Mathematics

An Electromagnetic Technique to Detect Defects at Interfaces

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joint work with

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Research Trend

Asymptotic methods in connection with qualitative methods

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Perturbation of transmission eigenvalues in presence of thin layer or small volume penetrable inclusions in a known inhomogeneous medium.

- CAKONI-CHAULET-HADDAR (2014) - *IMA J. Appl. Math.*
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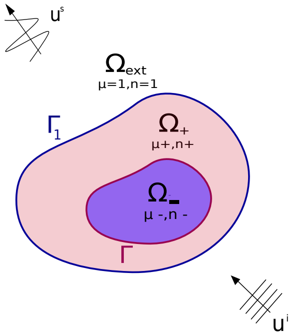
Scattering by periodic media – homogenization and transmission eigenvalues.

- CAKONI-HADDAR-HARRIS (2015) - *Inverse Problems and Imaging*
- CAKONI-GUZINA-MOSKOW (2016) - *SIAM J. Math. Anal.*

Healthy Material - Everything Known



$$\Omega := \Omega_- \cup \Omega_+ \subset \mathbb{R}^m, \quad m = 2, 3$$

$$\begin{aligned}
\Delta u^{ext} + k^2 u^{ext} &= 0 && \text{in} \quad \Omega_{\text{ext}} \\
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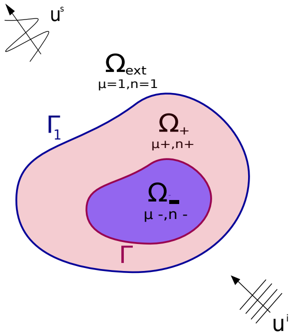

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with continuity of fields and conormal derivatives across interfaces

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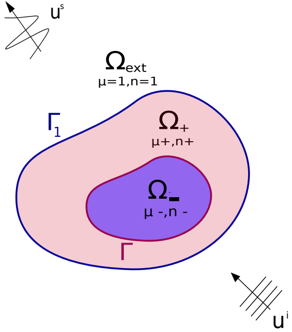

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$u^{ext} = u^s + u^i$ we take $u^i := e^{ikx\cdot d}$, d unit vector

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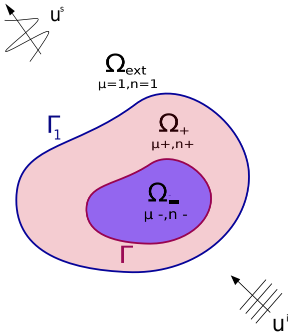

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$$u^{ext} = u^s + u^i \quad \text{we take } u^i := e^{ikx \cdot d}, \; d \text{ unit vector}$$

$$\lim_{r \to \infty} |x|^{\frac{m-1}{2}} \left(\frac{\partial u^s}{\partial |x|} - iku^s \right) = 0, \quad \text{uniformly in } \hat{x} = x/|x|$$

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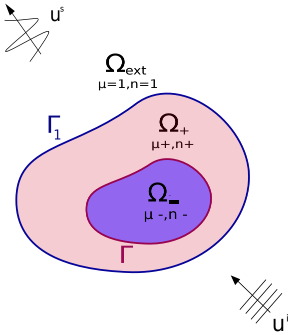

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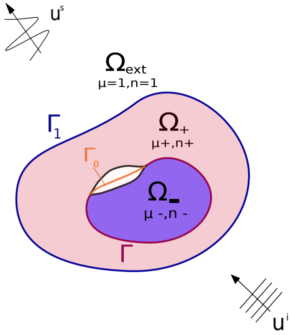
k is the wave number in Ω_{ext} ($k = \omega\sqrt{\epsilon_{ext}\mu_{ext}}$).

Material with Defect at the Interface



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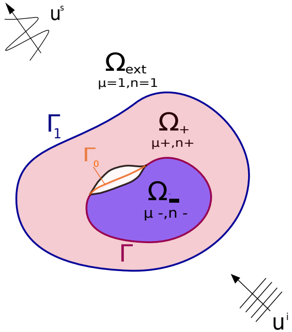

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\begin{aligned}
u^{ext} &= u^+ && \text{and} && \nabla u^{ext} \cdot \nu = 1/\mu_+ \nabla u^+ \cdot \nu && \text{on} \quad \Gamma_1 \\
u^+ &= u^- && \text{and} && 1/\mu_+ \nabla u^+ \cdot \nu = 1/\mu_- \nabla u^- \cdot \nu && \text{on} \quad \Gamma \backslash \overline{\Gamma}_0 \\
U &= u^+ && \text{and} && 1/\mu_0 \nabla U \cdot \nu = 1/\mu_+ \nabla u^+ \cdot \nu && \text{on} \quad \Gamma_+ \\
U &= u^- && \text{and} && 1/\mu_0 \nabla U \cdot \nu = 1/\mu_- \nabla u^- \cdot \nu && \text{on} \quad \Gamma_-.
\end{aligned}
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The Inverse Problem

Denote the unit sphere by $\mathbb{S}^{m-1} := \{x \in \mathbb{R}^m, \quad |x| = 1\}$

$$u^s(x,d) = \gamma_m \frac{e^{ik|x|}}{|x|^{(m-1)/2}} u_\infty(\hat{x},d) + O\left(\frac{1}{|x|}\right)$$

where $\gamma_m = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$, if $m = 2$ and $\gamma_m = \frac{1}{4\pi}$ if $m = 3$.

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$u_\infty(\hat{x},d)$ for incident directions d and observation directions $\hat{x}$, both on a nonzero measure subset of $\mathbb{S}^{m-1}$

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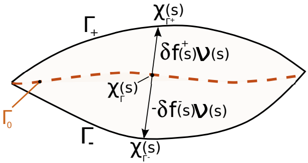
Determine the damaged part Γ_0 of the known interface Γ from the above (measured) data without knowing μ_0 and n_0

Asymptotic Model



Small parameter: the thickness of the opening is much smaller than interrogating wavelength $\lambda := 2\pi/k$ and the thickness of the layers.

- Introduces essential computational difficulty in the numerical solution of the forward problem.
- We use the linear sampling method to solve the inverse problem and want to probe along the known boundary Γ for the defective part Γ_0.

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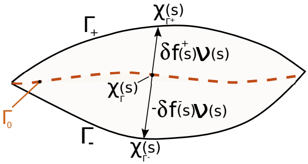

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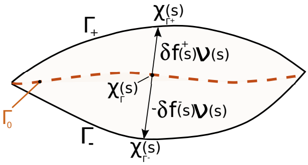
Replace the opening Ω_0 by appropriate jump conditions on u^+ and u^- across the exact part of the boundary Γ_0

Asymptotic Model

We use asymptotic method.

- B. ASLANYÜREK, H. HADDAR, AND H. SAHINTÜRK, Generalized impedance boundary conditions for thin dielectric coatings with variable thickness, *Wave Motion*, 48, 681700, 2011.
- B. DELOURME, H. HADDAR, AND P. JOLY, Approximate models for wave propagation across thin periodic interfaces, *J. Math. Pures Appl.*, 98:2871, 2012.
- B. DELOURME Modeles et asymptotiques des interfaces fines et periodiques en electromagnetisme, *PhD thesis, Universite Pierre et Marie Curie - Paris VI*, 2010.

Asymptotic Model



$$\Gamma_0 := \{\chi_\Gamma(s),\ s \in [0, L]\}$$

Neighborhood of Γ_0: $x = \chi_\Gamma(s) + \eta\nu(s)$, $\xi = \dfrac{\eta}{\delta}$

$$\Gamma_\pm = \left\{\chi_\Gamma(s) + \delta f^\pm(s)\nu(s), \quad s \in [0, L]\right\}$$

$$U(s,\xi) = \sum_{j=0}^{\infty} \delta^j U_j(s,\xi),\ u^\pm(s,\eta) = \sum_{j=0}^{\infty} \delta^j u_j^\pm(s,\eta)\ (*)$$

We expand each of the terms $u_j^\pm(s,\eta)$ in a power series with respect to the normal direction coordinate η around zero, i.e.

$$u_j^\pm(s,\eta) = u_j^\pm(s,0) + \eta\frac{\partial}{\partial\eta}u_j^\pm(s,0) + \frac{\eta^2}{2}\frac{\partial^2}{\partial\eta^2}u_j^\pm(s,0) + \ldots$$

and after plugging in $(*)$ we obtain

$$u^\pm(s,\eta) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \delta^j \frac{\eta^k}{k!}\frac{\partial^k}{\partial\eta^k}u_j^\pm(s,0).$$

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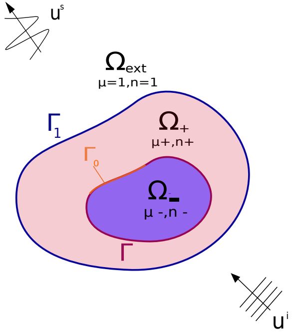
Remark

If we assume that $f^{\pm}(0) = f^{\pm}(L) = 0$ the next asymptotic model can be rigorously justified following the approach of Delourme's thesis for periodic interfaces.

Asymptotic Model



In Ω_{ext}, Ω_+ and Ω_- we have the same equations and on Γ_1 and $\Gamma \setminus \Gamma_0$ the same transmission conditions as for the healthy material.

Recalling the notation

$$[w] = w^+ - w^- \text{ and } \langle w \rangle = (w^+ + w^-)/2$$

on Γ_0 we have that

$$[u] = \alpha \left\langle \frac{1}{\mu} \frac{\partial u}{\partial \nu} \right\rangle \quad \text{and} \quad \left[\frac{1}{\mu} \frac{\partial u}{\partial \nu} \right] = \left(-\nabla_\Gamma \cdot \langle \beta f \rangle \nabla_\Gamma + \gamma \right) \langle u \rangle$$

where

$$\alpha = 2\delta \langle f(\mu_0 - \mu) \rangle , \quad \beta^\pm = 2\delta \left(\frac{1}{\mu_0} - \frac{1}{\mu^\pm} \right), \quad \gamma = 2\delta k^2 \langle f(n - n_0) \rangle$$

Well-posedness of Asymptotic Model

- Introduce $\mathcal{H} := \left\{ u \in H^1(B_R \backslash \overline{\Gamma_0}) \text{ such that } \sqrt{f^{\pm}}\, \nabla_\Gamma \langle u \rangle \in L^2(\Gamma_0) \right\}$

$$\|u\|^2_{\mathcal{H}} = \|u\|^2_{H^1(B_R \backslash \overline{\Gamma_0})} + \left\| \sqrt{f^+}\, \nabla_\Gamma \langle u \rangle \right\|^2_{L^2(\Gamma_0)} + \left\| \sqrt{f^-}\, \nabla_\Gamma \langle u \rangle \right\|^2_{L^2(\Gamma_0)}.$$

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- Assume that $\Re\left(\frac{1}{\mu^{\pm}}\right) \geq \epsilon_1 > 0$, and $\Re\left(\frac{1}{\mu_0} - \frac{1}{\mu^{\pm}}\right) \geq \epsilon_2 > 0$

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- $0 \leq \Im(n^\pm) \leq \Im(n_0)$ and $0 \leq \Im(\mu^\pm) \leq \Im(\mu_0)$
- $f^\pm$ go to zero at the boundary of Γ_0 in Γ such that $1/\langle f(\mu_0 - \mu)\rangle \in L^t(\Gamma_0)$ for $t = 1 + \epsilon$ in $\mathbb{R}^2$ and $t = 7/4 + \epsilon$ in $\mathbb{R}^3$ for arbitrary small $\epsilon > 0$.

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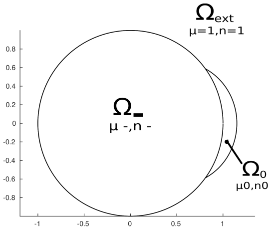
Theorem

Under the above assumptions the direct approximate model has a unique solution $u \in \mathcal{H}$ which depends continuously on the incident wave u^i with respect to the $\mathcal{H}$-norm.

Numerical Validation



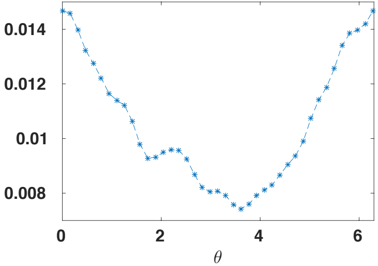
$$e(\delta, d) := \frac{\| u_\delta^{ext} - u^{ext} \|_{H^1(B_R \setminus \overline{\Omega})}}{\| u^{ext} \|_{H^1(B_R \setminus \overline{\Omega})}}$$

$$e^\infty(\delta, d) := \frac{\| u_\delta^{\infty} - u^{\infty} \|_{L^2(\mathbb{S}^1)}}{\| u^{\infty} \|_{L^2(\mathbb{S}^1)}}$$

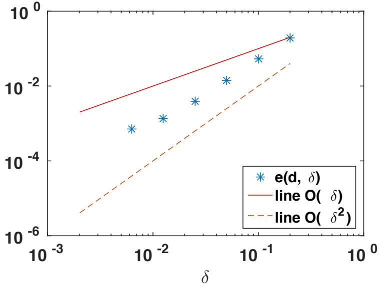
$$f^-(s) = 0, \;\; f^+(s) := -l^{-2}(s+l)(s-l) \text{ for } s \in (-l, l), \;\; \text{with } l = 0.2\pi,$$

on the interface $r = 1$. The material properties are chosen to be $n_- = 1, \mu_- = 1$ in Ω_-, $n_+ = 1, \mu^+ = 1$ in Ω_+, $n_0 = 0.2, \mu_0 = 0.9$ in Ω_0, and the wave number $k = 3$.

Numerical Validation



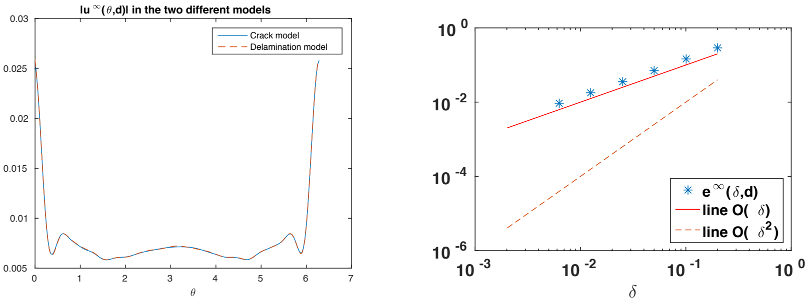
(a)



(b)

Panel (a) shows the H^1 relative error of total fields resulting from different incident direction. The maximum error is obtained for $d = (1, 0)$. Panel (b) the H^1 relative error for different values of δ and $d = (1, 0)$. The approximated rate of convergence is $O(\delta^{1.7})$.

Numerical Validation



(a) (b)

Panel (a) shows the plot of the absolute value of the far field for both models for $\delta = 0.05$. Panel (b) shows the far field L^2 relative error $e^\infty(\delta, d)$, for different values of δ and $d = (1, 0)$. The approximated rate of convergence is $O(\delta^1)$.

The Inverse Problem

u^s the scattered field due to the layered media and the flaw on the interface.

$$u^s(x,d) = \gamma_m \frac{e^{ik|x|}}{|x|^{(m-1)/2}} u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right), \qquad m = 2, 3$$

Data

$u_\infty(\hat{x}, d)$ for incident directions d and observation directions $\hat{x}$ in a nonzero measure subset of $\mathbb{S}^{m-1}$

The Inverse Problem

Determine the damaged part Γ_0 of the known interface Γ from the above (measured) data without knowing μ_0 and n_0

The Inverse Problem

Data defines the far field operator $F : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$

$$(Fg)(\hat{x}) = \int_{\mathbb{S}^{m-1}} u^\infty(\hat{x}, d) g(d) ds_d$$

By linearity $Fg = F_b g + F_d g$ with

$$(F_b g)(\hat{x}) = \int_{\mathbb{S}^{m-1}} u_b^\infty(\hat{x}, d) g(d) ds_d$$

where $u_b^\infty(\hat{x}, d)$ is the far field pattern of the scattered field $u_b^s(x, d)$ due to healthy material, i.e the unique solution $u_b = u_b^s + e^{ikx \cdot d} \in H^1_{loc}(\mathbb{R}^m)$ of

$$\nabla \cdot \left(\frac{1}{\mu} \nabla u_b \right) + k^2 n u_b = 0 \quad \text{in } \mathbb{R}^m$$

and u_b^s satisfies Sommerfeld radiation condition.

The Inverse Problem

Consider the far field equation

$$\left(F_d g\right)(\hat{x}) = \phi_L^\infty, \qquad L \subset \Gamma$$

where for some $(\alpha_L, \beta_L) \in L^2(L) \times \tilde{H}^1(L)$

$$\phi_L^\infty(x) = \gamma_m^{-1} \int_L \left\{ \alpha_L(y) G_b^\infty(x,y) + \beta_L(y) \frac{1}{\mu} \frac{\partial G_b^\infty(x,y)}{\partial \nu(y)} \right\} ds(y)$$

with $G_b^\infty(x,y)$ the far field of the radiating solution $G_b(\cdot, z)$ to

$$\nabla \cdot \left(\frac{1}{\mu} \nabla G_b(\cdot, z) \right) + k^2 n G_b(\cdot, z) = -\delta(\cdot - z), \qquad \text{in } \mathbb{R}^m \setminus \{z\}$$

The Inverse Problem

Lemma (Mixed reciprocity)

$$G_b^\infty(\hat{x}, z) = \gamma_m u_b(z, -\hat{x}) \quad \text{for all } z \in \mathbb{R}^m \text{ and } \hat{x} \in \mathbb{S}^{m-1}$$

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- $F_d g = GHg$

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$$Hg := (-\nabla_\Gamma \cdot \langle \beta f \rangle \nabla_\Gamma + \gamma)\, u_{b,g}, \;\; u_{b,g}(x) := \int_{\mathbb{S}^{m-1}} u_b(x, d) g(d)\, ds_d$$

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- For $L \subset \Gamma$

$$L \subset \Gamma_0 \iff \phi_L^\infty \in \text{Range}(G)$$

The Inverse Problem

Theorem (Linear Sampling Method)

1. For an arbitrary arc $L \subset \Gamma_0$ and $\epsilon > 0$, there exists a function $g_L^\epsilon \in L^2(\mathbb{S}^{m-1})$ such that

$$\|F_D g_L^\epsilon - \phi_\infty^L\|_{L^2(\mathbb{S}^{m-1})} < \epsilon$$

   and, as $\epsilon \to 0$, the corresponding solution u_{b,g_L^ϵ} to the background problem converges in $\mathcal{H}$.

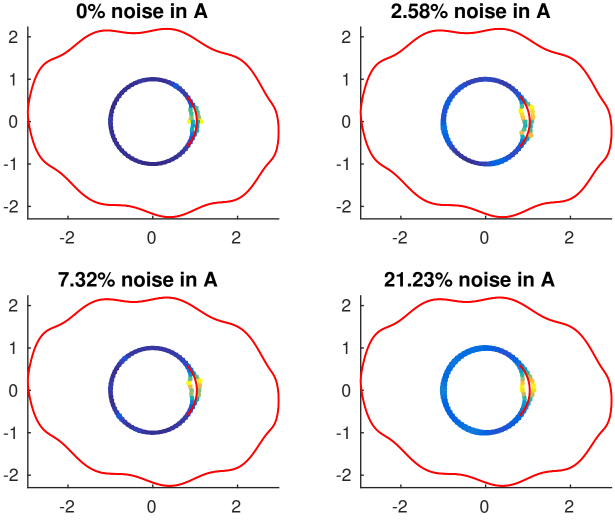
2. For $L \not\subset \Gamma_0$ and $\epsilon > 0$, every function $g_L^\epsilon \in L^2(\mathbb{S}^{m-1})$ such that

$$\|F_D g_L^\epsilon - \phi_\infty^L\|_{L^2(\mathbb{S}^{m-1})} < \epsilon$$

   is such that the corresponding solution u_{b,g_L^ϵ} to the background problem satisfies

$$\lim_{\epsilon\to 0} \|u_{b,g_L^\epsilon}\|_{\mathcal{H}} = \infty \quad \text{and} \quad \lim_{\epsilon\to 0} \|g_L^\epsilon\|_{L^2(\mathbb{S}^{m-1})} = \infty.$$

Example of Reconstruction



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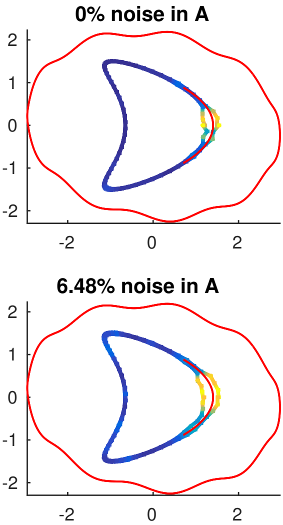

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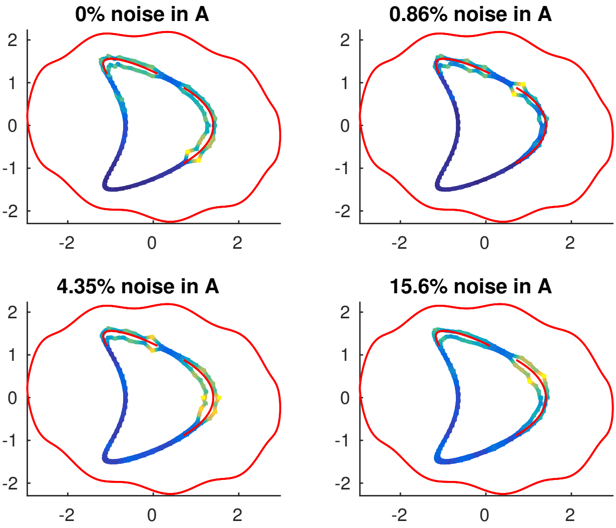

Remarks

F. CAKONI, I. DE TERESA TRUEBA, H. HADDAR, AND P. MONK, Nondestructive testing of the delaminated interface between two materials, *SIAM J. Appl. Math.* (accepted).

We are working on Maxwell's equation model for this problem.