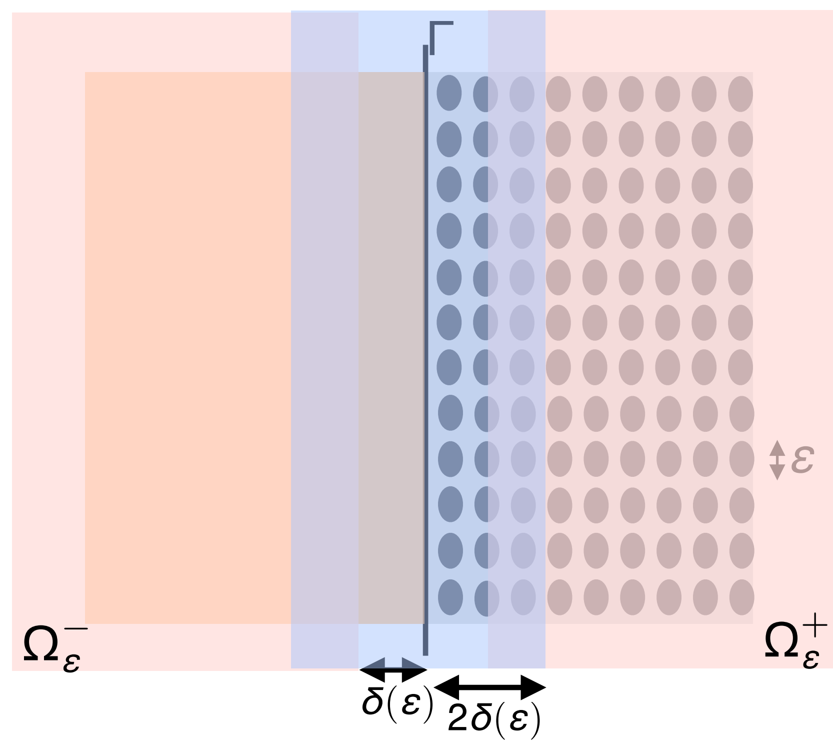
A simple case in 2D



The higher order transmission problem

Ideas for the proof of the error estimates :

1. Error estimates between the exact solution and the matched asymptotic expansion



$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$$

$$\lim_{\varepsilon \to 0} \frac{\delta(\varepsilon)}{\varepsilon} = +\infty$$

We use the stability of the original problem.

2. Error estimates between the matched asymptotic expansion and the approximate solution

We use the stability of the higher order transmission problem.