Recent progress in the Calderón problem

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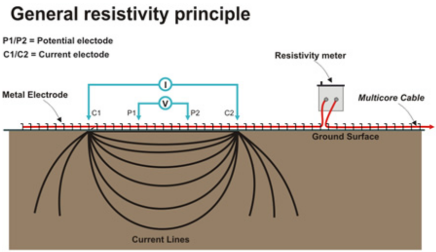
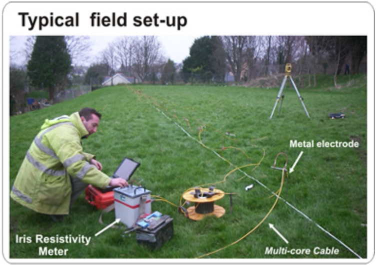
Outline

1. Calderón problem
2. Low regularity
3. Partial data and anisotropy

Calderón problem

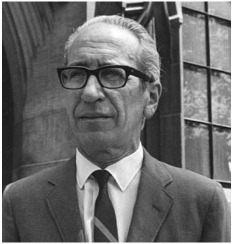
Electrical Resistivity Imaging in geophysics (1920's) [image: TerraDat]





A.P. Calderón (1980):

- mathematical formulation
- solution of the linearized problem
- exponential solutions



Calderón problem

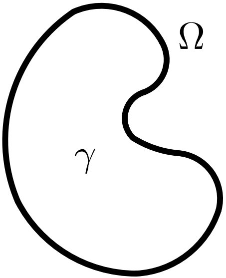
Conductivity equation

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain, $\gamma \in L^\infty(\Omega)$ positive scalar function (electrical conductivity).

Boundary measurements given by *Dirichlet-to-Neumann (DN) map*[1]

$$\Lambda_\gamma : f \mapsto \gamma \nabla u \cdot \nu|_{\partial\Omega}.$$



Inverse problem: given Λ_γ, determine γ.

[1] as a map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$

Calderón problem

Model case of inverse boundary problems for elliptic equations (Schrödinger, *Maxwell*, elasticity). Arises as the zero frequency limit of an inverse problem for Maxwell equations.

Related to:

- optical and hybrid imaging methods
- inverse scattering
- geometric problems (boundary rigidity)
- periodic Schrödinger operators
- invisibility

Calderón problem

Uniqueness results:

$n \geq 3$	linearized problem	Calderón 1980
	$\gamma \in C^2$	Sylvester-Uhlmann 1987
	$\gamma \in W^{1,\infty}$	Haberman-Tataru 2013, Caro-Rogers 2016
	$\gamma \in W^{1,n}$	Haberman 2016, n=3,4
$n = 2$	$\gamma \in C^2$	Nachman 1996
	$\gamma \in L^\infty$	Astala-Päivärinta 2006

Calderón problem

Techniques:

$n \geq 3$	linearized problem		exponential solutions
		$\gamma \in C^2$	L^2 Carleman estimates
		$\gamma \in W^{1,\infty}$	Bourgain space estimates + averaging
		$\gamma \in W^{1,n}$	L^p harmonic analysis, n=3,4
$n = 2$		$\gamma \in C^2$	$\overline{\partial}$-scattering theory
		$\gamma \in L^\infty$	quasiconformal methods

Similarities with the *unique continuation principle* (u vanishes in a ball $\implies$ $u \equiv 0$)!

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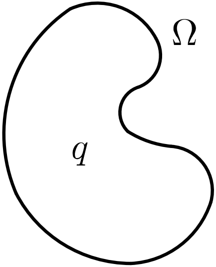
Schrödinger equation

Substitute $u = \gamma^{-1/2} v$, conductivity equation $\mathrm{div}(\gamma \nabla u) = 0$ reduces to Schrödinger equation $(-\Delta + q)v = 0$ where

$$q = \frac{\Delta(\gamma^{1/2})}{\gamma^{1/2}}.$$

If $q \in L^\infty(\Omega)$, consider Dirichlet problem

$$\begin{cases} (-\Delta + q)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$



The DN map is $\Lambda_q : f \mapsto \partial_\nu u|_{\partial\Omega}$.

Inverse problem: given Λ_q, determine q.

Integration by parts

Recall that $\Lambda_q : u|_{\partial\Omega} \mapsto \partial_\nu u|_{\partial\Omega}$ when $(-\Delta + q)u = 0$.

Lemma (Integration by parts)

$$\Lambda_{q_1} = \Lambda_{q_2} \iff \int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0$$

whenever $u_j \in H^1(\Omega)$ *solve* $(-\Delta + q_j)u_j = 0$ *in* Ω.

Need to show that products $\{u_1 u_2\}$ are complete!

Complex geometrical optics

Exponential solutions for $\rho \in \mathbb{C}^n$ [Calderón 1980]

$$\Delta u = 0, \quad u = e^{\rho \cdot x}, \quad \rho \cdot \rho = 0.$$

If $q \in L^\infty(\Omega)$, CGO solutions [Sylvester-Uhlmann 1987]

$$(-\Delta + q)u = 0, \quad u = e^{\rho \cdot x}(1 + r),$$

where $\|r\|_{L^2} \to 0$ as $|\rho| \to \infty$.

Need solvability for the conjugated Laplacian

$$\Delta_\rho := e^{-\rho \cdot x} \circ \Delta \circ e^{\rho \cdot x} = \Delta + 2\rho \cdot \nabla.$$

Estimates for $\Delta_\rho = \Delta + 2\rho \cdot \nabla$

Theorem. If $f \in L^2(\Omega)$, there is $u = \Delta_\rho^{-1} f$ with

$$\Delta_\rho u = f, \qquad \|\Delta_\rho^{-1} f\|_{L^2(\Omega)} \leq \frac{C}{|\rho|} \|f\|_{L^2(\Omega)}.$$

Proof. Taking Fourier transforms, we have

$$\Delta_\rho u = f \iff \underbrace{(-|\xi|^2 + 2i\rho \cdot \xi)}_{:= p_\rho(\xi)} \hat{u} = \hat{f} \iff u = \mathscr{F}^{-1}\left\{ \frac{1}{p_\rho} \hat{f} \right\}.$$

Characteristic set is a *codim 2 sphere*: for $\rho = \tau(e_n - ie_1)$

$$p_\rho^{-1}(0) = \{\xi \in \mathbb{R}^n \,;\, |\xi - \tau e_1| = \tau, \ \xi_n = 0\}.$$

Microlocally $\frac{1}{p_\rho(\xi)} \sim \frac{1}{|\rho|(\eta_1 + i\eta_2)}$, use L^2 estimates for $\bar{\partial}$. □

Sylvester-Uhlmann (1987), $n \geq 3$

Theorem. If $q_1, q_2 \in L^\infty(\Omega)$ satisfy $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$.

Proof. Show that $\{u_1 u_2\}$ is complete where

$$(-\Delta + q_j)u_j = 0, \qquad u_j = e^{\rho_j \cdot x}(1 + r_j).$$

Need $(\Delta_{\rho_j} - q_j)r_j = q_j$. Trying $r_j = \Delta_{\rho_j}^{-1} f_j$ leads to

$$(\mathrm{Id} - \underbrace{q_j \Delta_{\rho_j}^{-1}}_{\|\cdot\|_{L^2 \to L^2} \leq \|q_j\|_{L^\infty} \frac{C}{|\rho_j|}})f_j = \underbrace{q_j}_{\|\cdot\|_{L^2} \leq C}.$$

Solve by Neumann series for $|\rho_j|$ large. If $n \geq 3$, then for any $\xi \in \mathbb{R}^n$ find $\rho_j \in \mathbb{C}^n$, $\rho_j \cdot \rho_j = 0$, to recover *Fourier transform*

$$u_1 u_2 \approx e^{(\rho_1 + \rho_2)\cdot x} = e^{ix\cdot\xi} \quad \text{as } |\rho_j| \to \infty. \qquad \square$$

Low regularity

If γ is $W^{1,\infty}$, then $q \in W^{-1,\infty}$. Need [Haberman-Tataru 2013]

- *Bourgain type spaces* $\dot{X}^s_\rho$ adapted to the equation:

$$\|u\|_{\dot{X}^s_\rho} = \||\Delta_\rho|^s u\|_{L^2}$$

- substitute of L^2 estimate (trivial): $\|\Delta_\rho^{-1} f\|_{\dot{X}_\rho^{-1/2} \to \dot{X}_\rho^{1/2}} = 1$
- *averaged estimate*

$$\|q\|_{\dot{X}_\rho^{-1/2}} = o(1) \text{ on average as } |\rho| \to \infty.$$

Here $\hat{q}$ cannot concentrate on all *codim 2 spheres* $p_\rho^{-1}(0)$!
Caro-Rogers (2016) proved uniqueness for $\gamma \in W^{1,\infty}$ using *Bourgain spaces with two large parameters*.

Unbounded potentials

If $q \in L^\infty$ (i.e. $\gamma \in W^{2,\infty}$), we used the estimate

$$\|q\Delta_\rho^{-1}\|_{L^2 \to L^2} \leq \|q\|_{L^\infty} \|\Delta_\rho^{-1}\|_{L^2 \to L^2},$$

hence L^2 estimates for Δ_ρ^{-1} suffice.

Multiplication by $q \in L^{n/2}$ maps $L^{\frac{2n}{n-2}}$ to $L^{\frac{2n}{n+2}}$, thus require L^p estimates for Δ_ρ^{-1}. More generally, consider $q \in W^{-1,n}$ (i.e. $\gamma \in W^{1,n}$).

L^p estimates

Theorem (Kenig-Ruiz-Sogge 1987)

If $\rho \cdot \rho = 0$, then

$$\|u\|_{L^{\frac{2n}{n-2}}} \lesssim \|\Delta_\rho u\|_{L^{\frac{2n}{n+2}}}, \qquad u \in C_c^\infty(\mathbb{R}^n).$$

Proof. Characteristic set is a *codim 2 sphere* + Stein-Tomas Fourier restriction estimates. □

Implies unique continuation for $-\Delta + q$ for $q \in L^{n/2}$, and the uniqueness result: [Chanillo, Jerison-Kenig 1990, Lavine-Nachman 1991]

$$q_j \in L^{n/2}, \quad \Lambda_{q_1} = \Lambda_{q_2} \implies q_1 = q_2.$$

L^p estimates

Theorem (Haberman 2016)

Uniqueness in the Calderón problem holds for the equations

$$\mathrm{div}(\gamma \nabla u) = 0, \quad \gamma \in W^{1,n}, \quad n = 3, 4,$$
$$((D + \vec{b})^2 + q)u = 0, \quad \|\vec{b}\|_{W^{\varepsilon,n}} \textit{ small}, \ q \in W^{-1,n}, \quad n = 3.$$

Related to unique continuation for $-\Delta + \vec{b} \cdot \nabla$ with $\vec{b} \in L^n$ [Wolff 1992]. Main ideas:

- frequency localized KRS (Strichartz) estimates
- Bourgain spaces, averaging over suitable ρ
- paradifferential and Littlewood-Paley methods

The two-dimensional case

If $\mathbb{D} \subset \mathbb{R}^2$ and $\gamma \in L^\infty(\mathbb{D})$, have

$$\mathrm{div}(\gamma \nabla u) = 0 \quad \underset{u=\mathrm{Re}(f)}{\leftrightsquigarrow} \quad \overline{\partial} f = \mu \overline{\partial f}$$

where $\mu = \frac{1-\gamma}{1+\gamma}$, $\|\mu\|_{L^\infty} < 1$. Reduce *conductivity equation* to *Beltrami equation*, requires no derivatives for γ! Employ

- CGO solutions $f(z,k) = e^{ikz}(1 + \eta(z,k))$ *for all $k \in \mathbb{C}$*
- $\overline{\partial}$-scattering theory [Beals-Coifman 1988]
- quasiconformal methods

to determine $\gamma \in L^\infty(\mathbb{D})$ from Λ_γ [Astala-Päivärinta 2006].

Open questions

1. (Low regularity, $n \geq 3$) Can one solve the Calderón problem for the operator $Pu = -\mathrm{div}(a\nabla u) + \vec{b} \cdot \nabla u + cu$ where

$$a \in W^{1,n}, \quad \vec{b} \in L^n, \quad c \in W^{-1,n}$$

up to natural gauges?

2. (Counterexamples, $n \geq 3$) Can one find $\gamma_1, \gamma_2 \in C^\alpha$ with $0 < \alpha < 1$ so that

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \text{but} \quad \gamma_1 \neq \gamma_2?$$

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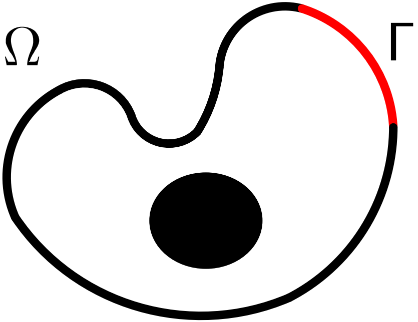
Local data problem

Prescribe voltages on Γ, measure currents on Γ:



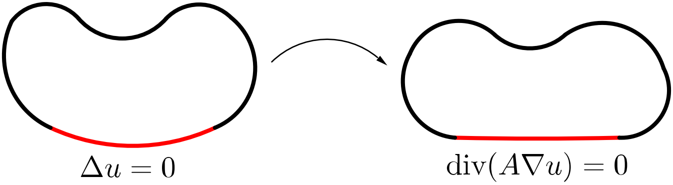
Measure $\Lambda_\gamma f|_\Gamma$ for any f with $\mathrm{supp}(f) \subset \Gamma$. Reduces to showing density of products $\{u_1 u_2\}$ where $\mathrm{supp}(u_j|_{\partial\Omega}) \subset \Gamma$.

Local data problem

Uniqueness known

- if $n = 2$ for any $\Gamma \subset \partial\Omega$ [Imanuvilov-Uhlmann-Yamamoto 2010]
- if $n \geq 3$ and inaccessible part has a conformal symmetry (e.g. flat, cylindrical or part of a surface of revolution) [Kenig-S 2013, Isakov 2007, Kenig-Sjöstrand-Uhlmann 2007]

Flattening the boundary results in *matrix conductivities*:



Calderón problem for $\mathrm{div}(A\nabla u) = 0$, $A = (a^{jk})$, open if $n \geq 3$!

Geometric formulation

Let (M, g) be a compact smooth Riemannian manifold with boundary ∂M. The *Laplace-Beltrami operator* Δ_g on M is given by

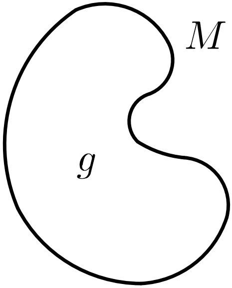
$$\Delta_g u = \sum_{j,k=1}^{n} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_j} \left(\sqrt{\det g}\, g^{jk} \frac{\partial u}{\partial x_k} \right),$$

where $g = (g_{jk})$, $g^{-1} = (g^{jk})$.

If $n = \dim(M) \geq 3$, one has

$$\Delta_g u = 0 \quad \Longleftrightarrow \quad \mathrm{div}(A \nabla u) = 0$$

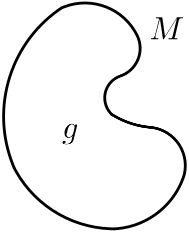
upon taking $a^{jk} = \sqrt{\det g}\, g^{jk}$.



Anisotropic problem

(M, g) compact C^∞ mfld with boundary, $q \in C^\infty(M)$. Consider

$$\begin{cases} (-\Delta_g + q)u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases}$$



Here $\Delta_g \leftrightsquigarrow \mathrm{div}(A\nabla \cdot\,)$. Consider DN map

$$\Lambda_q : f \mapsto \partial_\nu u|_{\partial M}.$$

Recover q from Λ_q. As before, enough to show that the set

$$\{u_1 u_2\,;\, (-\Delta_g + q_j)u_j = 0\}$$

is complete in $L^1(M)$.

Complex geometrical optics

Recall CGO solutions [Sylvester-Uhlmann 1987]

$$(-\Delta + q)u = 0 \text{ in } \mathbb{R}^n, \quad u = e^{\rho \cdot x}(1 + r).$$

Geometric version [Dos Santos-Kenig-S-Uhlmann 2009]:

$$(-\Delta_g + q)u = 0 \text{ in } M, \quad u = e^{\pm\varphi/h}(a + r),$$

where φ is a weight, $h \ll 1$ and $\|r\|_{L^2} \to 0$ as $h \to 0$.
Need estimates for the conjugated Laplacian ($\sim \Delta_\rho$)

$$P_\varphi = e^{\varphi/h}(-h^2\Delta_g)e^{-\varphi/h}.$$

Fourier transforms are not enough, need *"variable coefficient Fourier analysis"* (=microlocal analysis) to study P_φ!

Solvability

L^2 estimates for $P_{\pm\varphi} \Longleftrightarrow$ *principal symbol* p_φ of P_φ satisfies:

Definition (Kenig-Sjöstrand-Uhlmann 2007, Dos Santos et al 2009)

If $(M, g) \subset\subset (U, g)$, we say that $\varphi \in C^\infty(U)$ with $d\varphi \neq 0$ is a *limiting Carleman weight* (LCW) if

$$\{\overline{p}_\varphi, p_\varphi\} = 0 \text{ in the set where } p_\varphi = 0.$$

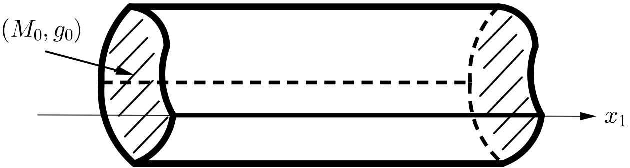
Examples in $\mathbb{R}^3$: $\varphi(x) = x_1$ and $\varphi(x) = \log|x|$. Questions:

1. Which (M, g) have LCWs?
2. If (M, g) has LCWs, can one solve the Calderón problem?

1. Existence of LCWs

LCWs require a certain conformal symmetry, such as:

(M, g) is *conformally transversally anisotropic (CTA)* if $(M, g) \subset\subset (\mathbb{R} \times M_0, g)$ where $g = c(e \oplus g_0)$.



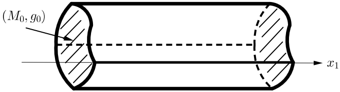
- corresponds to $A(x_1, x') = c(x_1, x') \begin{pmatrix} 1 & 0 \\ 0 & A_0(x') \end{pmatrix}$
- a 3D manifold has an LCW $\leftrightsquigarrow$ *det(Cotton-York)* $= 0$ [Angulo-Guijarro-Faraco-Ruiz 2016]

2. Solving the Calderón problem

Dos Santos-Kenig-S-Uhlmann (2009): uniqueness results when

(a) (M, g) is *CTA*

(b) transversal mfld (M_0, g_0) is *simple*[2].



Dos Santos-Kurylev-Lassas-S (2016) weakened (b) to

(b') transversal mfld (M_0, g_0) has injective X-ray transform.

Difficulty: need a replacement for Fourier transform in $\mathbb{R}^n$!

[2] strictly convex ball with *no conjugate points*

Solving the Calderón problem

Equality of DN maps $\implies$ a certain *transform* of $q_1 - q_2$ vanishes (replaces Fourier transform in $\mathbb{R}^n$):

Theorem (S 2016)

(M, g) compact with LCW φ. Then

$$\Lambda_{q_1} = \Lambda_{q_2} \implies \int_\Gamma (q_1 - q_2)\Psi \, dS = 0$$

if Γ is a *good bicharacteristic leaf* for P_φ and $\Psi \in \mathrm{Holom}(\Gamma)$.

In $\mathbb{R}^n$ any 2-plane is a good leaf [Greenleaf-Uhlmann 2001]. In general they are curved 2-manifolds (related to geodesics).

Complex involutive operators

Write $p_\varphi = a + ib$ $(= |\xi|^2 - |d\varphi|^2 + 2i\langle d\varphi, \xi\rangle)$. Then

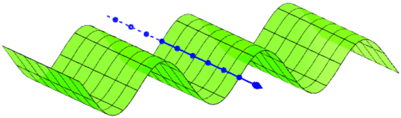
$$\{a, b\} = 0 \text{ on the } \textit{characteristic set } \Sigma = \{a = b = 0\}.$$

Complex involutive symbol [Duistermaat-Hörmander 1972]:

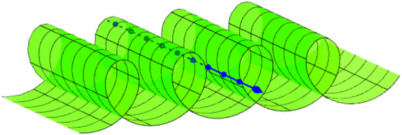
- Σ is an involutive $(2n-2)$-dim. submanifold of T^*U
- Σ is foliated by 2-dim. manifolds (*bicharacteristic leaves*) generated by integral curves of H_a and H_b
- singularities for P_φ propagate along bicharacteristic leaves

A leaf Γ is *good* if it "straightens to a domain in $\mathbb{R}^2$" and supports quasimodes. Then $P_\varphi \approx \overline{\partial}$ microlocally near Γ.

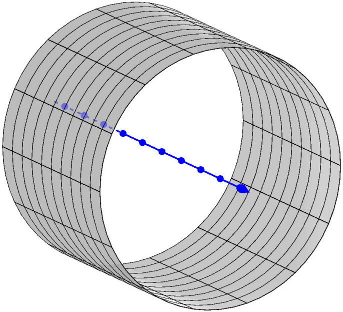
Bicharacteristic leaves



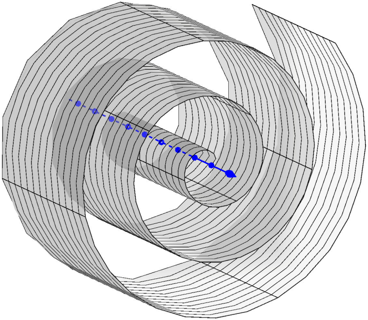
Good



Good



? (holonomy)



Not good (trapped)

Normal form

Microlocal reduction to normal form: a good bicharacteristic leaf can be "straightened" in phase space.

Theorem. If Γ is a *good bicharacteristic leaf*, there is a canonical transformation χ near Γ with

$$\chi(\Gamma) \subset \{((x_1, x_2, 0), e_n)\}, \quad \chi^* p_\varphi = \xi_1 + i\xi_2.$$

There is a semiclassical Fourier integral operator F associated with the graph of χ such that

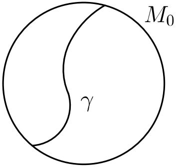
$$F^* P_{\pm\varphi} F \sim h(D_1 \pm iD_2) + \text{lower order}.$$

Thus enough to study the operator $h(D_1 \pm iD_2)$.

Transform

If $(M, g) \subset\subset (\mathbb{R} \times M_0, g)$ is *CTA*, Calderón problem solvable if the *geodesic X-ray transform* on M_0,

$$If(\gamma) = \int_\gamma f\, dt, \quad \gamma \text{ maximal geodesic,}$$



is invertible. This holds on compact strictly convex M_0, if M_0:

- is *simple* (ball with no conjugate points) [Mukhometov 1978]
- has *negative curvature* [Guillarmou 2016]
- has *nonnegative curvature* and $\dim(M_0) \geq 3$ [Uhlmann-Vasy 2016, Paternain-S-Uhlmann-Zhou 2016]
- embeds in a *product* of non-closed manifolds [S 2016]

Open questions

3. (Local data, $n \geq 3$) If $\Omega \subset \mathbb{R}^n$ is bounded and $\Gamma \subset \partial\Omega$ is nonempty, can one solve the Calderón problem with measurements on Γ?

4. (Linearized problem, $n \geq 3$) If (M, g) is compact with boundary, are products of harmonic functions dense in L^1?