On a discretization of confocal quadrics

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joint work with W. Schief, Yu. Suris, J. Techter

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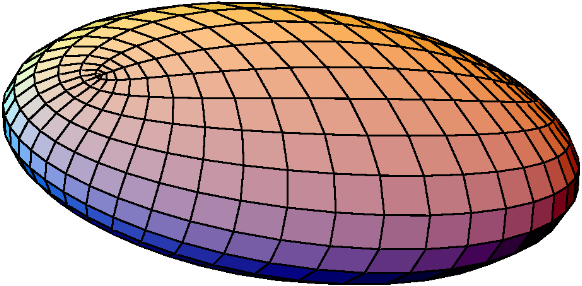
Ellipsoid



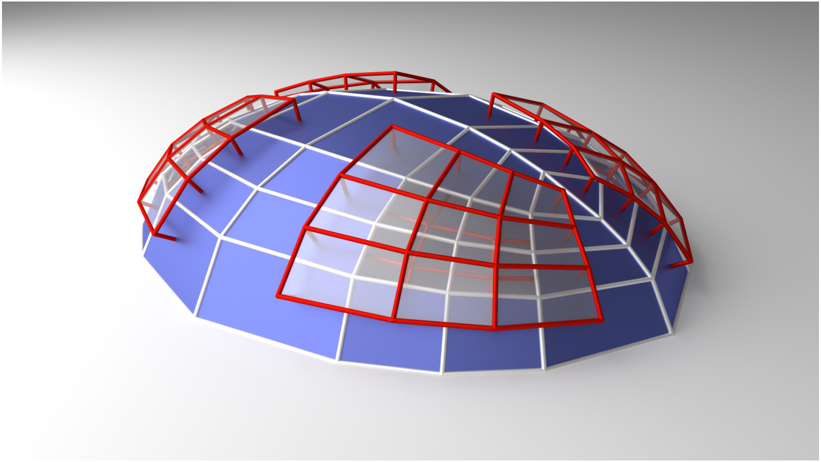
Conformal curvature line parametrized ellipsoid

Open problem. Discrete ellipsoid

Open problem. Discrete confocal quadrics

As a referee stated:
"Confocal quadrics is an ubiquitous subject that goes back to Jacobi and Chasles; it is also an evergreen topic, studied, in the 20th century, by J. Moser and V. Arnold, among others. Quadrics provide basic examples of continuous- and discrete-time integrable systems, namely, the geodesic flows and billiard ball maps."
"I expect this [...] to generate much more research: one cannot help wondering which of the numerous features of conics and quadrics, described in the classic geometry literature, have discrete analogs, and what these analogs may look like."

Discrete ellipsoid (and confocal quadrics) in this talk

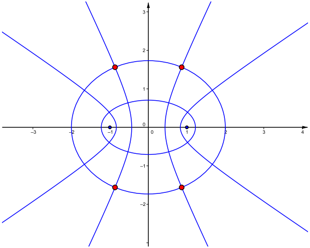


Confocal quadrics

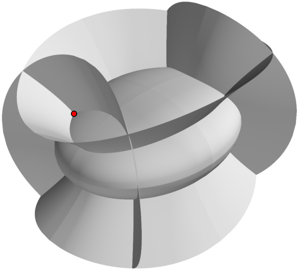
For any numbers $a_1 > \cdots > a_N > 0$, the one-parameter (λ) family of quadrics defined by

$$\frac{x_1^2}{\lambda + a_1} + \cdots + \frac{x_N^2}{\lambda + a_N} = 1$$

is known as a family of confocal quadrics in $\mathbb{R}^N$.



$N = 2$



$N = 3$

Confocal (elliptic) coordinates

Through each point $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ subject to $x_1 \cdots x_N \neq 0$, there pass exactly N orthogonal quadrics (different signatures) corresponding to some values $\lambda = u_1, \dots, \lambda = u_N$. Obtained by solving the equations

$$\sum_{k=1}^{N} \frac{x_k^2}{u_i + a_k} = 1, \qquad i = 1, \dots, N,$$

$-a_1 < u_1 < -a_2 < \cdots < -a_N < u_N$.
The parameters $u_1, \dots, u_N$ are known as confocal (or elliptic) coordinates and represent an orthogonal coordinate system in each of the 2^N hyperoctants via

$$x_k^2 = \frac{\prod_{i=1}^{N}(u_i + a_k)}{\prod_{i \neq k}(a_k - a_i)}, \qquad k = 1, \dots, N. \qquad \text{(Discretisation?)}$$

Algebraic properties

Confocal coordinates (in the first hyperoctant)

$$\boldsymbol{x} : \mathcal{U} \to \mathbb{R}^N_+, \qquad x_k = \frac{\prod_{i=1}^{k-1} \sqrt{-(u_i + a_k)} \prod_{i=k}^{N} \sqrt{(u_i + a_k)}}{\prod_{i=1}^{k-1} \sqrt{a_i - a_k} \prod_{i=k+1}^{N} \sqrt{a_k - a_i}}$$

$$\mathcal{U} = \{(u_1, \ldots, u_N) : -a_1 < u_1 < -a_2 < \cdots < -a_N < u_N\}$$

enjoy the following properties:

(1) $x_k = \rho_k^1(u_1) \cdots \rho_k^N(u_N)$ (separability)

(2) $x_k(u_k \nearrow -a_k) = x_k(u_{k-1} \searrow -a_k) = 0$ (boundary conditions)

Algebraic properties

(3) $\boldsymbol{x}$ is a solution of the Euler-Poisson-Darboux equations

$$\frac{\partial^2 \boldsymbol{x}}{\partial u_i \partial u_j} = \frac{\gamma}{u_i - u_j}\left(\frac{\partial \boldsymbol{x}}{\partial u_j} - \frac{\partial \boldsymbol{x}}{\partial u_i}\right), \qquad \gamma = \frac{1}{2}$$

($i \neq j$) which are multi-dimensionally consistent. The coordinate lines on the surfaces $\boldsymbol{x}(u_i, u_j)$ are therefore conjugate.

(4) $\left\langle \frac{\partial \boldsymbol{x}}{\partial u_i}, \frac{\partial \boldsymbol{x}}{\partial u_j} \right\rangle = 0$ (orthogonality)

Conjugacy and orthogonality means that the confocal coordinates (u_i, u_j) are curvature coordinates on the surfaces $\boldsymbol{x}(u_i, u_j)$.

All two-dimensional coordinate surfaces are isothermic

The properties (1) - (4) characterise confocal coordinates

Formulas

Theorem. Separable solutions (1) of the Euler-Poisson-Darboux equations (3) subject to the boundary conditions (2) are given by

$$x_k = D_k \prod_{i=1}^{k-1} \sqrt{-(u_i + a_k)} \prod_{i=k}^{N} \sqrt{(u_i + a_k)}.$$

The orthogonality condition (4) is satisfied if and only if (up to a global scaling)

$$D_k^{-1} = \prod_{i=1}^{k-1} \sqrt{a_i - a_k} \prod_{i=k+1}^{N} \sqrt{a_k - a_i}$$

so that $(u_1, \ldots, u_N)$ constitute confocal coordinates.

What are the discrete analogues of the properties (1) - (4)?

Discrete Euler-Poisson-Darboux equations

For some $\mathcal{U} \subset \mathbb{Z}^N$, we consider discrete nets

$$\boldsymbol{x} : \mathcal{U} \to \mathbb{R}^N, \qquad (n_1, \ldots, n_N) \mapsto (u_1, \ldots, u_N),$$

satisfying the discrete Euler-Poisson-Darboux equations

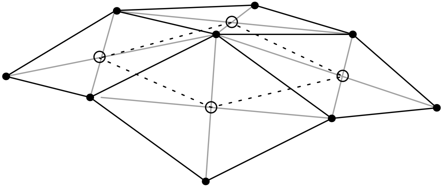
$$\Delta_i \Delta_j \boldsymbol{x} = \frac{\gamma}{n_i + \epsilon_i - n_j - \epsilon_j} (\Delta_j \boldsymbol{x} - \Delta_i \boldsymbol{x}), \qquad \gamma = \frac{1}{2},$$

where $i \neq j$ and $\Delta_i f(n_i) = f(n_i + 1) - f(n_i)$.

- The discrete EPD equations are multi-dimensionally consistent and define particular discrete conjugate nets, i.e. the discrete surfaces $\boldsymbol{x}(n_i, n_j)$ are composed of planar quadrilaterals.
- The discrete EPD equations were introduced by Konopelchenko and Schief (2014).
- All two-dimensional subnets are Koenigs.

Discrete Koenigs nets

- A discrete surface $f : \mathbb{Z}^2 \to \mathbb{R}^3$ with planar faces and non-planar vertices is a discrete Koenigs net if the intersection points of diagonals of any four quadrilaterals sharing a vertex are co-planar. [B., Suris '09]
- Koenigs + orthogonal = isothermic



Separability

Introduce the “Pochhammer symbol” (Gelfand *et al.*)

$$(u)_{1/2} = \frac{\Gamma(u + \frac{1}{2})}{\Gamma(u)}$$

which (up to rescaling) may be regarded as a discretisation of $\sqrt{u}$ since

$$\lim_{\epsilon \to 0} \epsilon^{1/2} \left(\frac{u}{\epsilon}\right)_{1/2} = u^{1/2}.$$

Theorem. A separable function

$$x(n_1, \ldots, n_N) = \rho^1(n_1) \cdots \rho^N(u_n)$$

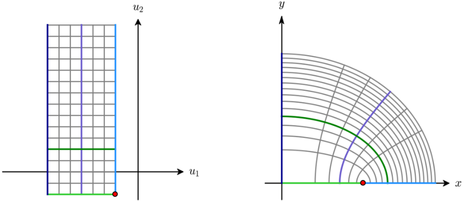
is a solution of the discrete EPD equations if and only if

$$\rho^i(n_i) = d_i(n_i + \epsilon_i + c)_{1/2} = \tilde{d}_i(-n_i - \epsilon_i - c + \tfrac{1}{2})_{1/2},$$

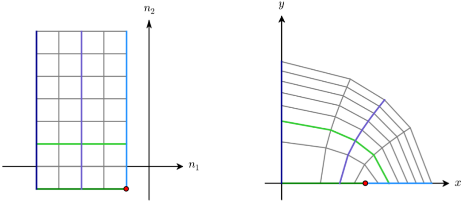
where c is a constant of separation.

Combinatorics

Classical case: $\mathcal{U} = \{(u_1, u_2) : -a_1 < u_1 < -a_2 < u_2\}$



Discrete case: $\mathcal{U} = \{(n_1, n_2) : -\alpha_1 \leq n_1 \leq -\alpha_2 \leq n_2\}$



Boundary conditions

Consider the region

$$\mathcal{U} = \{(n_1, \dots, n_N) \in \mathbb{Z}^N : -\alpha_1 \le n_1 \le -\alpha_2 \le \cdots \le \alpha_N \le n_N\}$$

for some positive integers $\alpha_1 > \cdots > \alpha_N$ and discrete nets $\boldsymbol{x} : \mathcal{U} \to \mathbb{R}_+^N$. Then, the parameters ϵ_i and the constants of separation c_k may be adjusted in the following manner:

Theorem. Separable solutions of the discrete EPD equations subject to the $2N-1$ boundary conditions

$$x_k(n_k = -\alpha_k) = x_k(n_{k-1} = -\alpha_k) = 0$$

are given by

$$x_k = D_k \prod_{i=1}^{k-1} (-u_i - a_k + \tfrac{1}{2})_{1/2} \prod_{i=k}^{N} (u_i + a_k)_{1/2},$$

$$u_i = n_i - \frac{i}{2}, \quad a_k = \alpha_k + \frac{k}{2}.$$

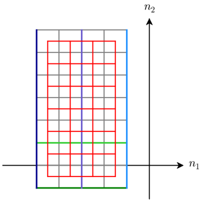
Orthogonality

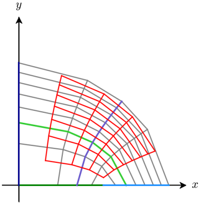
The standard notion of discrete orthogonality (+ conjugacy), that is, circularity turns out to be incompatible!
Instead, we extend the discrete net $\boldsymbol{x}$ to

$$\boldsymbol{x} : \mathcal{U} \cup \mathcal{U}^* \to \mathbb{R}^N_+$$

$$\mathcal{U}^* = \{(n_1, \ldots, n_N) \in (\mathbb{Z} + \tfrac{1}{2})^N : -\alpha_1 \leq n_1 \leq -\alpha_2 \leq \cdots \leq \alpha_N \leq n_N\}$$

and demand that any edge of $x(\mathcal{U})$ be orthogonal to the dual facet of $x(\mathcal{U}^*)$.



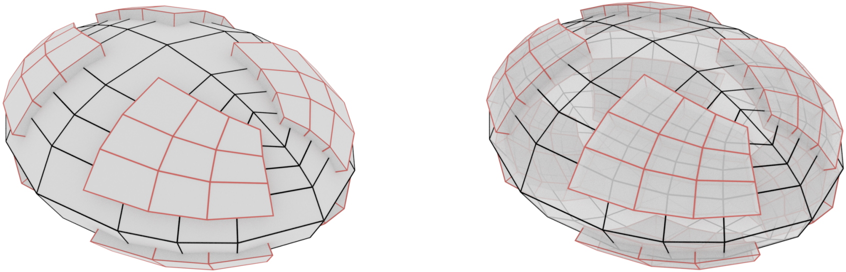


Discrete confocal quadrics

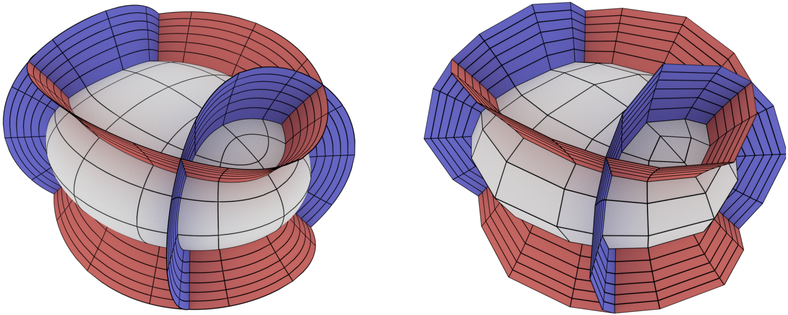
Theorem. The discrete orthogonality condition is satisfied if and only if (up to a global scaling)

$$D_k^{-1} = \prod_{i=1}^{k-1} \sqrt{a_i - a_k} \prod_{i=k+1}^{N} \sqrt{a_k - a_i}$$

so that discrete confocal quadrics are uniquely defined.



Discrete vc. Continuous



Three confocal quadrics and their discrete counterparts

Algebraic identities

A lattice point $\boldsymbol{x}(\boldsymbol{n})$ and its nearest neighbours $\boldsymbol{x}(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})$ are related by

$$\frac{x(\boldsymbol{n})x(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{u_1+a_1}+\frac{y(\boldsymbol{n})y(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{u_1+a_2}+\frac{z(\boldsymbol{n})z(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{u_1+a_3}=1$$

$$\frac{x(\boldsymbol{n})x(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{u_2+a_1}+\frac{y(\boldsymbol{n})y(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{u_2+a_2}+\frac{z(\boldsymbol{n})z(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{u_2+a_3}=1$$

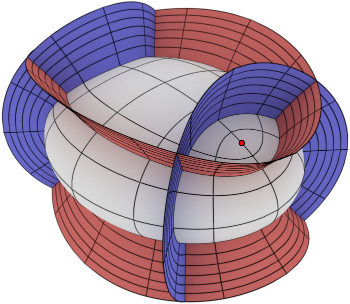
$$\frac{x(\boldsymbol{n})x(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{u_3+a_1}+\frac{y(\boldsymbol{n})y(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{u_3+a_2}+\frac{z(\boldsymbol{n})z(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{u_3+a_3}=1,$$

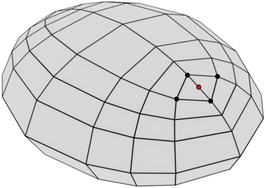
where $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$, $\sigma_i=\pm1$ and

$$u_1=n_1+\tfrac{1}{4}\sigma_1-\tfrac{3}{4},\quad u_2=n_2+\tfrac{1}{4}\sigma_2-\tfrac{5}{4},\quad u_3=n_3+\tfrac{1}{4}\sigma_3-\tfrac{7}{4}.$$

This discretisation of the defining equations for confocal quadrics exists for any N.

Discrete umbilics





The umbilics ("spherical" points) on confocal ellipsoids lie on the focal hyperbola

$$\frac{x^2}{a_1 - a_2} - \frac{z^2}{a_2 - a_3} = 1, \qquad y = 0.$$

The discrete umbilics (verices of valence 2; $n_1 = n_2 = -\alpha_2$) likewise lie on a discrete focal hyperbola.

Where to go from here

We can discretize confocal quadrics parametrised in terms of arbitrary curvature coordinates. For instance, we can discretise the following classical parametrizations:

$N = 2$:

$$\boldsymbol{x} = \begin{pmatrix} \cos u \cosh v \\ \sin u \sinh v \end{pmatrix}$$

$N = 3$:

$$\boldsymbol{x} = \begin{pmatrix} \mathrm{sn}(u,k)\,\mathrm{dn}(v,\hat{k})\,\mathrm{ns}(w,k) \\ \mathrm{cn}(u,k)\,\mathrm{cn}(v,\hat{k})\,\mathrm{ds}(w,k) \\ \mathrm{dn}(u,k)\,\mathrm{sn}(v,\hat{k})\,\mathrm{cs}(w,k) \end{pmatrix}$$

$$k^2 = \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}, \quad \hat{k}^2 = 1 - k^2.$$

Discrete confocal quadrics