Numerical study of 2+1 dimensional nonlinear dispersive PDEs

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Outline

- Introduction
- Dispersive shocks and blow-up in gKdV
- Blow-up in gKP
- Semiclassical DS II
- Dispersive shocks and blow-up in DS II
- Blow-up in solutions to the NV equation
- Outlook

Korteweg-de Vries equation

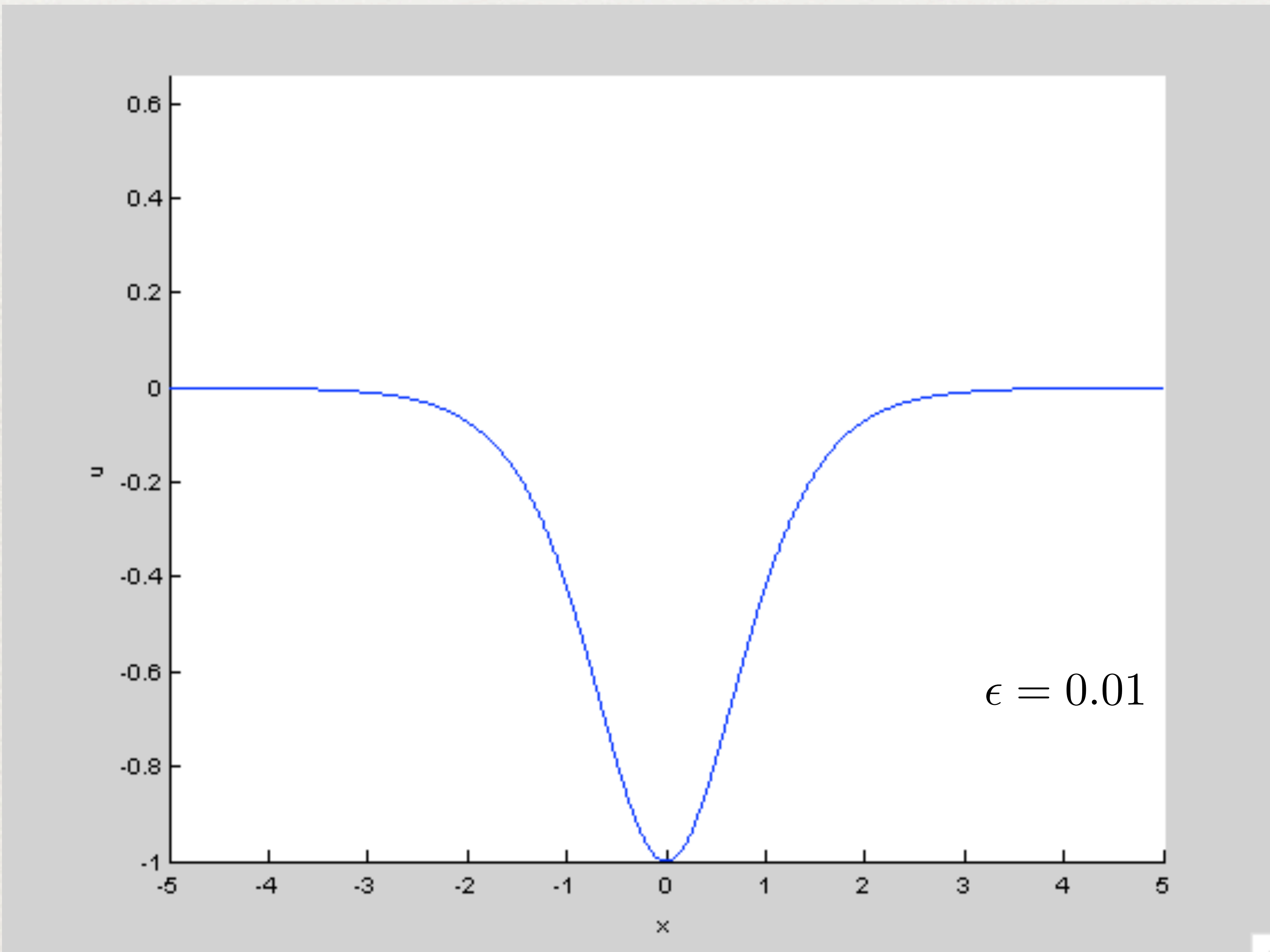
$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0$$

$$u_0 = -\mathrm{sech}^2 x$$

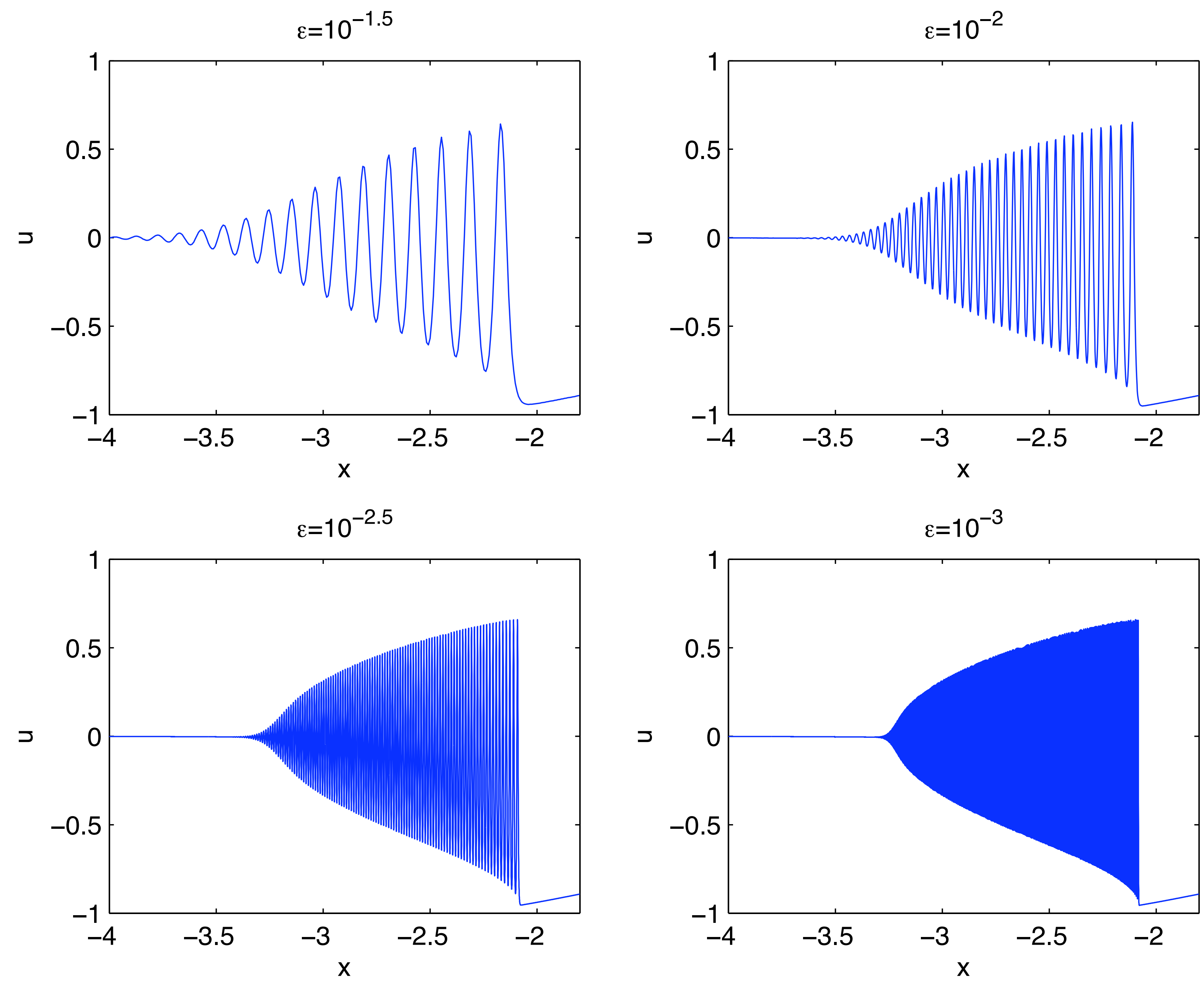
$\epsilon = 0.01$

Korteweg-de Vries equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0 \qquad u_0 = -\mathrm{sech}^2 x$$



Different values of ε



$t = 0.4$

gKdV, small dispersion

$$u_t + \epsilon^2 u_{xxx} + u^n u_x = 0$$

$$u_0 = \mathrm{sech}^2 x, \quad \epsilon = 0.1 \qquad n = 4$$

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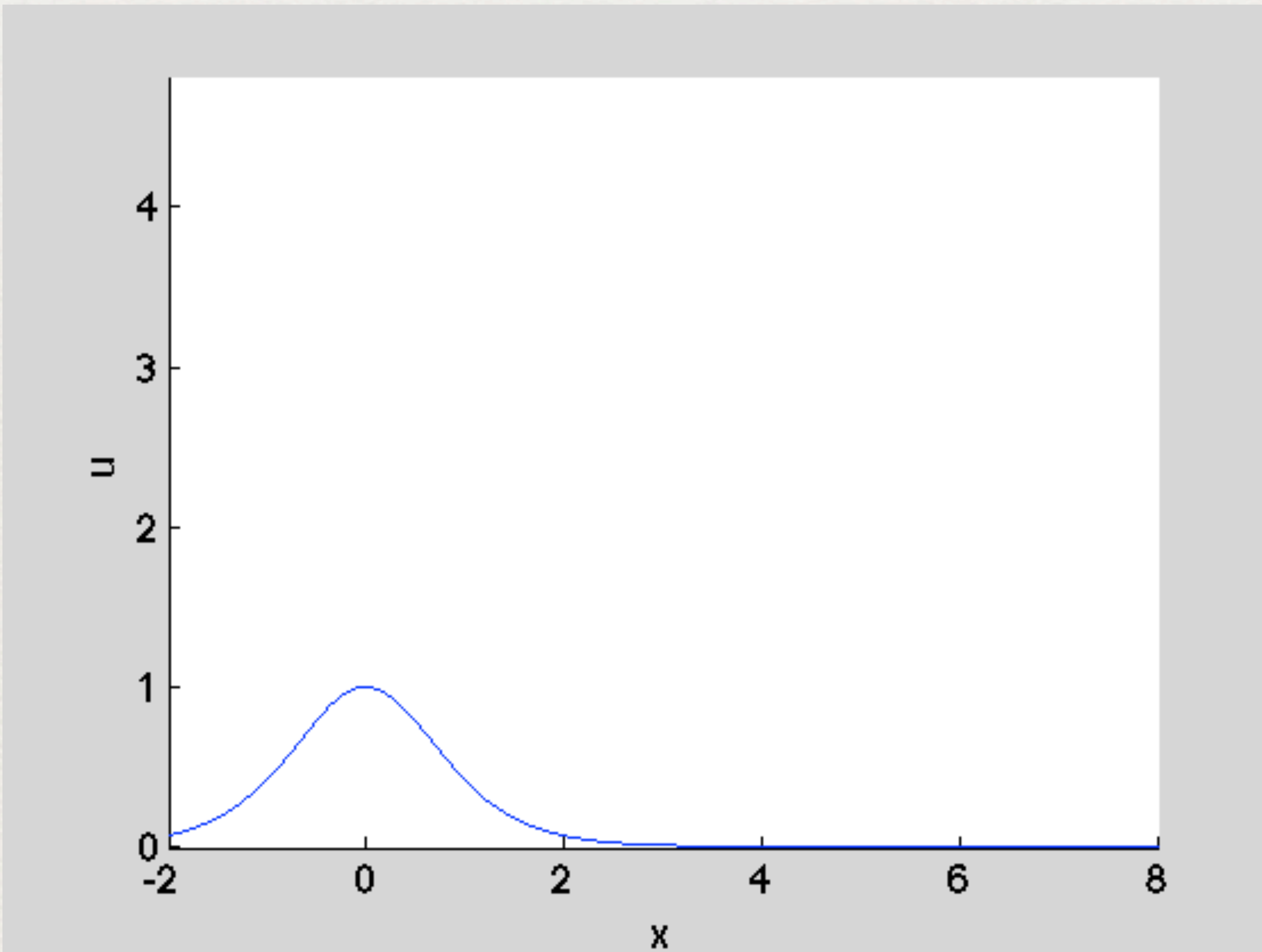

gKdV, small dispersion

$$u_0 = \text{sech}^2 x, \quad \epsilon = 0.01 \qquad n = 4$$

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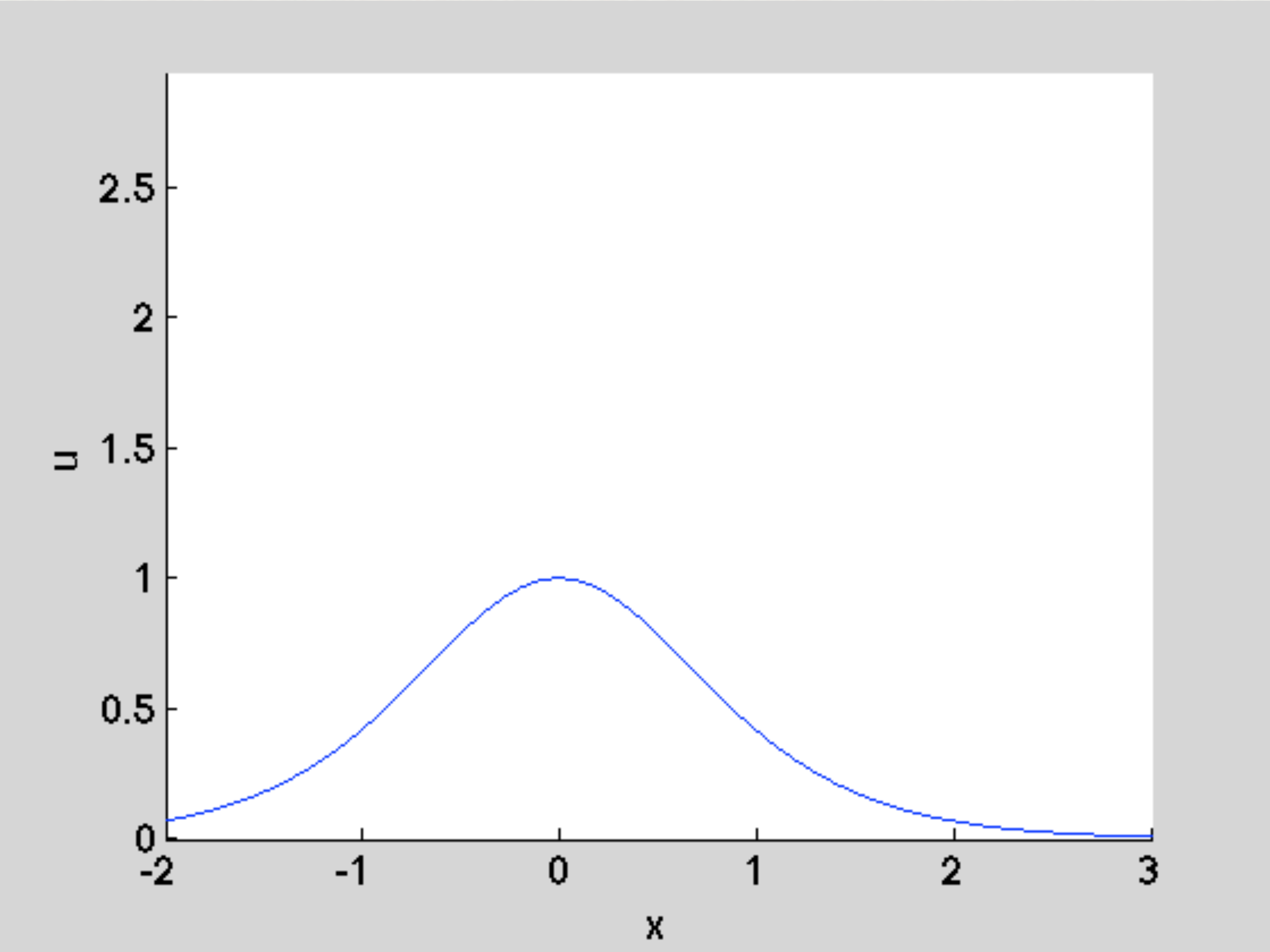
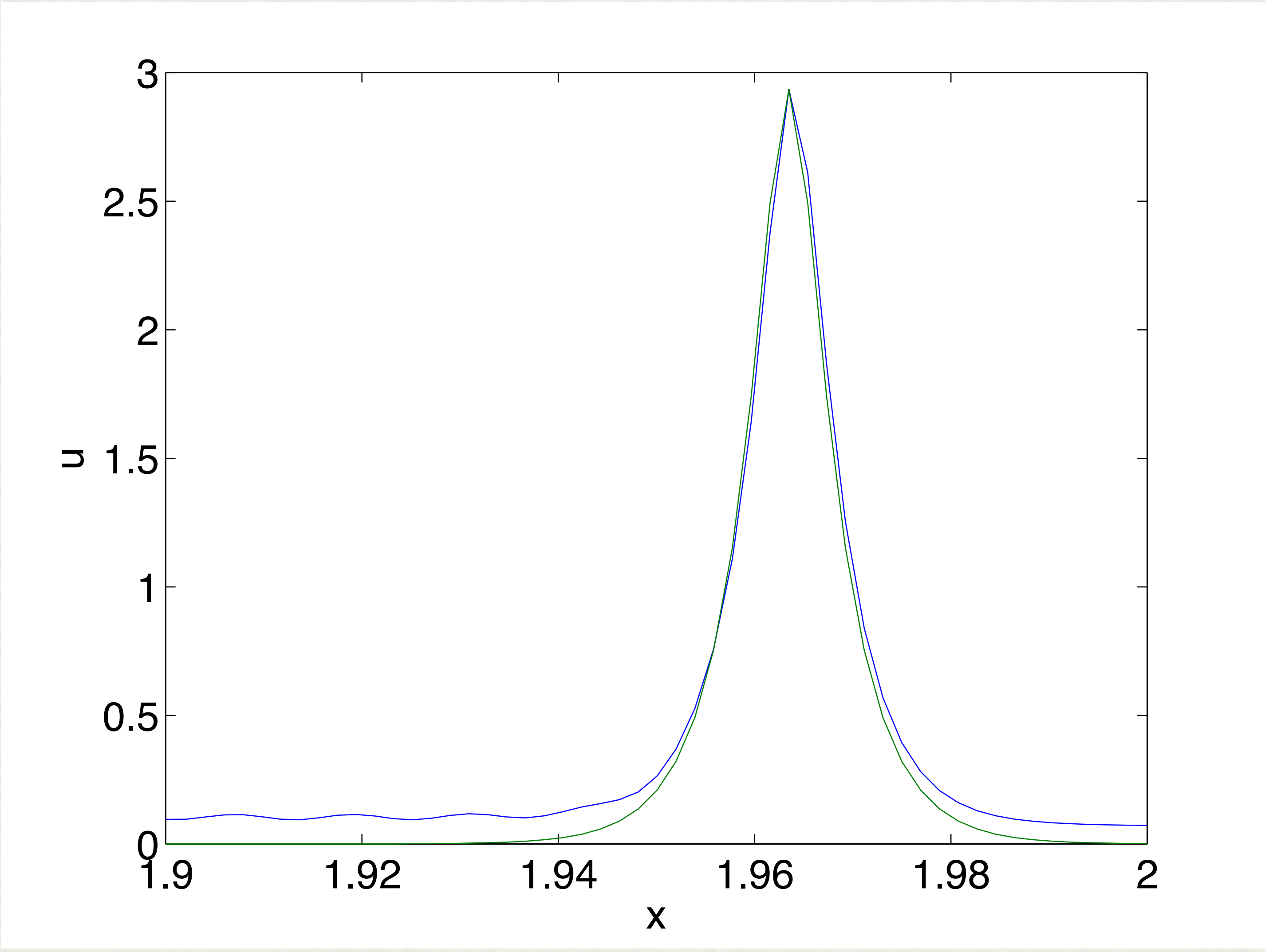


Fitting to rescaled soliton

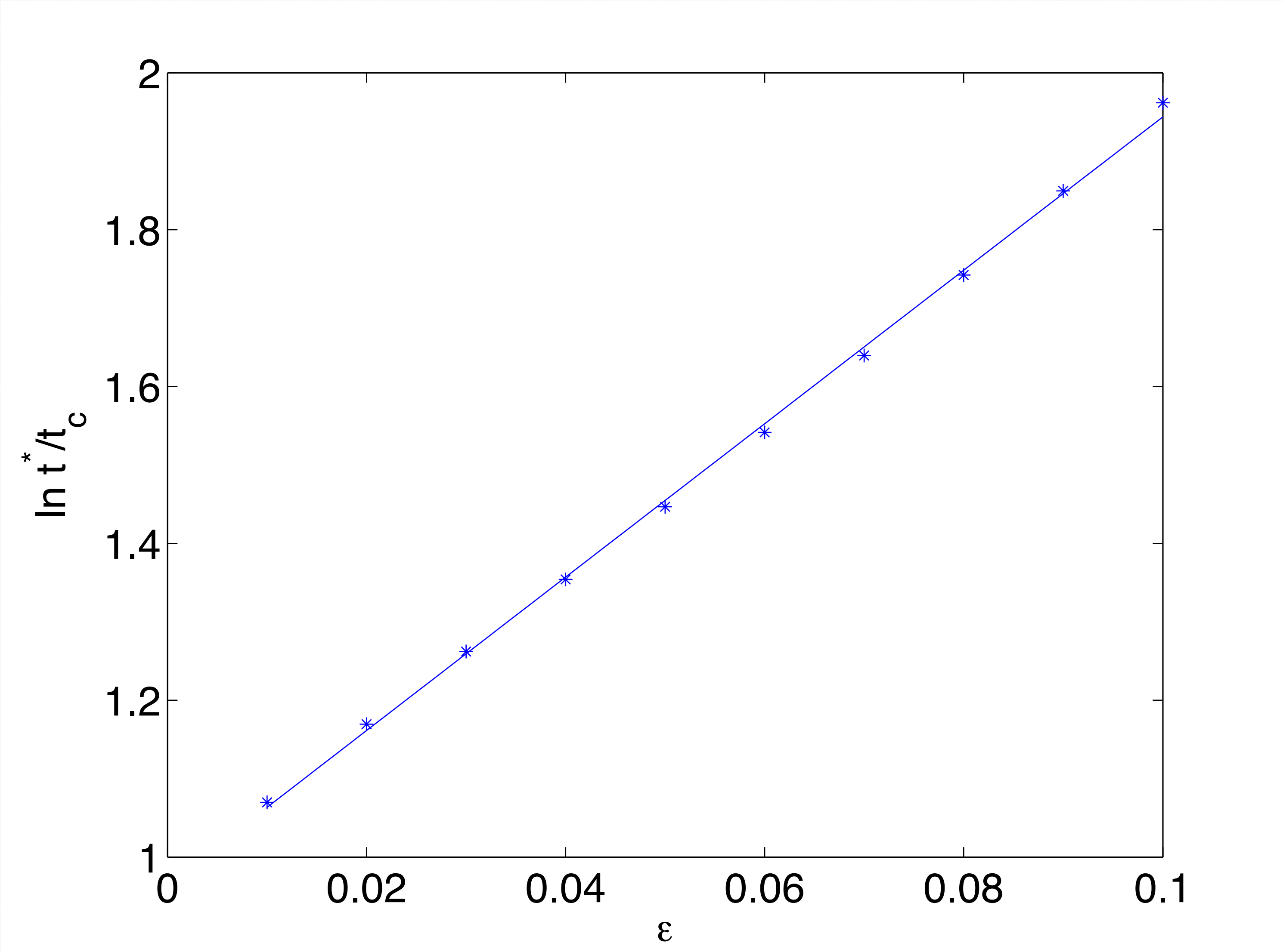
- Martel, Merle, Raphaël 2012: selfsimilar blow-up, blow-up profile dynamically rescaled soliton

C. Klein and R. Peter, *Numerical study of blow-up in solutions to generalized Korteweg-de Vries equations*, Physica D 304-305 (2015), 52-78



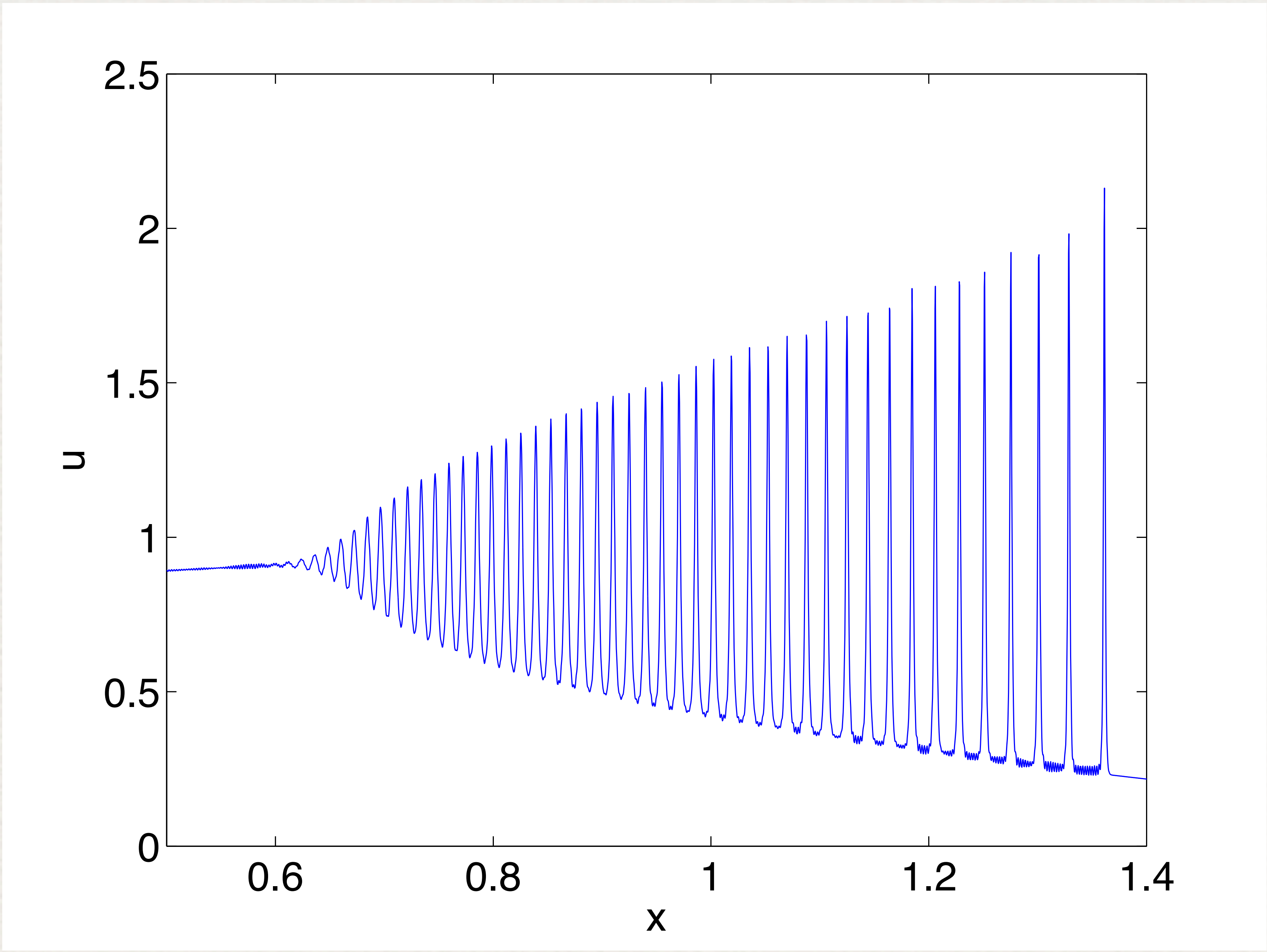
Scaling

- blow-up time t^* always greater than critical time t_c of Hopf ($\epsilon = 0$)
- exponential dependence of blow-up time t^* on ϵ, finite number of solitons appear before blow-up, fastest blows up
- universality?



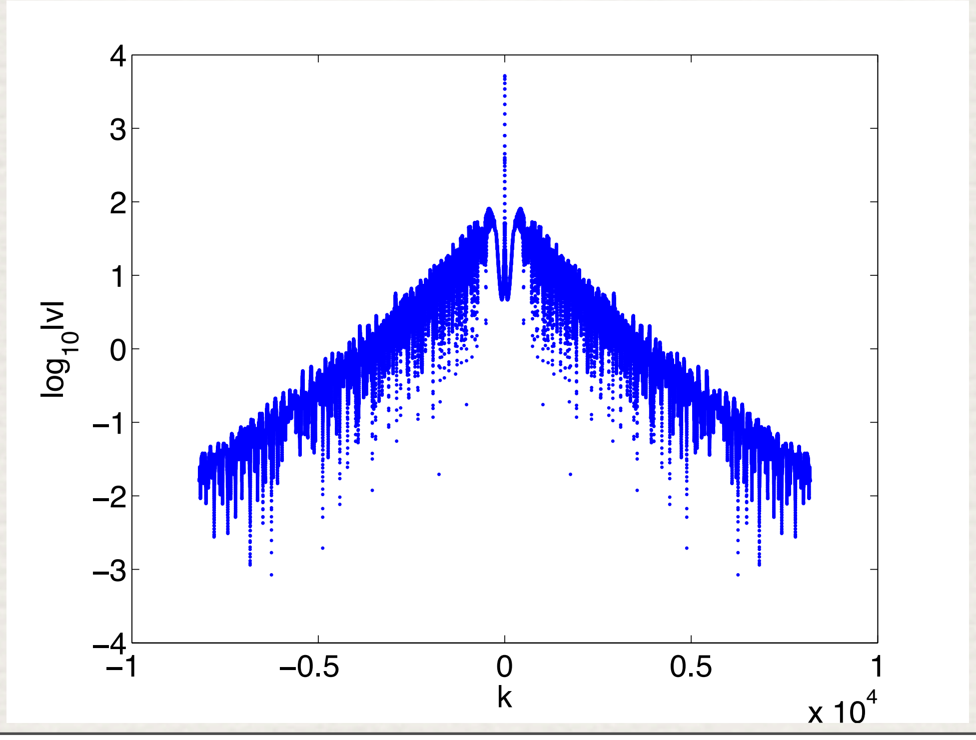
gKdV, small dispersion

$$u_0 = \text{sech}^2 x, \quad \epsilon = 0.001 \qquad n = 4$$



Numerical issues

- task: resolve rapid modulated oscillations, blow-up, high resolution in space and time needed
- spatial discretization: discrete Fourier series;
  advantage: excellent approximation properties for smooth functions, minimizes introduction of numerical viscosity
- stiff ODE system, exponential integrators (ETD)
- slowly decreasing dispersive oscillations, reenter the computational domain; lead to instabilities in the rescaled code, therefore tracing of the norms of the solution of the direct computation
- oscillations in the Fourier coefficients:
  poles in the complex plane at $z_j = \alpha_j - i\delta_j$
  of the form $u \sim (z - z_j)^{\mu_j}$,
  $\hat{u} \sim \sqrt{2\pi}\mu_j^{\mu_j+\frac{1}{2}} e^{-\mu_j} \frac{(-i)^{\mu_j+1}}{k^{\mu_j+1}} e^{-ik\alpha_j - k\delta_j}$



Generalized Kadomtsev-Petviashvili equations

- generalized Kadomtsev-Petviashvili (gKP) equation, $\lambda = -1$ gKP I, $\lambda = 1$ gKP II

$$u_t + u^n u_x + u_{xxx} + \lambda \partial_x^{-1} u_{yy} = 0$$

- nonlocal equation, algebraic decrease towards infinity of the solution even for rapidly decreasing initial data

- constraint

$$\int_{\mathbb{R}} \partial_{yy} u(x, y, t)\, dx = 0, \quad \forall t > 0$$

if not satisfied by the initial condition, solution not regular in t

- numerical study of blow-up by Wang, Ablowitz, Segur (1994)

- gKP I solitons (de Bouard, Saut 1997), unstable for $n \geq 4/3$

$$-cQ_{zz} + \frac{1}{n+1}(Q^{n+1})_{zz} + Q_{zzzz} + \lambda Q_{yy} = 0$$

Dynamic rescaling

C. Klein and R. Peter, *Numerical study of blow-up in solutions to generalized Kadomtsev-Petviashvili equations*, Discr. Cont. Dyn. Syst. B 19(6), (2014) doi:10.3934/dcdsb.2014.19.1689

- coordinate change

$$\xi = \frac{x - x_m}{L}, \quad \eta = \frac{y - y_m}{L^2}, \quad \frac{d\tau}{dt} = \frac{1}{L^3}, \quad U = L^{2/n} u$$

$||u||_2$ invariant for $n = 4/3$

- rescaled equation

$$U_\tau - a\left(\frac{2}{n}U + \xi U_\xi + 2\eta U_\eta\right) - v_\xi U_\xi - v_\eta U_\eta + U^n U_\xi + U_{\xi\xi\xi} + \lambda \int_{-\infty}^{\xi} U_{\eta\eta}\, d\xi = 0$$

- blow-up

$$-a^\infty\left(\frac{2}{n}U^\infty + \xi U^\infty_\xi + 2\eta U^\infty_\eta\right) - v^\infty_\xi U^\infty_\xi - v^\infty_\eta U^\infty_\eta + (U^\infty)^n U^\infty_\xi + \epsilon^2 U^\infty_{\xi\xi\xi} + \lambda \int_{-\infty}^{\xi} U^\infty_{\eta\eta}\, d\xi = 0$$

- numerical instabilities due to algebraic decay of the solutions

gKP I, critical case

$$n = 4/3, \quad u_0 = 12\partial_{xx}\exp(-x^2 - y^2)$$

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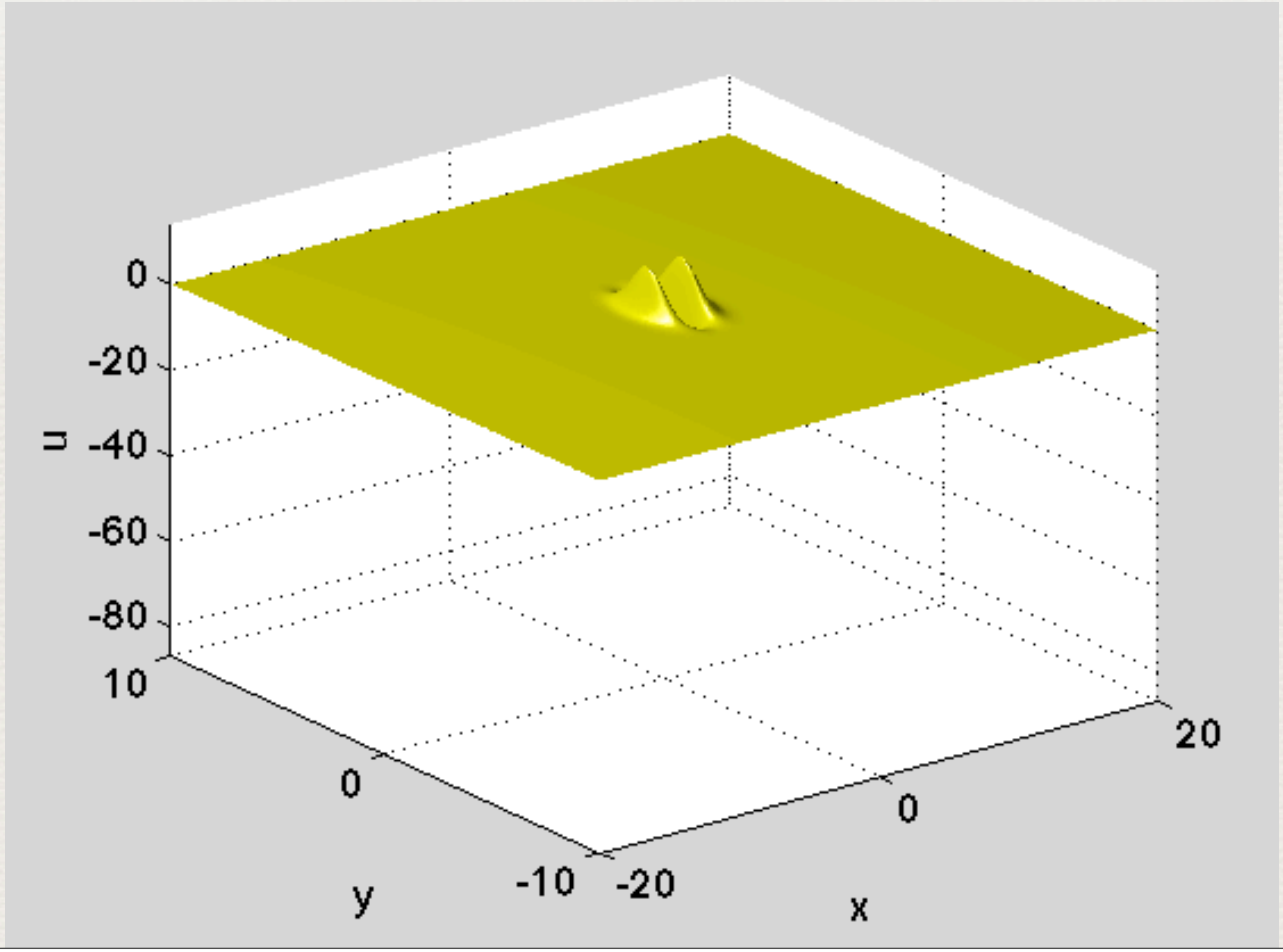
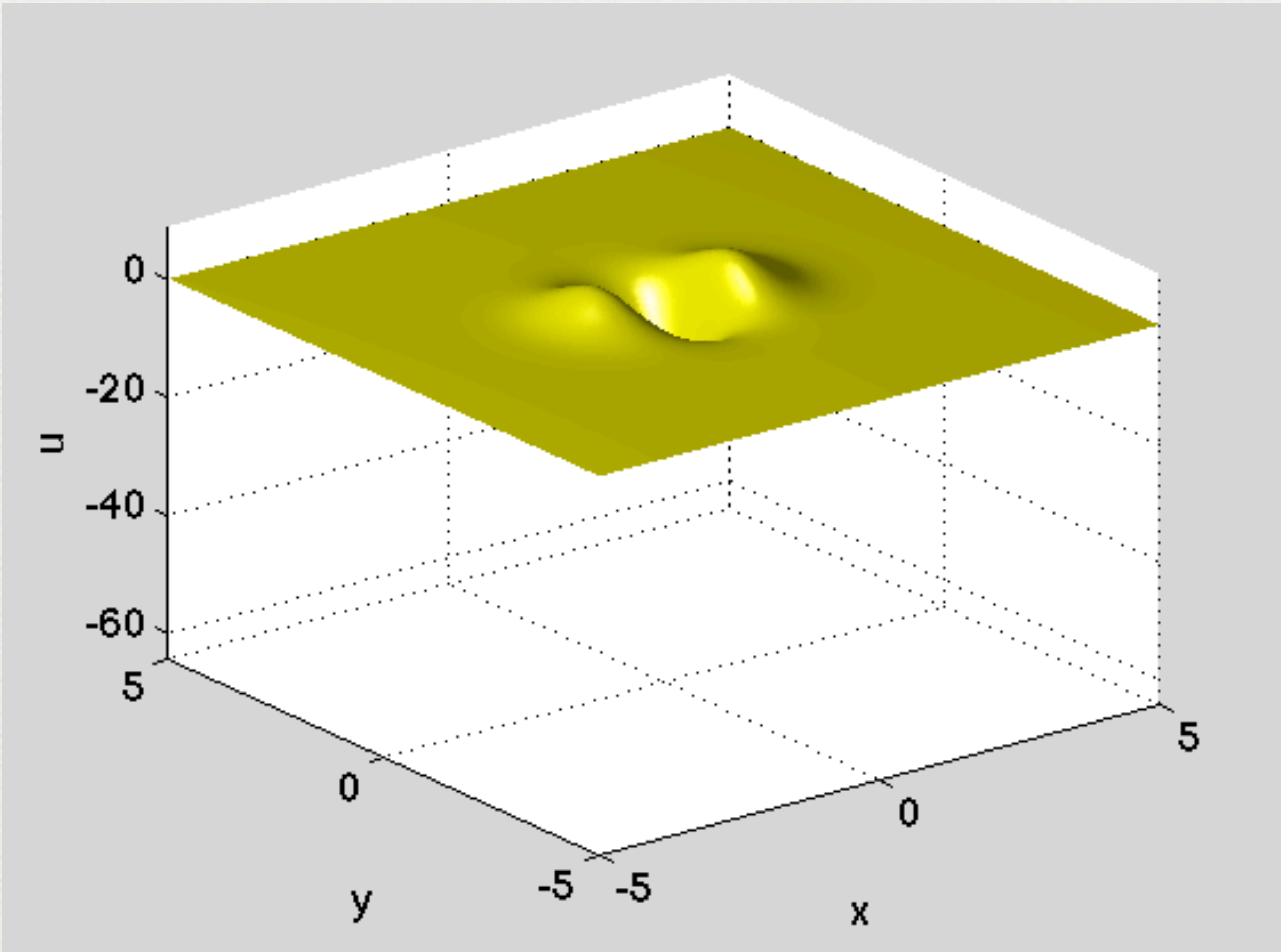


gKP I, supercritical case

$$n = 2, \quad u_0 = 6\partial_{xx}\exp(-x^2 - y^2)$$

gKP I, supercritical case

$$n = 2, \quad u_0 = 6\partial_{xx}\exp(-x^2 - y^2)$$



Conjecture

- for $n < 4/3$, the solution is smooth for all t.
- for gKP II, the solution is smooth for all t for $n \leq 2$.
- for gKP I with $n = 4/3$, initial data with sufficiently small energy and sufficiently large mass lead to blow-up at $t^* < \infty$; asymptotically for $t \sim t^*$, the solution is given by a rescaled soliton where the scaling factor $L \propto 1/\tau$ for $\tau \to \infty$. This implies the blow-up is characterized by

$$||u||_\infty \propto \frac{1}{(t^* - t)^{3/4}}, \quad ||u_y||_2 \propto \frac{1}{t^* - t}. \tag{1}$$

- for gKP I with $n > 4/3$ and gKP II with $n > 2$, initial data with sufficiently small energy and sufficiently large mass lead to blow-up at $t^* < \infty$; asymptotically for $t \sim t^*$, the solution is given by a localized solution to the asymptotic PDE, which is conjectured to exist and to be unique, after rescaling where the scaling factor $L \propto \exp(\kappa\tau)$ for $\tau \to \infty$ with κ a negative constant. This implies the blow-up is characterized by

$$||u||_\infty \propto \frac{1}{(t^* - t)^{2/(3n)}}, \quad ||u_y||_2 \propto \frac{1}{(t^* - t)^{(1+4/n)/6}}. \tag{2}$$

Semiclassical limit of NLS

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + V(|\psi|^2)\psi = 0\,.$$

Introducing the slow variables

$$u = |\psi|^2, \quad v = \frac{\epsilon}{2i}\left(\frac{\psi_x}{\psi} - \frac{\bar{\psi}_x}{\bar{\psi}}\right)$$

the NLS can be written in the form

$$u_t + (uv)_x = 0 \tag{1}$$

$$v_t + vv_x - \partial_x V(u) + \frac{\epsilon^2}{4}\left(\frac{u_x^2}{2u^2} - \frac{u_{xx}}{u}\right)_x = 0. \tag{2}$$

Riemann Invariants $r_\pm = v \pm Q(u)$, $Q'(u) = \sqrt{\frac{-V'(u)}{u}}$,
Characteristic velocity: $\lambda_\pm = v \pm \sqrt{-V'(u)u}$

Defocusing NLS

$$\psi_0(x) = \exp(-x^2),$$

$$\epsilon = 0.5,$$

$$0 \leq t \leq 1,$$

$$u = |\psi|^2$$

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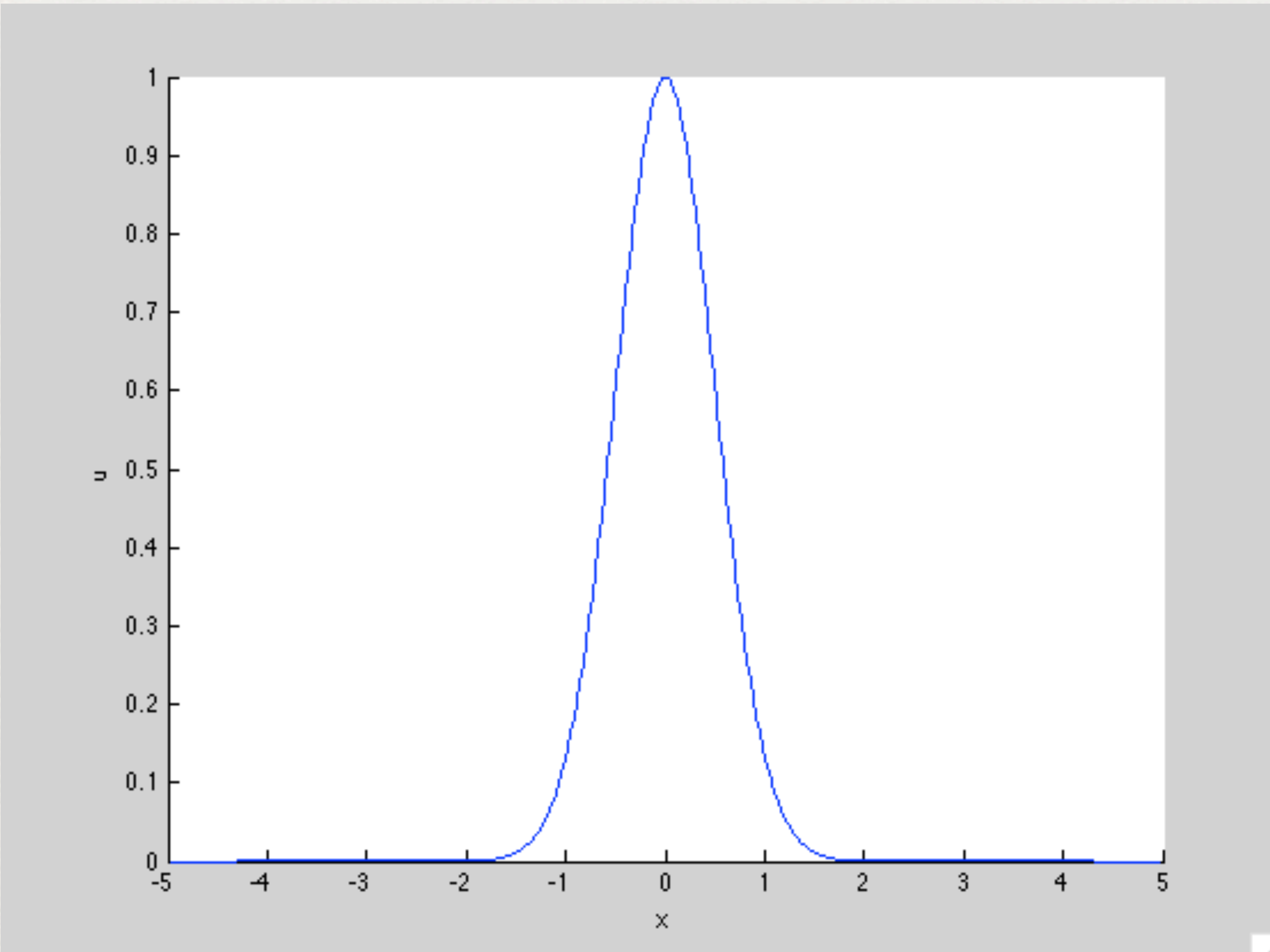


Focusing NLS

$$\psi_0(x) = \exp(-x^2),$$

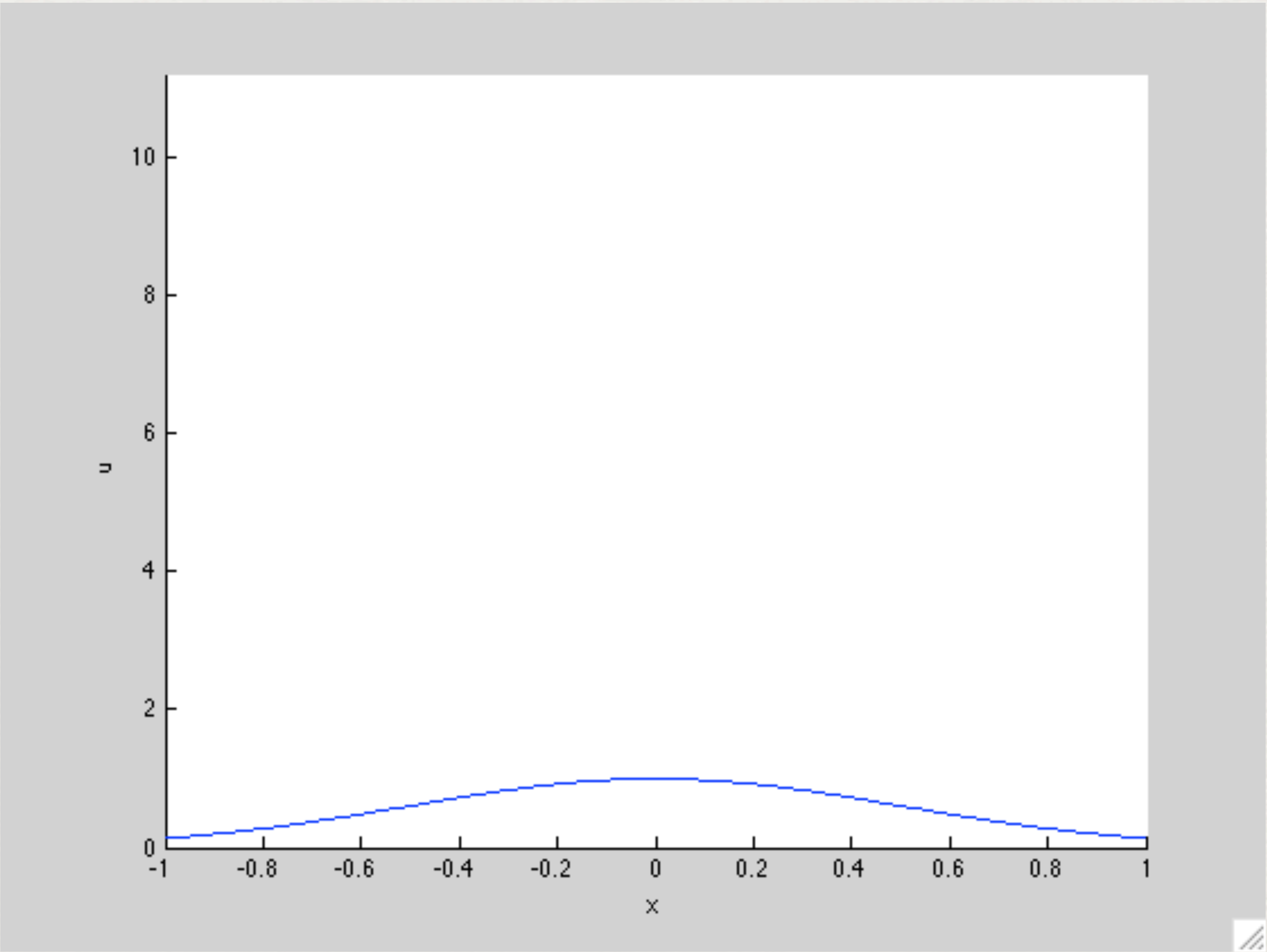
$$\epsilon = 0.1,$$

$$0 \leq t \leq 0.8,$$

$$u = |\psi|^2$$

Focusing NLS

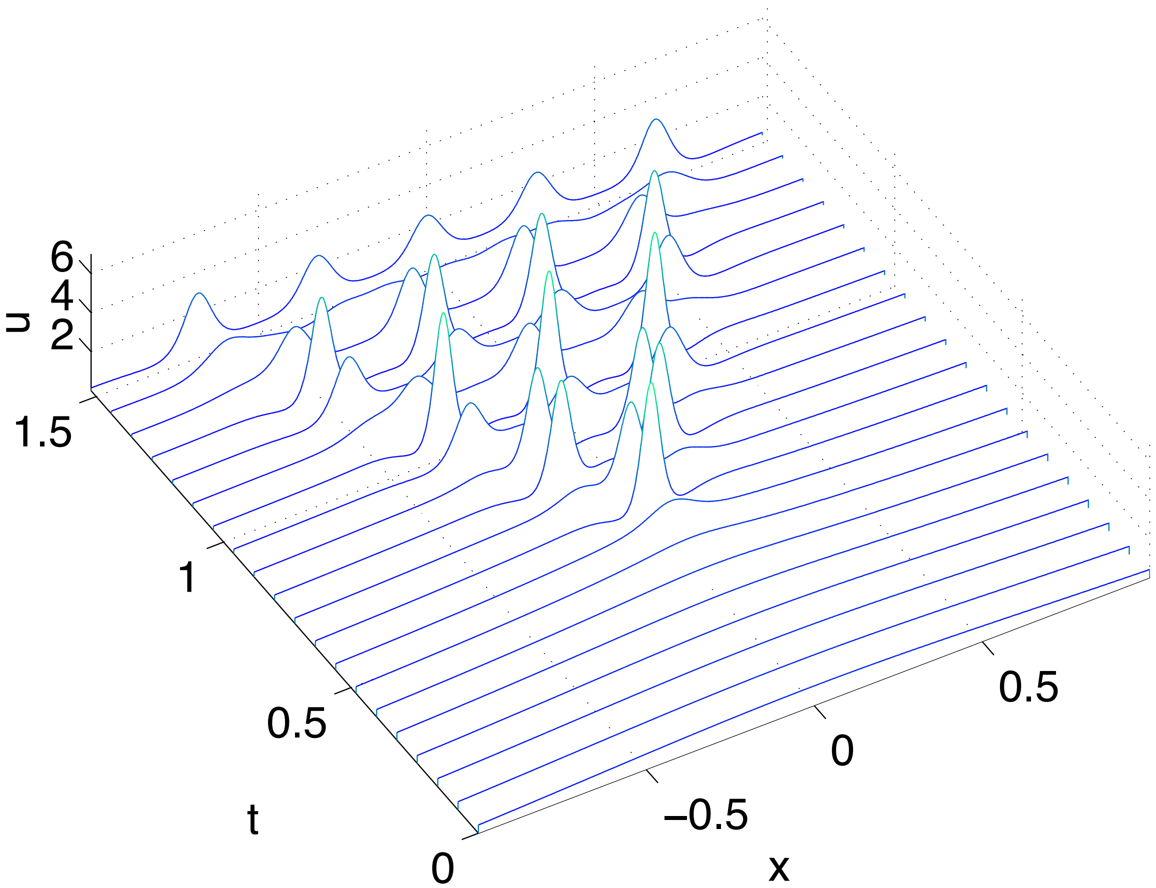
$$\psi_0(x) = \exp(-x^2),$$
$$\epsilon = 0.1,$$
$$0 \leq t \leq 0.8,$$
$$u = |\psi|^2$$

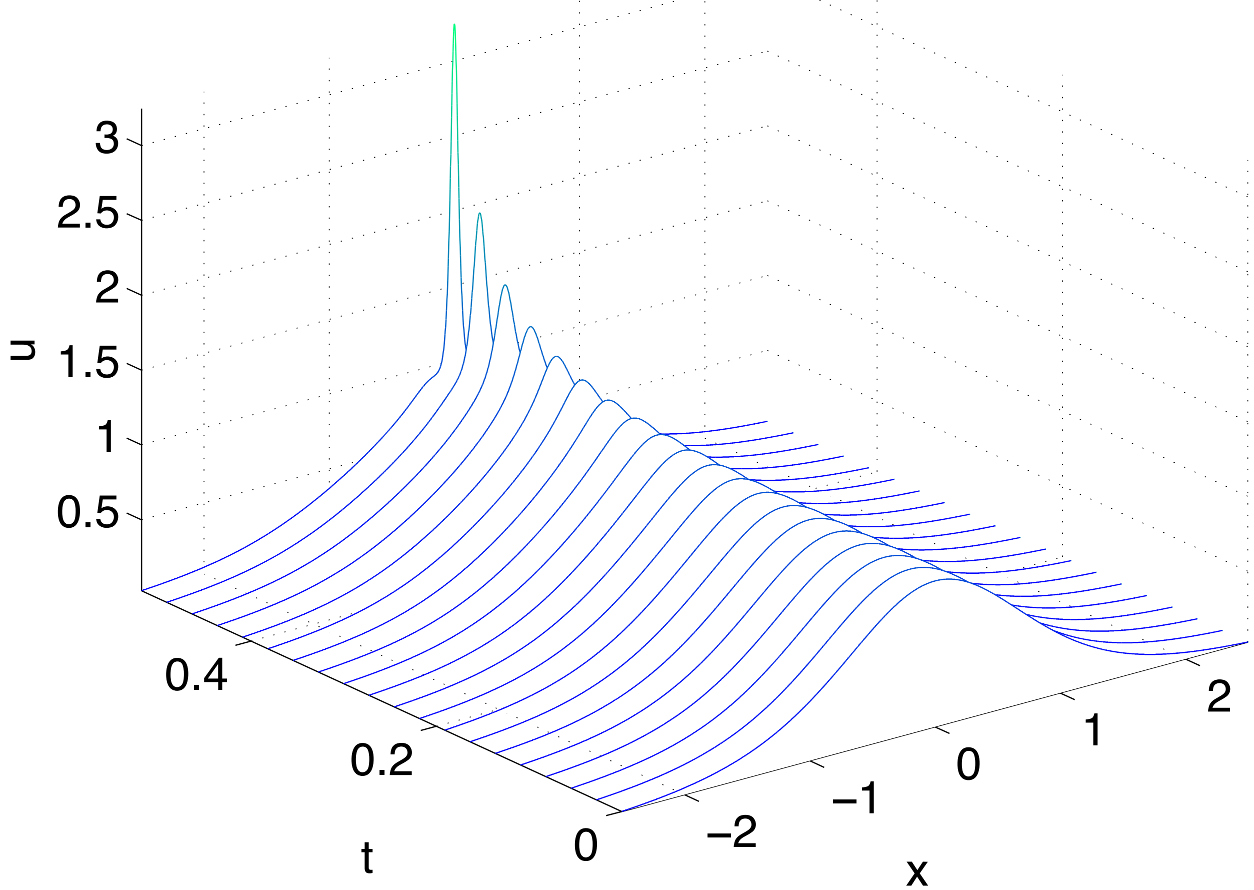


Blow-up

- unstable blow-up: $||\Psi||_\infty \propto 1/(t^* - t)$, stable blow-up: $||\Psi||_\infty \propto 1/\sqrt{t^* - t}$
- $t^* - t^c = 0(\epsilon^{4/5})$
- pole for $\epsilon \to 0$ given by pole of tritronquée solution

cubic NLS $\qquad\qquad$ $\psi_0 = \mathrm{sech} x$ $\qquad\qquad$ quintic NLS





Davey-Stewartson equation

C. Klein and K. Roidot, *Numerical Study of the semiclassical limit of the Davey-Stewartson II equations*, Nonlinearity 27, 2177-2214 (2014).

- $$\begin{aligned} i\epsilon u_t + \epsilon^2 u_{xx} - \alpha\epsilon^2 u_{yy} + 2\rho\left(\Phi + |u|^2\right) u &= 0 \\ \Phi_{xx} + \alpha\Phi_{yy} + 2\,|u|^2_{xx} &= 0 \end{aligned}$$

- integrable cases: $\alpha = \pm 1$, $\rho = \pm 1$
  - DS I, $\alpha = -1$
  - DS II, hyperbolic-elliptic, $\alpha = 1$

- y-independent potential plus boundary condition at infinity: reduction to NLS

- first numerical studies: White-Weideman (1994), Besse, Mauser, Stimming (2004), McConnell, Fokas, Pelloni (2005)

DS II

- mean field Φ: defocusing ($\rho = -1$) and focusing case ($\rho = 1$) different
- elliptic operator for Φ can be inverted with periodic boundary conditions
- Sung 1995: initial data $\psi_0 \in L^p$, $1 \leq p < 2$ with Fourier transform $\hat{\psi}_0 \in L^1 \cap L^\infty$, smallness condition

$$||\hat{\psi}_0||_{L^1}||\hat{\psi}_0||_{L^\infty} < \frac{\pi^3}{2}\left(\frac{\sqrt{5}-1}{2}\right)^2$$

  no condition for defocusing case

- initial data $u_0 = \exp(-x^2 - \eta y^2)$: Sung condition

$$\frac{1}{\epsilon^2\eta} \leq \frac{1}{8}\left(\frac{\sqrt{5}-1}{2}\right)^2 \sim 0.0477.$$

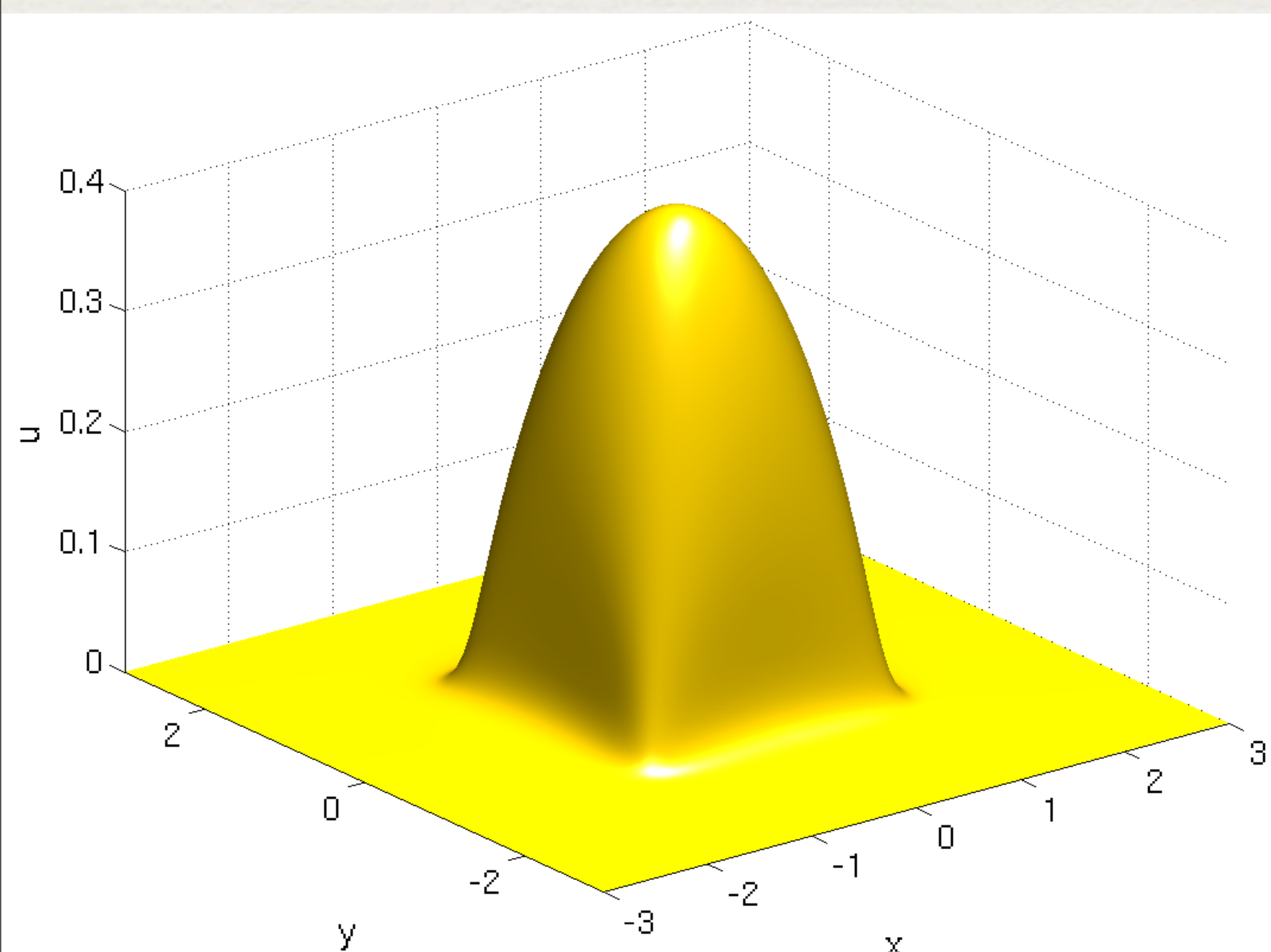
- Ozawa 1992: exact blowup solution for lump-like initial data

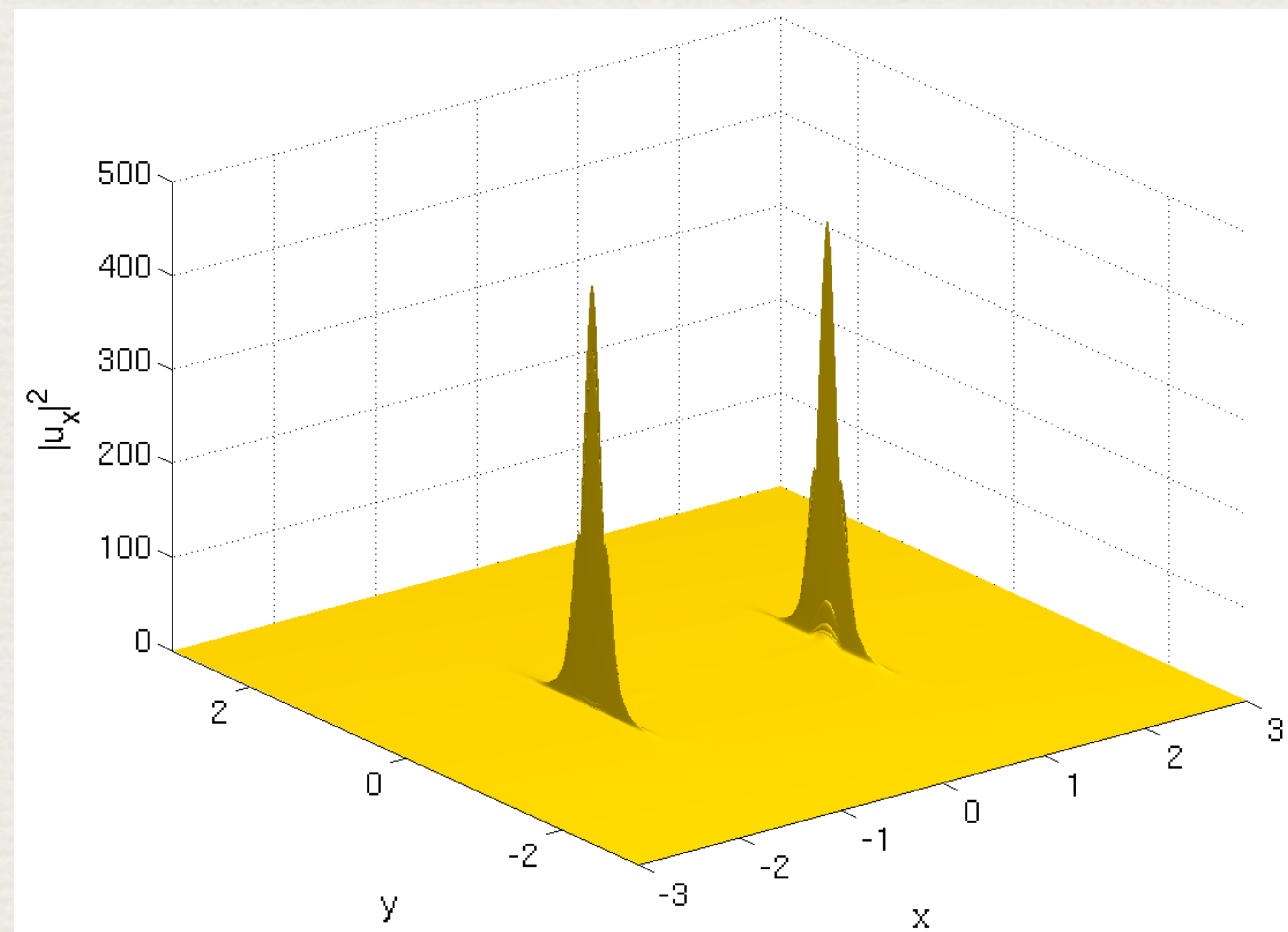
Semiclassical limit

- semiclassical limit ($\Psi = \sqrt{u}e^{iS/\epsilon}$, $\epsilon \to 0$, $\mathcal{D}_\pm = \partial_x^2 \pm \partial_y^2$)

$$\begin{cases} S_t + S_x^2 - S_y^2 + 2\rho\mathcal{D}_+^{-1}\mathcal{D}_-(u) & = & \frac{\epsilon^2}{2}\left(\frac{u_x x}{u} - \frac{u_x^2}{u^2} - \frac{u_y y}{u} + \frac{u_y^2}{u}\right) \\ u_t + 2\left(S_x u\right)_x - 2\left(S_y u\right)_y & = & 0 \end{cases},$$

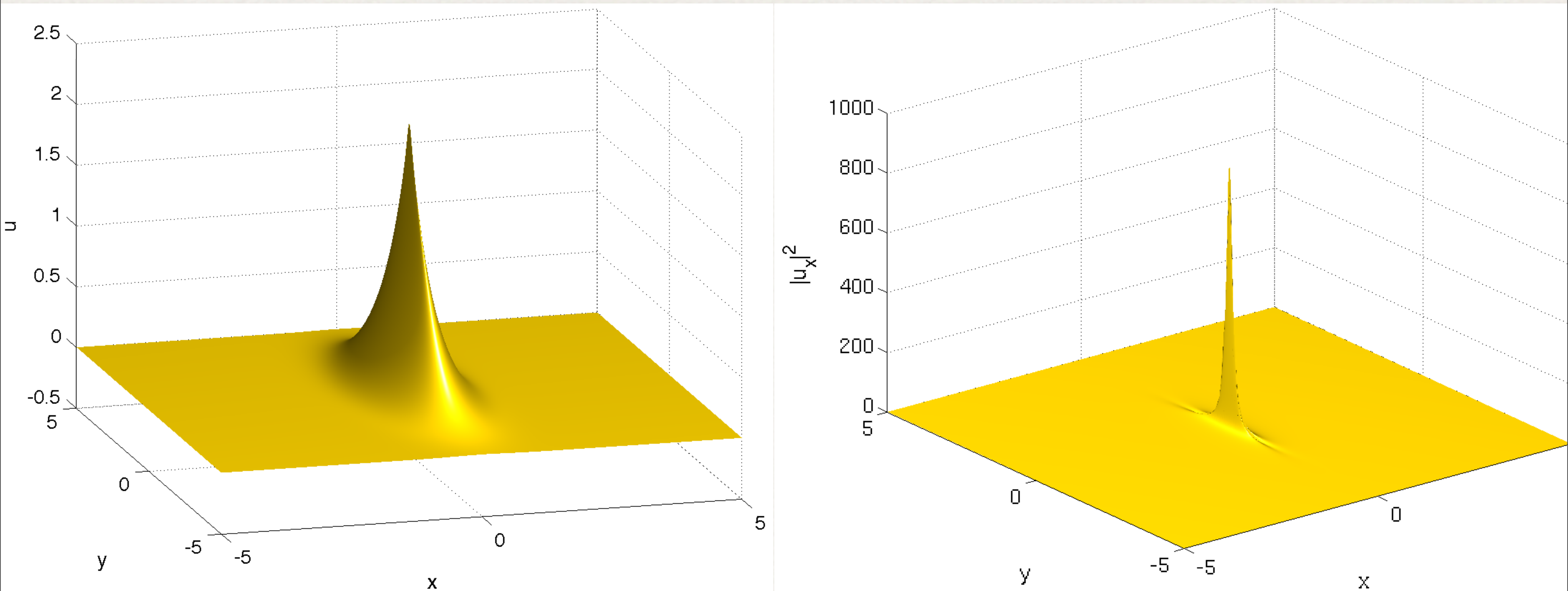
- defocusing case, $u_0 = \exp(-2(x^2+y^2))$, $S_0 = 0$



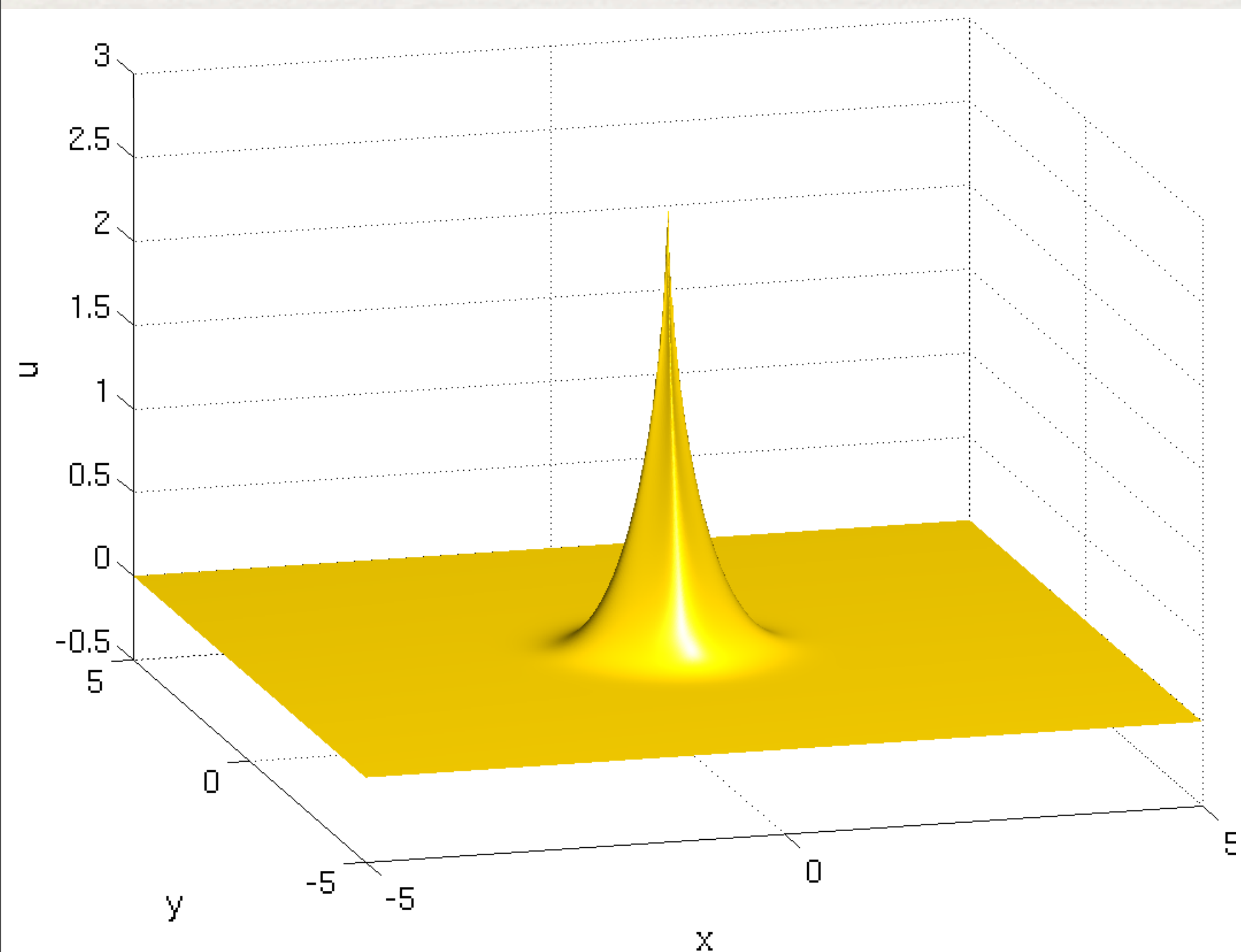


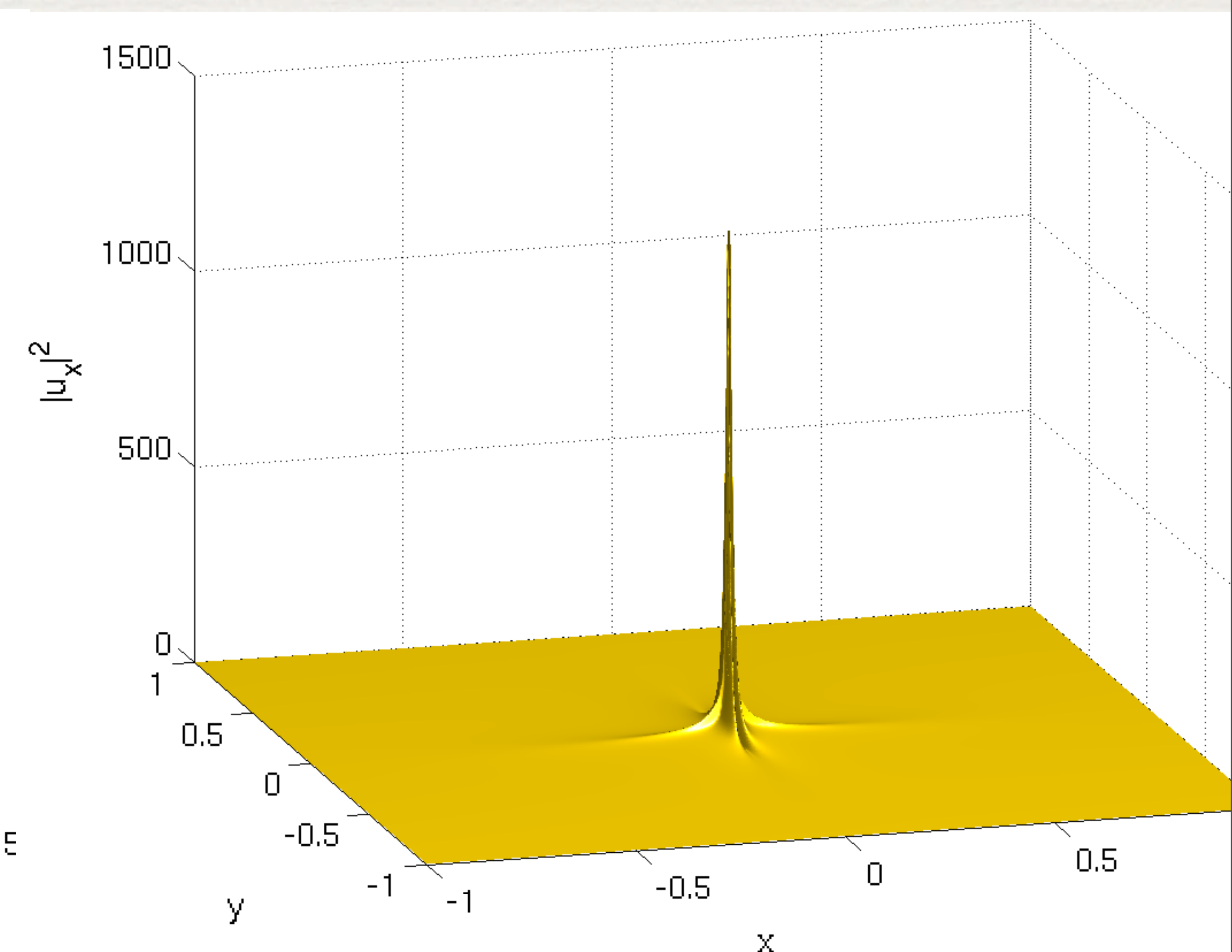
Focusing semiclassical DS II system

- $u_0 = \exp(-2(x^2 + 0.1y^2))$, $S_0 = 0$



Symmetric initial data





Defocusing DS II

$u_0 = \exp(-x^2 - y^2)$

$\epsilon = 0.1$

Defocusing DS II

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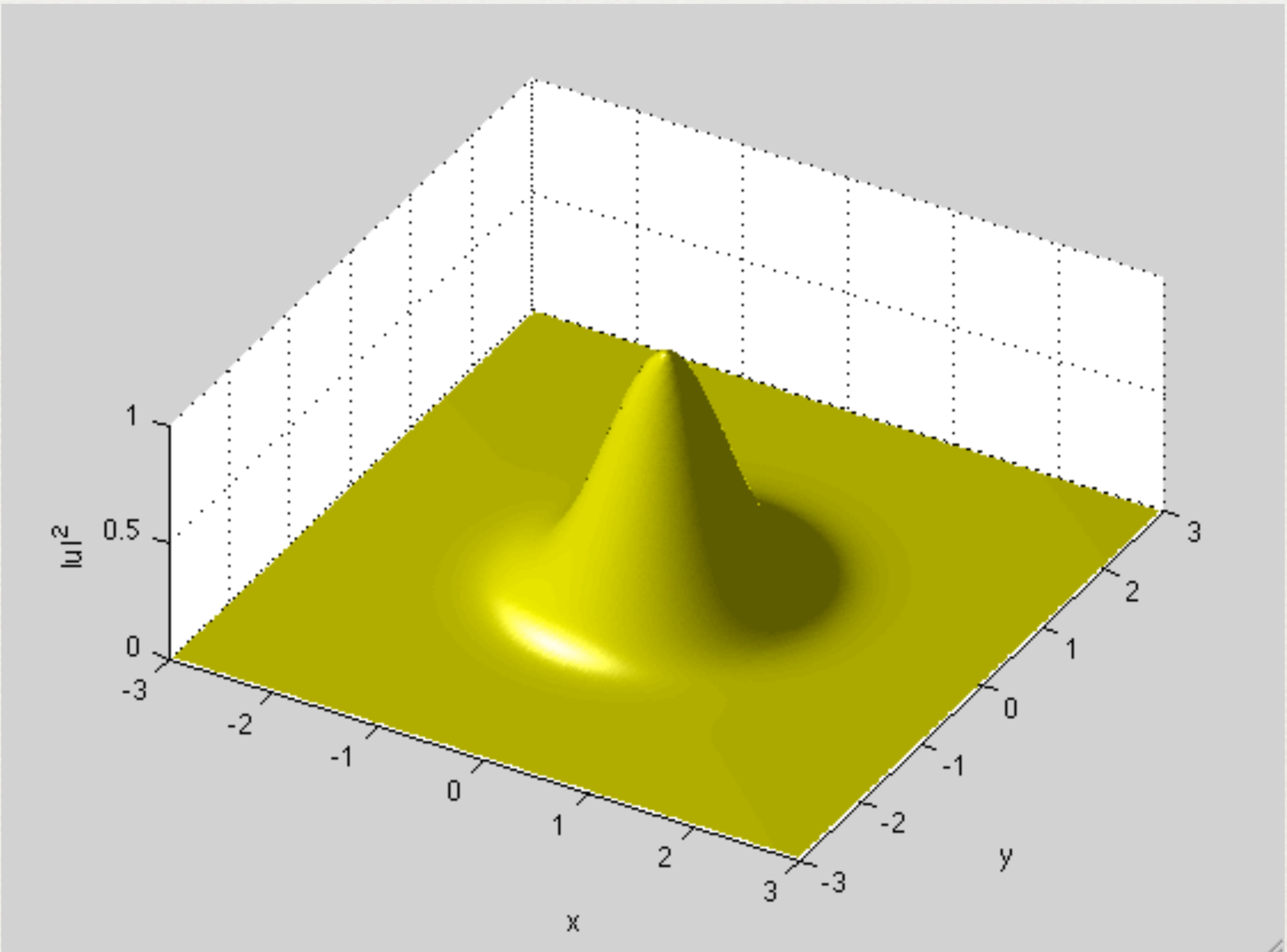
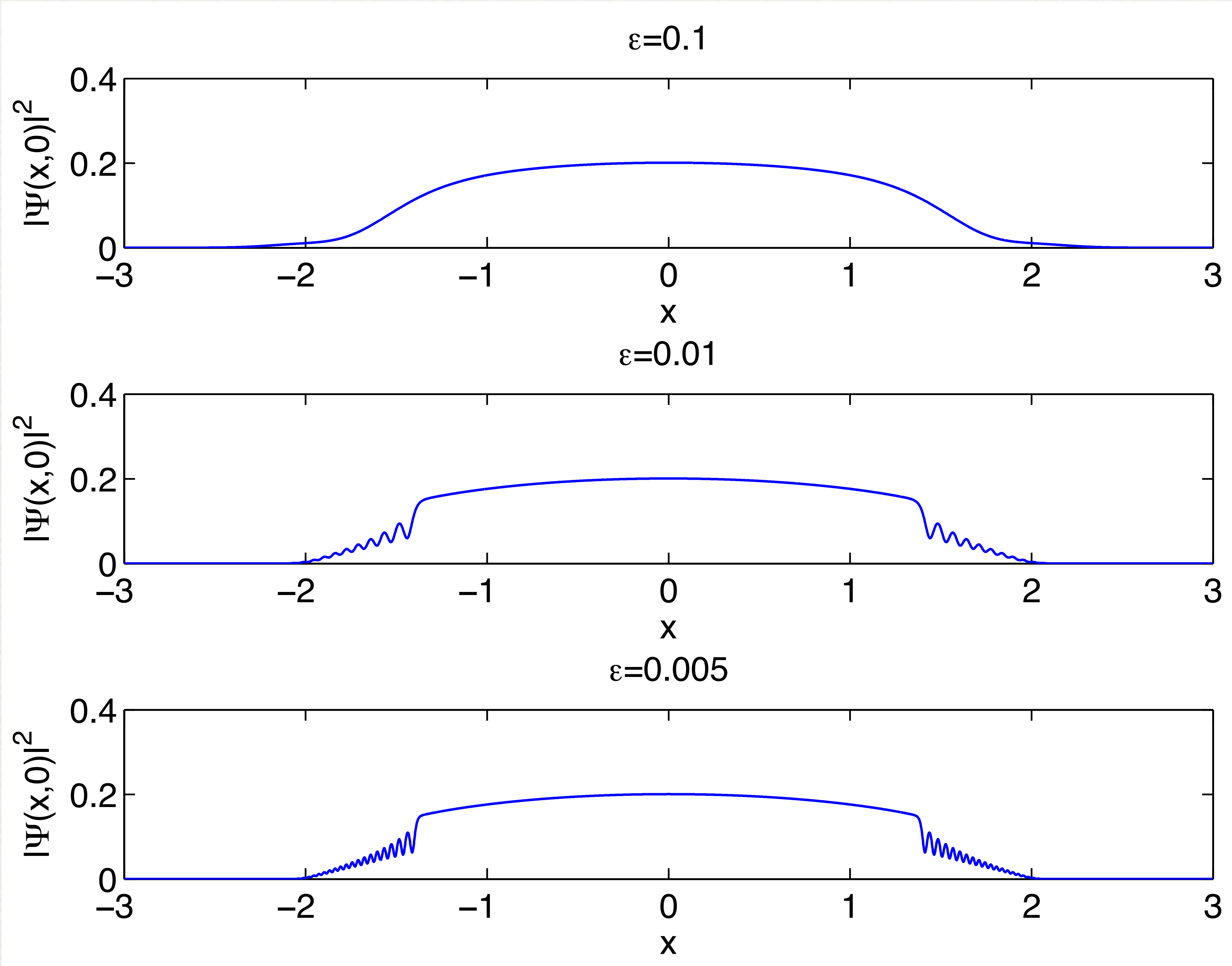


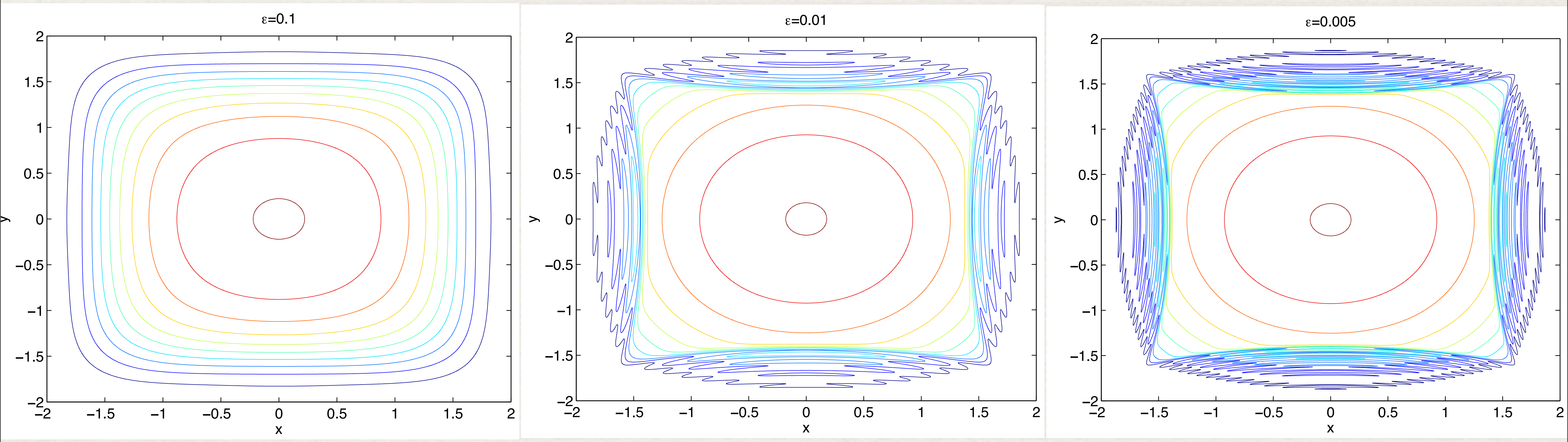
$$\epsilon = 0.1$$

Defocusing DS II



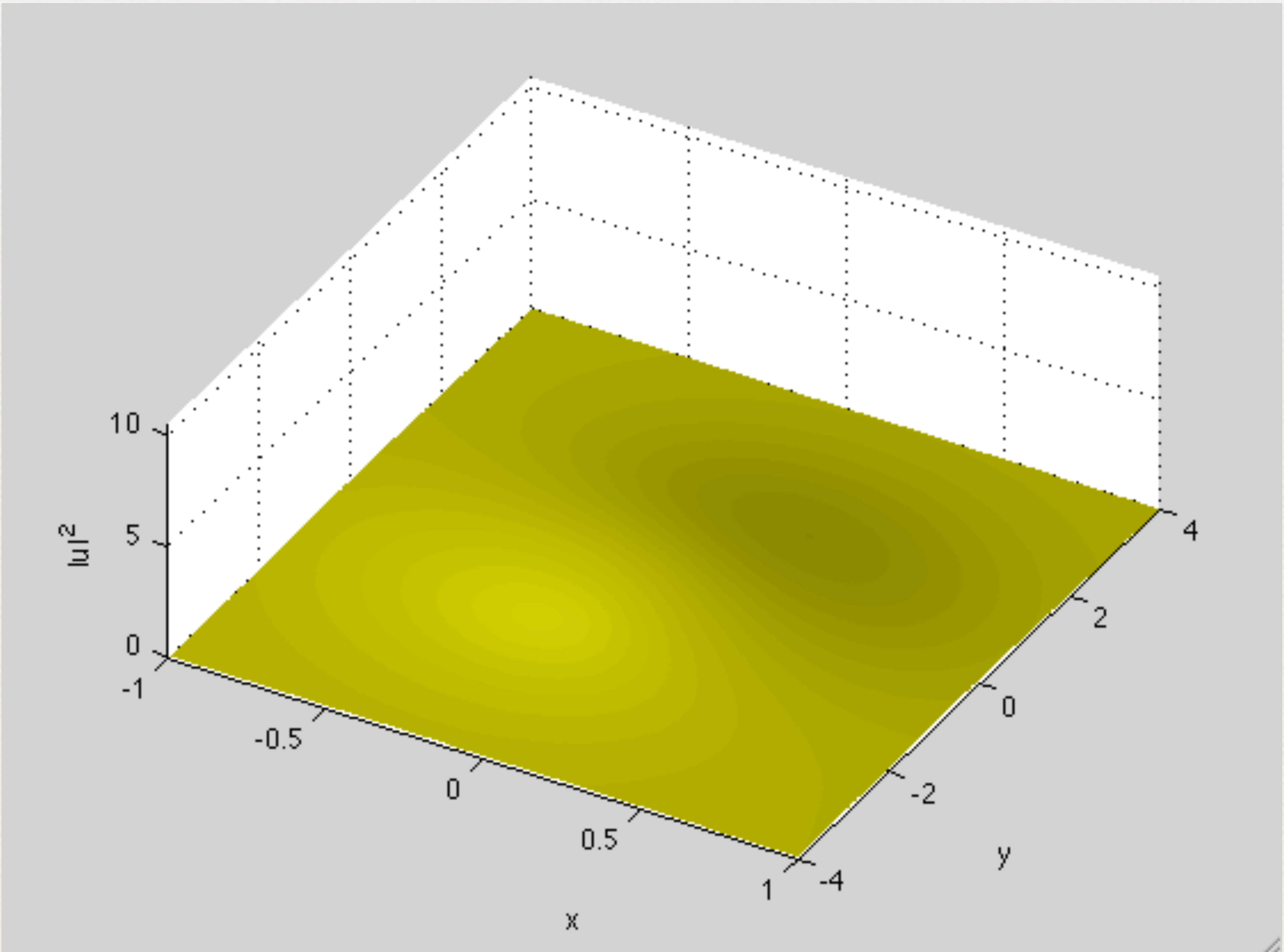
Defocusing DS II

- $t = t_c$: scaling of the difference between semiclassical and DS II solution proportional to $\epsilon^{2/7}$
- $t \gg t_c$: dispersive shock



Focusing DS

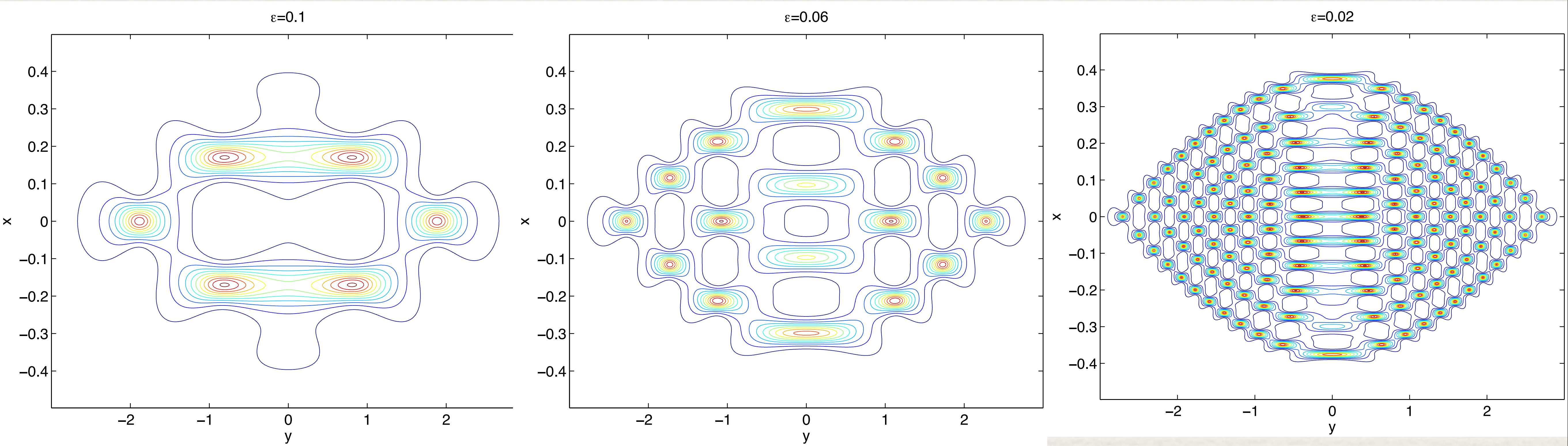
$u_0 = \exp(-x^2 - 0.1y^2)$



$\epsilon = 0.1$

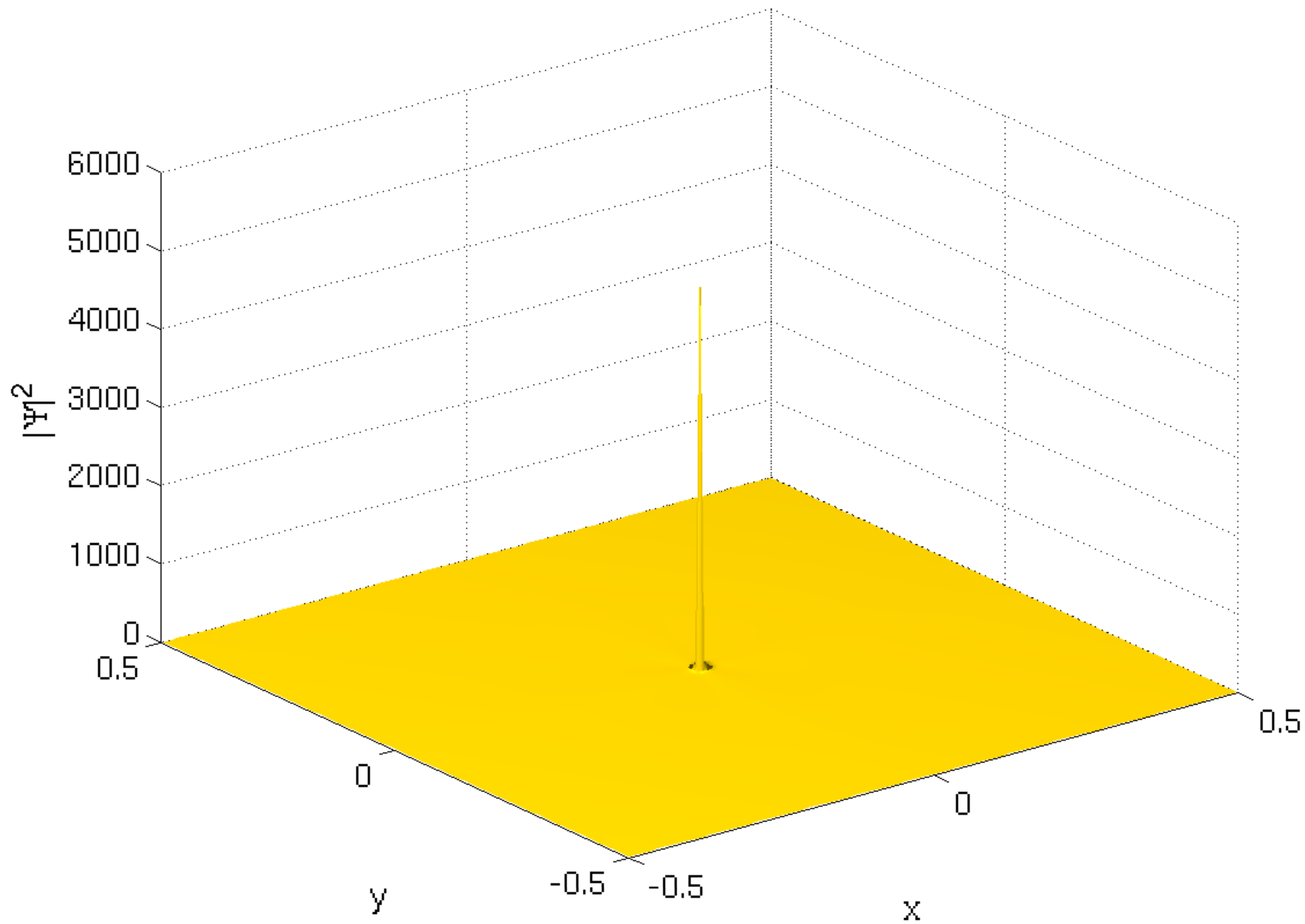
Focusing DS II

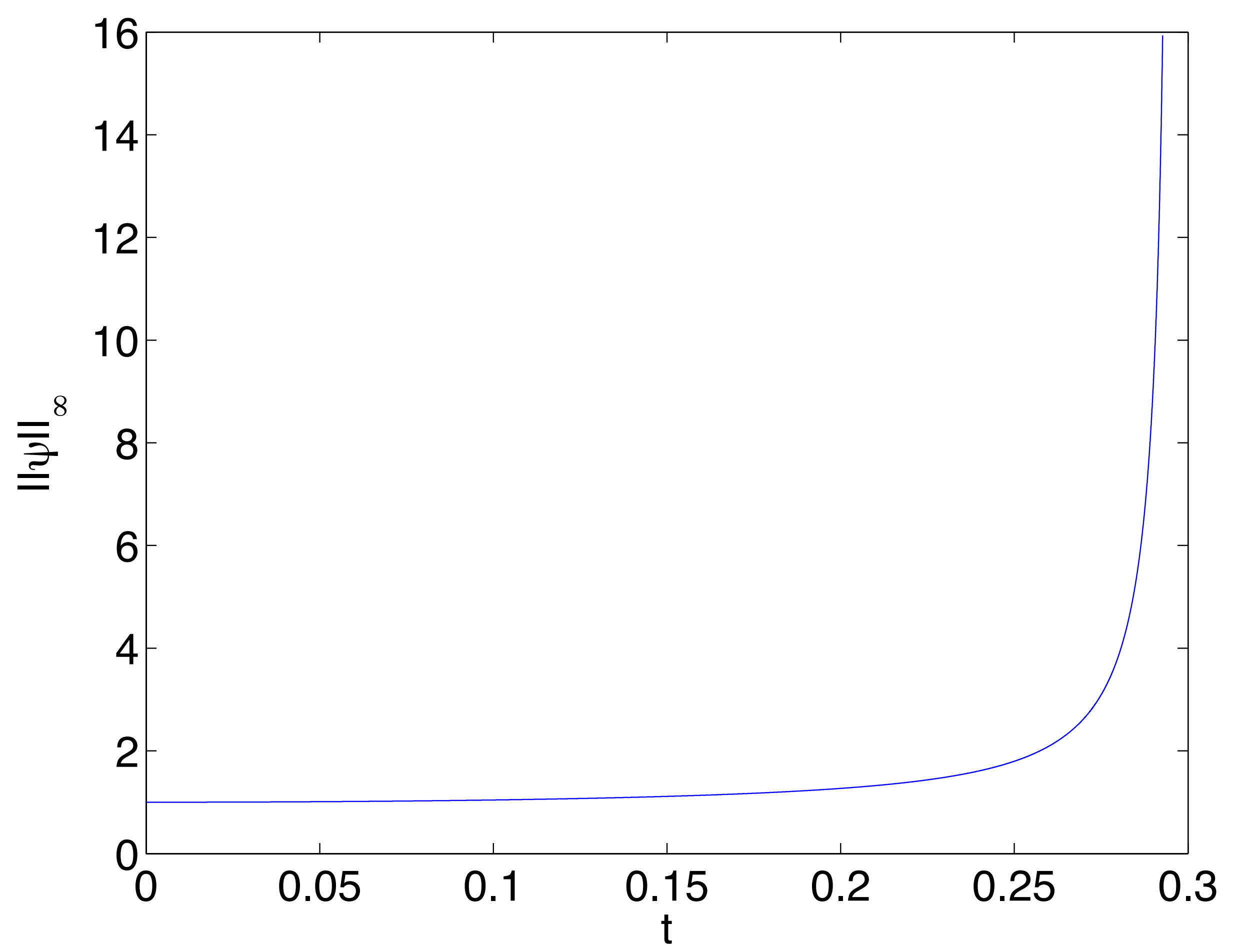
- $t = t_c$: scaling of the difference between semiclassical and DS II solution proportional to $\epsilon^{2/5}$
- $t \gg t_c$: dispersive shock for non-symmetric initial data



Blow-up

- finite time blow-up for symmetric initial data
- not as in Ozawa ($||\Psi||_\infty \propto 1/(t^* - t)$, but as in the stable blow-up for NLS ($||\Psi||_\infty \propto 1/\sqrt{t^* - t}$)
- $t^* - t^c = 0(\epsilon)$





Novikov-Veselov equation

- generalization of KdV, no application known yet

$$\partial_t v = 4\Re(4\partial_z^3 v + \partial_z(vw) - E\partial_z w),$$
$$\partial_{\bar{z}} w = -3\partial_z v, \quad v = \bar{v}, \text{ i.e. } v \text{ is a real-valued function}, \quad E \in \mathbb{R},$$
$$v = v(x,y,t), \quad w = w(x,y,t), \quad z = x + iy, \quad (x,y) \in \mathbb{R}^2, \quad t \in \mathbb{R},$$

- nonlocal equation:

$$\hat{w}_1(\xi_1,\xi_2) = \frac{3(\xi_2^2 - \xi_1^2)}{\xi_1^2 + \xi_2^2}\hat{v}(\xi_1,\xi_2), \quad \hat{w}_2(\xi_1,\xi_2) = \frac{6\xi_1\xi_2}{\xi_1^2 + \xi_2^2}\hat{v}(\xi_1,\xi_2),$$

where $\hat{w}_i$, $i = 1,2$ (or $\mathcal{F}w_i$) denotes the two-dimensional Fourier transform of w_i, and where (ξ_1,ξ_2) are the dual variables of (x,y).

- completely integrable, lump solutions known
- KP limit: $E = \pm\kappa^2$, $y = \kappa Y$, $\nu(x,Y,t) = v(x,\kappa Y,t)$, $\kappa \to \infty$: $u(x,y,t) = -\nu(-x,2y,\frac{1}{2}t)$ satisfies KP I $(+)$ or KP II $(-)$
- intermediate values of $|E|$, blow-up possible, exact blow-up solutions known

Dynamic rescaling

- exact symmetry for constant L and z_m:

$$\zeta = \frac{z - z_m}{L}, \quad \xi = \frac{x - x_m}{L}, \quad \eta = \frac{y - y_m}{L}, \quad \frac{d\tau}{dt} = \frac{1}{L^3}, \quad V = L^2 v, \quad W = L^2 w$$

where $z_m(t) = x_m(t) + i y_m(t)$

- dynamically rescaled NV equation

$$\partial_\tau V = a\,(2V + \xi\partial_\xi V + \eta\partial_\eta V) + 2\Re(c\partial_\zeta V) + 4\Re\left(4\partial_\zeta^3 V + \partial_\zeta(VW) - EL^2\partial_\zeta W\right),$$

with

$$a = \partial_\tau(\ln L), \quad c = \frac{\partial_\tau z_m}{L}.$$

- blow-up: $\tau \to \infty$, $L \to 0$, V, W, a, c assumed independent of τ in the limit

$$0 = a^\infty\,(2V^\infty + \xi\partial_\xi V^\infty + \eta\partial_\eta V^\infty) + 2\Re(c^\infty\partial_\zeta V^\infty) + 4\Re\left(4\partial_\zeta^3 V^\infty + \partial_\zeta(V^\infty W^\infty)\right)$$

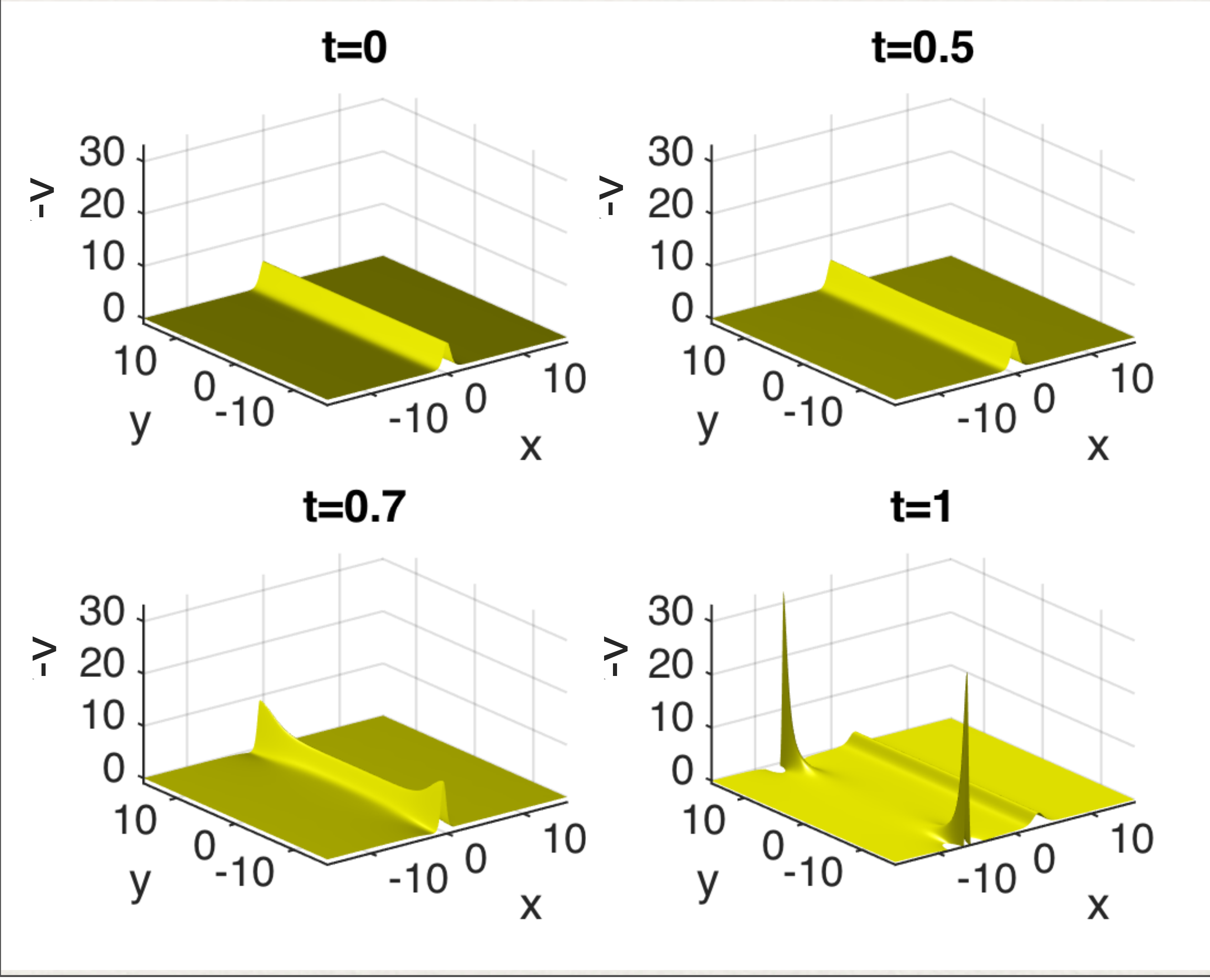
- if $L(\tau) = C_1\tau$ with constant C_1, and thus $a^\infty = 0$:

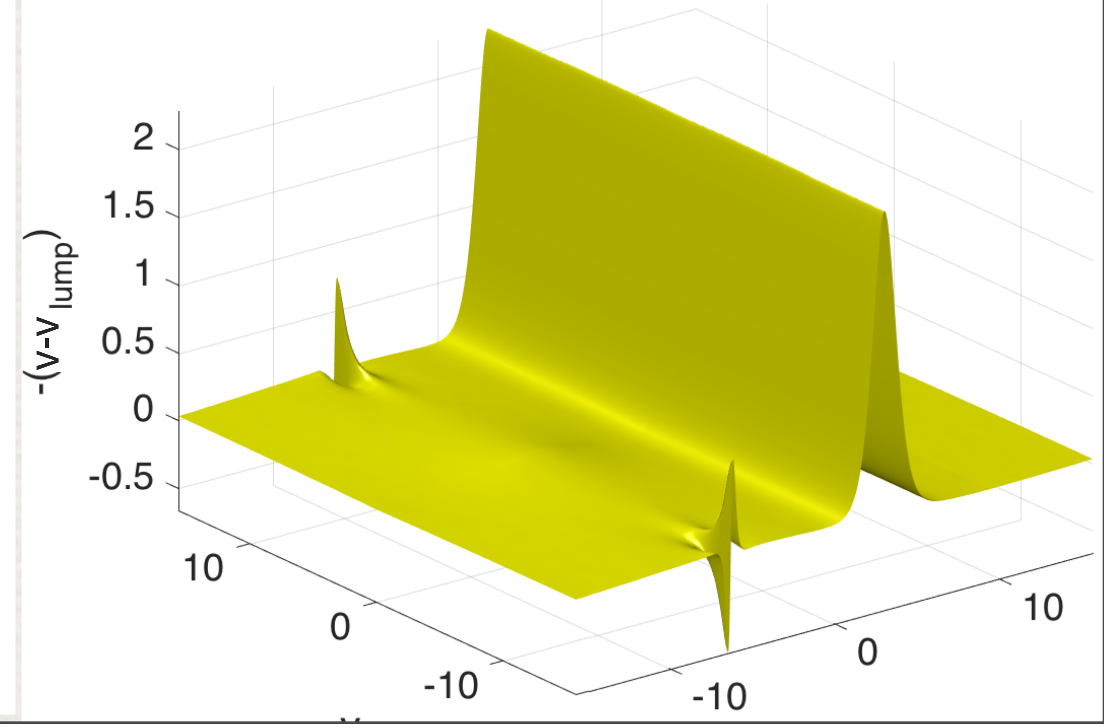
$$L(t) \propto (t^* - t)^{1/2}$$

exact blow-up solutions: $L(t) \propto (t^* - t)^{1/3}$

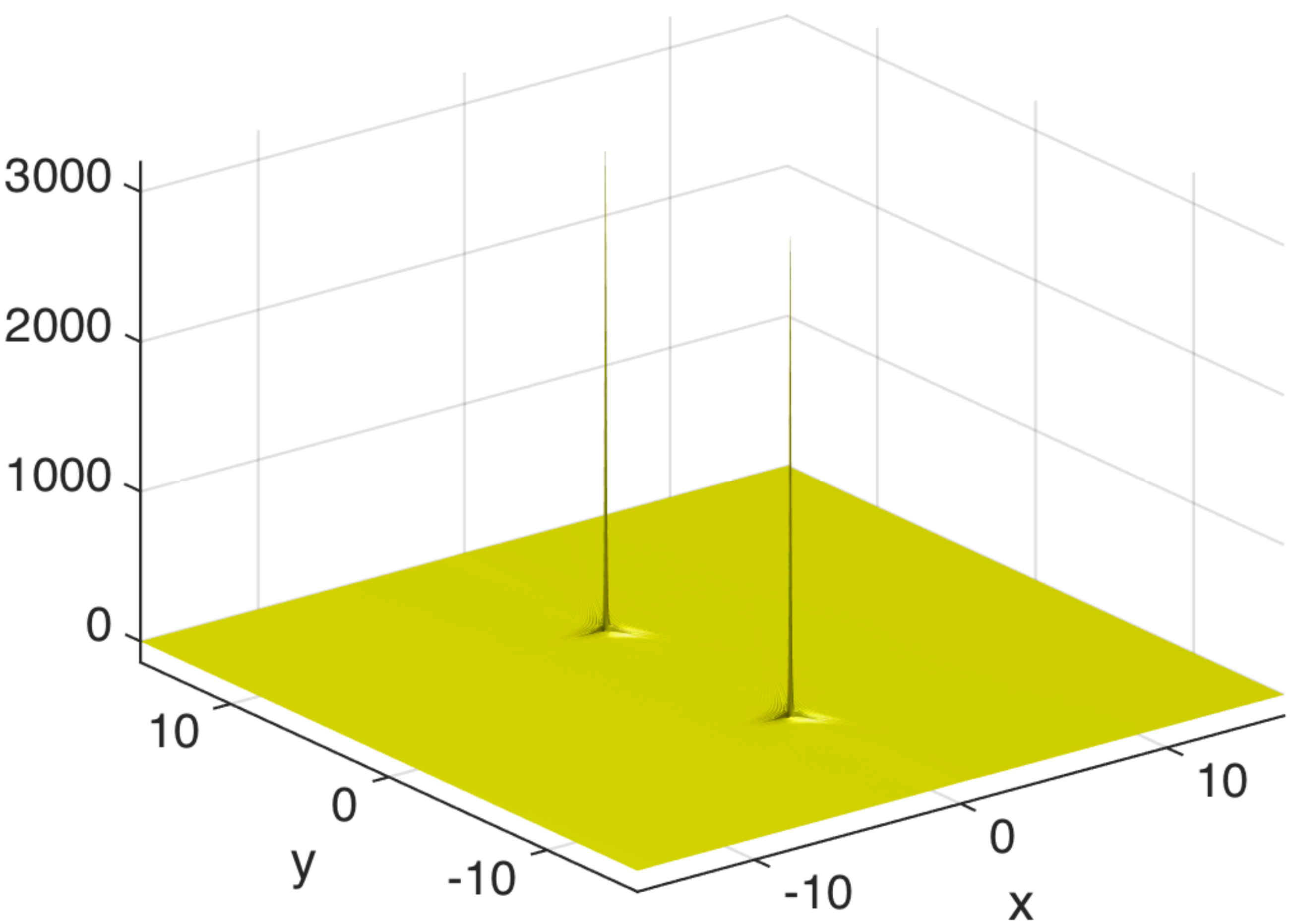
KdV soliton stability

- KP II limit $E \ll -1$: stable
- KP I limit $E \gg 1$: large solitons unstable against lump formation

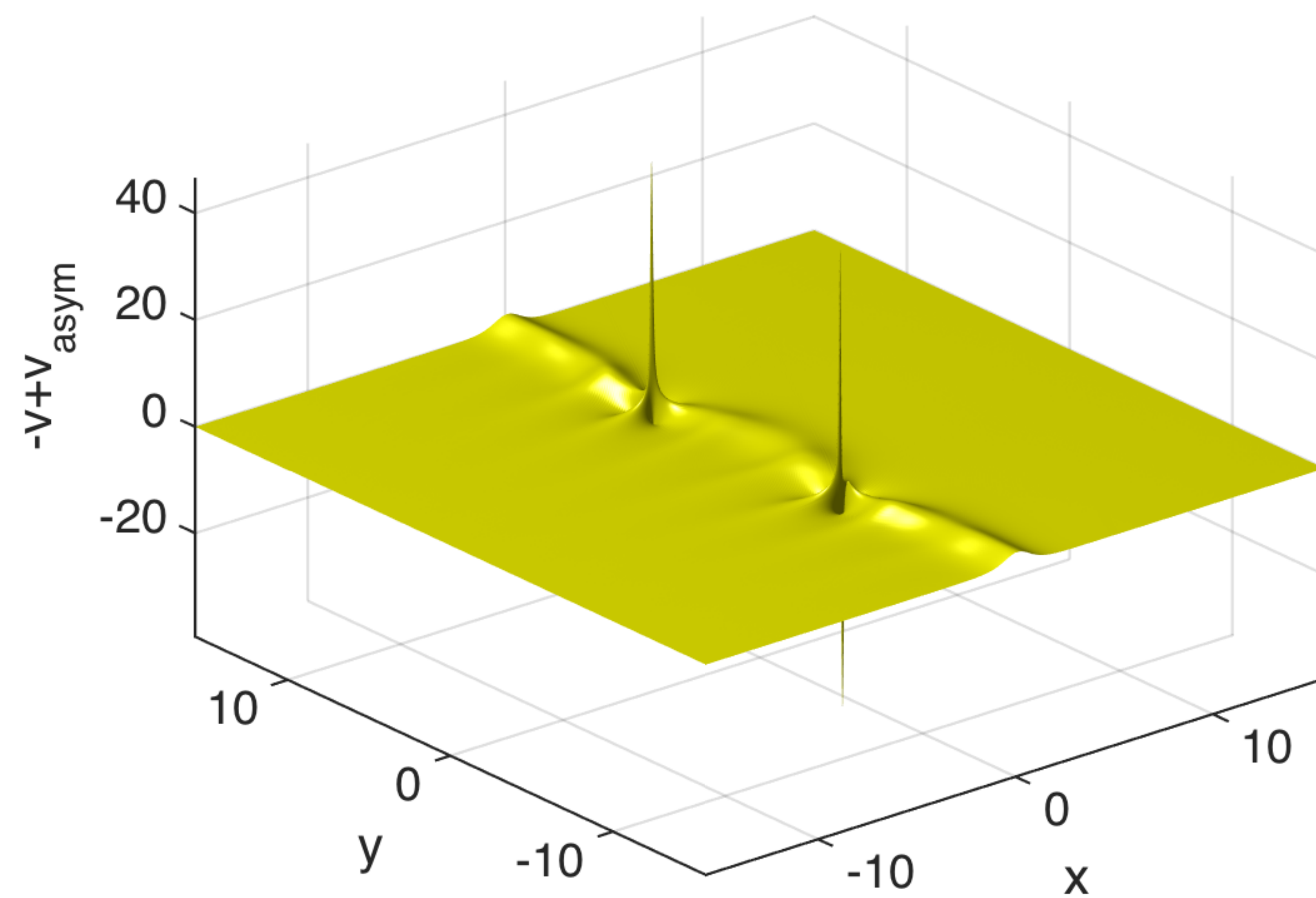




- intermediate values of E ($E = 0$)



fitted lump



Conjecture

- Let $|E| \ll 10$.
  - The KdV soliton for small a is stable under NV dynamics.
  - The KdV soliton for large a is unstable under NV dynamics against an L^∞ blow-up in finite time.
  - NV solutions corresponding to localized initial data of sufficiently small L^2 norm are global in time. Localized initial data of sufficiently large L^2 norm will blow-up in finite time.
  - A blow-up at time t^* is self similar according to the scaling for $t \sim t^*$,

$$v \sim \frac{1}{L^2} Q\left(\frac{z - z^*}{L}\right), \quad L = \sqrt{t^* - t},$$

where z^* is the location of the blow-up which appears to be finite, and where Q is the lump.