Jordan hydrodynamic type systems. Parabolicity, integrability, confluence and regularization.

B. Konopelchenko

Salento University, Lecce, Italy

LMS EPSRC Durham Symposium
"Geometric and algebraic aspects of integrability"
Durham 27/07-04/08, 2016.

based in part on:

1. Y. Kodama, B. Konopelchenko, "Confluence of hypergeometric functions ...", arXiv: 1510.0159
2. B. Konopelchenko, G. Ortenzi, "Jordan form, parabolicity...", arXiv: 1510.0249.

Systems of quasilinear PDEs of the first order

$$u_{it} = \sum_{k=1}^{n} V_{ik}(u) u_{kx} \qquad i = 1, \dots, n.$$

Types: eigenvalues of $V(u)$

1. Real, distinct - hyperbolic
2. Complex, ... - elliptic
3. Equal ... - parabolic

Systems of one (single) type.

Systems of mixed type:

Transitions of change of type:

Transition line $\Omega(t, x) = 0$

Parabolicity on $\Omega = 0$.

Change of physics.

Text-book examples:

math:	hyperbolic	elliptic	parabolic
	$u_{xx} - u_{yy} = 0$	$u_{xx} + u_{yy} = 0$	$u_{xx} + u_y = 0$
	wave equation	Laplace eq.	heat eq.
phys.	propagation of waves	electrostatic	prop. of heat diffusion

nonlinear phenomena,
nonlinear equations.

e.g.

$$u_{xx} + u\,u_{yy} + u_y = 0$$

1. $u(x,y) < 0$ – hyperbolic, waves
2. $u(x,y) > 0$ – elliptic, eff. electostat.
3. $u(x,y) = 0$ transition line

parabolic on and near line.

Text-book examples from hydrodynamics

1. General stationary potential, two-dimensional flow of compressible gas (L.-L., Hydrodynamics, § 114).

$\vec{v} = \operatorname{grad} \varphi$.

Euler equation

$$(c^2 - \varphi_x^2)\varphi_{xx} + (c^2 - \varphi_y^2)\varphi_{yy} - 2\varphi_x \varphi_y \varphi_{xy} = 0$$

$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ - sound speed.

discriminant $\Delta = c^2(\varphi_x^2 + \varphi_y^2 - c^2) =$
$= c^2(\vec{v}^2 - c^2).$

$c = c(v)$

for polytropic gas $c^2 = \frac{\gamma+1}{2} c_*^2 - \frac{\gamma-1}{2} \vec{v}^2$

$\Rightarrow \quad \Delta = \frac{\gamma+1}{2}(\vec{v}^2 - c_*^2). \quad \Rightarrow$

1. $|v| < c_*$ - elliptic - ?
2. $|v| > c_*$ - hyperbolic - waves
3. $|v| = c_*$ - sonic (transition) line.

2. Chaplygin's equation

hodograph space (L-L, §§ 116, 118)

$$\Phi = -\varphi + x v_x + y v_y$$

$\Rightarrow$

$$\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{v^2}{1 - \frac{v^2}{c^2}} \frac{\partial^2 \Phi}{\partial v^2} + v \frac{\partial \Phi}{\partial v} = 0$$

where $v_x = v\cos\theta$, $v_y = v\sin\theta$.

sonic line $v = c_*$

near the sonic line

$$\frac{\partial^2 \Phi}{\partial \eta^2} - \eta \frac{\partial^2 \Phi}{\partial \theta^2} = 0$$ Euler-Tricomi eq.

$$\eta = (2\alpha_*)^{1/3} \frac{v - c_*}{c_*}$$

1. $v < c_*$, $\eta < 0$ – elliptic (subsonic)
2. $v > c_*$, $\eta > 0$ – hyperbolic (supersonic)
3. $v = c_*$, $\eta = 0$ – sonic line.

3. Simplest example

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u & 1 \\ v & u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x$$

Eigenvalues (characteristic speeds)

$$\lambda_{\pm} = u \pm \sqrt{v}$$

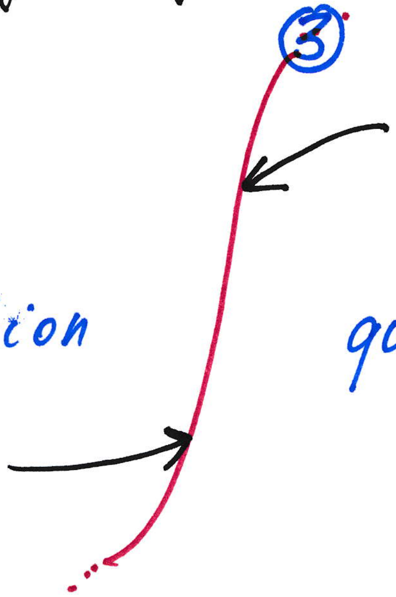
1. $v > 0$ - hyperbolic (one-layer Benney system)
2. $v < 0$ - elliptic (quasicl. Da Rios system $v = -R^2$)
3. $v(x,t) = 0$ - transition line

On the transition line (and nearby)

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \underbrace{\begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}}_{\text{Jordan block}} \begin{pmatrix} u \\ v \end{pmatrix}_x$$

Hamiltonian systems,
Nonlinear wave type equations...
Konopelchenko & Ortenzi (2015).

Change of type transitions



① Hyperbolic

Wave motion

② Elliptic

quasi-conformal mappings

(M. Lavrentiev, 1947)

$$\Gamma_{\bar z} = \mu(z, \bar z)\,\Gamma_z$$

$1 \to 3$ - development of wave's instability (Rayleigh-Taylor instab.)

$2 \to 3$ - quasi-conformal mapping becomes singular ($|\mu| \to 1$)

(Konop, Ortenzi, 2019)

$2 \to 1$ - subsonic → supersonic flow. formation of shock wave

(L.-L. §120)

$1 \to 2$ - supersonic → subsonic

3 - ? - parabolic

Transition criterium

Equations ?

Properties ?

Parabolic quasi-linear systems

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}_t = \begin{pmatrix} u_1 & 1 & 0 & \dots & 0 \\ 0 & u_1 & 1 & 0 \dots & 0 \\ \dots & & & \ddots & \\ & & 0 & u_1 & 1 \\ 0 & 0 & & 0 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}_x$$

whole plane(x,t)!

or

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_x$$

Jordan blocks

Nondiagonalizable <u>parabolic</u> systems

vs. nondiagonalizable strictly hyperbolic systems (Ferapontov, Mokhov, 1993-1996)

No generalized hodograph equations a la' Tsarev.

...

Parabolic systems as a limit of generic one.

1. dNLS
$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u & 1 \\ v & u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x$$

Riemann invariants
$$r_\pm = u \pm 2\sqrt{v}$$

Characteristic speeds
$$\lambda_\pm = u \pm \sqrt{v}$$

Diagonal form $\quad r_{\pm t} = \lambda_\pm r_{\pm x}$ –

$$(u \pm 2\sqrt{v})_t = (u \pm \sqrt{v})(u \pm 2\sqrt{v})_x \Rightarrow$$

$$u_t \pm \frac{1}{\sqrt{v}} v_t = u u_x \pm \cancelto{0}{\sqrt{v}}\, u_x \pm \frac{1}{\sqrt{v}} u v_x + v_x$$

$\Rightarrow$

$\to \quad u_t = u u_x + v_x,$

$\to \quad v_t = u v_x$

different orders!

2. Strong coupling constants limit

ε- systems (Pavlov, 2003)

$$\frac{\partial u_i}{\partial t} = \left(u_i + \varepsilon(u_1+u_2)\right)\frac{\partial u_i}{\partial x} \qquad i=1,2$$

$\varepsilon = 0$ - decoupled Burgers-Hopf eqs

interpretation: ε- coupling constant

Double scaling limit.

$$\varepsilon = \frac{1}{\delta^2}$$

$$u_1 = \delta\sqrt{v} + \frac{1}{2}\delta^2 u + \dots \qquad \delta \to 0$$

$$u_2 = -\delta\sqrt{v} + \frac{1}{2}\delta^2 u + \dots$$

Limiting system (orders δ, δ^2)!

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial t} = u\frac{\partial v}{\partial x}.$$

n-component ε-system

$$u_i = \delta a_{1i} + \delta^2 a_{2i} + \dots + \delta^n a_{ni} + \dots$$

$$\varepsilon = \frac{1}{\delta^n} \qquad \delta \to 0 \qquad a_{ki}(v_1, \dots, v_n)$$

Limiting system (orders $\delta, \delta^2, \dots, \delta^n$)

$$\frac{\partial v_k}{\partial t} = \sum_{l=1}^{n} J_{kl} \frac{\partial v_l}{\partial x}$$

J - $n \times n$ Jordan block.

3. Confluence process for Lauricella type functions

multidimensional generalization of Gauss hypergeometric function

Kodama & Konopelchenko (2015)

Generic integrable hydrodynamic type systems

=

Dynamics of critical points of functions

Konopelchenko, Martinez-Alonso, Medina $(\varepsilon = \pm\frac{1}{2})$ (2010-2014)

Konopelchenko, Ortenzi $(\varepsilon = \frac{1}{2})$ (2013-2014)

Kodama, Konopelchenko, Schief (2015)

Dubrovin (in different setting) (2008).

Function class -

- solutions of linear Darboux system

$$\frac{\partial^2 W}{\partial x_i \partial x_k} = A_{ik}(x) \frac{\partial W}{\partial x_i} + A_{ki}(x) \frac{\partial W}{\partial x_k},$$

$i \neq k$,

$i, k = 1, \dots, n$.

Superpositions

$$\mathbb{W}(x;t) = \sum_{k=1}^{N} t_k W_k(x) + \hat{W}.$$

Critical points of W

$$\left.\frac{\partial W}{\partial x_i}\right|_{x=u} = 0$$

$\Rightarrow$ semi-Hamiltonian integrable systems

$$\frac{\partial u_i}{\partial t} = \lambda_i(u)\frac{\partial u_i}{\partial x} \qquad i=1,\dots,n$$

$x = t_1,\ t = t_2$

with

$$\lambda_i = \frac{\dfrac{\partial W_2}{\partial u_i}}{\dfrac{\partial W_1}{\partial u_i}}.$$

Includes multiphase Whitham equation for KdV and NLS.

Particular class - EPD systems

$$(x_i - x_k)\frac{\partial^2 W}{\partial x_i \partial x_k} = \varepsilon_k \frac{\partial W}{\partial x_i} - \varepsilon_i \frac{\partial W}{\partial x_k}$$

$i \neq k$

$i,k = 1,\dots,n$

Lauricella (1893), Pavlov (2003)

Various integrable hydrodynamic type systems ... s- system

Lauricella type functions (1893)

$$W(x) = \int_\Gamma dz\, f(z) \prod_{i=1}^{n} (1 - z x_i)^{-\varepsilon_i}.$$

Limiting behaviour of $\prod_{i=1}^{n} (1 - z x_i)^{-\varepsilon_i}$

<u>$n=1$</u> classical formula

$$x_1 = \delta y_1, \quad \varepsilon_1 = \frac{1}{\delta}$$

$$\lim_{\delta \to 0} (1 - z x_1)^{-\varepsilon_1} = \lim_{\delta \to 0} (1 - z \delta y_1)^{-\frac{1}{\delta}} = e^{z y_1}.$$

<u>$n=2$</u>

$$\varepsilon_1 = \varepsilon_2 = \frac{1}{\delta^2}$$

$$x_1 = \delta y_2^{1/2} + \frac{1}{2} \delta^2 y_1,$$

$$x_2 = -\delta y_2^{1/2} + \frac{1}{2} \delta^2 y_1$$

?

$$\lim_{\delta \to 0} [(1 - z x_1)(1 - z x_2)]^{-\varepsilon} = \underline{e^{y_1 z + y_2 z^2}}.$$

Classical confluence process for Gauss hypergeometric function:

$$\begin{tikzcd}[row sep=small] & & \text{Hermite} \arrow[dr] & \\ \text{Gauss} \arrow[r] & \text{Kummer} \arrow[ur] \arrow[dr] & & \text{Airy} \\ & & \text{Bessel} \arrow[ur] & \end{tikzcd}$$

Gauss -

$$F(a,b,c;x) \doteq \int_C dz\, z^{a-1}(1-z)^{c-a-1}(1-zx)^{-b}.$$

regular singular points $\{0, 1, \frac{1}{x}, \infty\}$.

Confluence $\frac{1}{x} \to \infty$:

$x = \delta y$, $b = \frac{1}{\delta}$, $\delta \to 0$. $\Rightarrow$

Kummer -

$$F_K(a,c;y) \doteq \int_C dz\, z^{a-1}(1-z)^{c-a-1}\, \underline{e^{yz}}.$$

. . .

Airy :

$$F_A = \int_c dz\, e^{-\frac{z^3}{3}}\, \underline{e^{yz}}.$$

confluence hypergeometric equations.

Multidimensional generalizations of hypergeometric function

Appell (1882) – $n=2$, Lauricella (1893) – $n\geq 3$.

$F_D(a, \varepsilon_1, \dots, \varepsilon_n, b; x_1, \dots, x_n) =$

$$= \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 dz\, z^{a-1} (1-z)^{b-a-1} \prod_{i=1}^{n} (1-zx_i)^{-\varepsilon_i}.$$

– EPD system + equations $\frac{\partial^2 F}{\partial x_i^2} + \sum A_i \frac{\partial F}{\partial x_i} + BF = 0$ (Lauricella).

General hypergeometric functions and finite-dimensional Grassmannians –

Gelfand ... (1986-).

Aomoto & Kita (2011).

Confluence of general hypergeometric systems.

Gelfand, Retach, Serganova (1988)

Kimura & Takano (2006).

Grassmannian $Gr(m,n)$ $\quad m \le n$

points ξ - $m \times n$ matrices of rank m.

General hypergeometric functions

$$F(\xi;\mu) = \int_\Delta \chi(\xi;\mu)\,d\omega$$

χ - character of centralizer
μ - weight $\quad d\omega$ - measure in CP^1.
Stratification, partition $n = \{i_1 \dots i_s\}$

Our case $\quad Gr(2, n+3)$.

Lauricella functions $\eta(x;z)$

$$\chi(\xi;\mu)\,d\omega = \eta(x;z)\,dz.$$

<u>Examples.</u> $\quad Gr(2,5) \quad n=2.$

$$\xi = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -x_1 & -x_2 \end{pmatrix}$$

with $x_1 x_2 (x_1 - 1)(x_2 - 1)(x_1 - x_2) \neq 0$.

partition $\underline{5 = (1,1,1,1,1)}$ - top cell.

Character $\qquad (z_0, z_1) \in \mathbb{C}P^1$

$$\chi(\xi, \mu)\, d\omega = z_0^{\mu_0^{(1)}} z_1^{\mu_0^{(2)}} (z_0 - z_1)^{\mu_0^{(3)}} (z_0 - x_1 z_1)^{\mu_0^{(4)}} (z_0 - x_2 z_1)^{\mu_0^{(5)}} d\omega$$

$$= z^{\mu_0^{(2)}} (1-z)^{\mu_0^{(3)}} (1 - x_1 z)^{\mu_0^{(4)}} (1 - x_2 z)^{\mu_0^{(5)}} dz =$$

$$= \eta(x; z)\, dz$$

Setting $\mu_0^{(4)} = -\varepsilon_1$, $\mu_0^{(5)} = -\varepsilon_2$, $\mu_0^{(2)} = -\varepsilon_3$, $\mu_0^{(3)} = -\varepsilon_4$ one gets Lauricella differential with

$$\eta(x; z) = z^{-\varepsilon_3} (1-z)^{-\varepsilon_4} \underbrace{(1 - x_1 z)^{-\varepsilon_1} (1 - x_2 z)^{-\varepsilon_2}}$$

with regular singular points

$$\left\{ 0, 1, \frac{1}{x_1}, \frac{1}{x_2}, \infty \right\}.$$

2. Partition $5 = (2, 1, 1, 1)$

$$\xi = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & x_1 & 1 & -1 & -x_2 \end{pmatrix}$$

cell with co-dimension <u>one</u>.

$\Rightarrow$ character

$$\chi(\zeta;\mu)\,d\omega = z_0^{\mu_0^{(1)}} z_1^{\mu_0^{(2)}} e^{\mu_1^{(1)} x_1 \frac{z_1}{z_0}} (z_0 - z_1)^{\mu_0^{(3)}} (z_0 - x_2 z_1)^{\mu_0^{(4)}} d\omega$$

$$= e^{\mu_1^{(1)} x_1 z} \; z^{\mu_0^{(2)}} (1-z)^{\mu_0^{(3)}} (1 - x_2 z)^{\mu_0^{(4)}} dz =$$

$$= \eta(x;z)\,dz$$

Singular points - $\{0, 1, \frac{1}{x_2}, \infty\}$

and ∞ - irregular singular point due to the confluence of points $z = \frac{1}{x_1}$ and $z = \infty$.

$\Rightarrow$ Lauricella function

$$\eta(x;z) = z^{-\varepsilon_3}(1-z)^{-\varepsilon_4} \underbrace{e^{\varepsilon_1 z x_1}(1 - x_2 z)^{-\varepsilon_2}}.$$

3. Partition $5 = \{3, 1, 1\}$

$$\zeta = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & x_1 & 1 - x_2 \end{pmatrix}$$

- co-dimension two

Lauricella function - similar to case 2.

4. Partition $5 = (4, 1)$

$$\xi = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -x_1 & -x_2 \end{pmatrix}$$

cell with co-dimension <u>three</u>

Lauricella function similar to case 2.

5. Partition $5 = \{3, 2\}$

$$\xi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_1 & 1 & x_2 \end{pmatrix}$$

co-dimension <u>three</u>

Lauricella function

$$\eta(x; z) = z^{-\varepsilon_3} e^{-\frac{\varepsilon_1}{2} z^2 + \varepsilon_4 z} \, e^{(\varepsilon_1 x_1 + \varepsilon_2 x_2) z}.$$

6. Partition $5 = \{5\}$.

$$\xi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x_1 & x_2 \end{pmatrix}$$

– cell of co-dimension <u>four</u>

and Lauricella function

$$\eta(x; z) = e^{\varepsilon_3 z + \varepsilon_4 z^2 - \frac{\varepsilon_1}{3} z^3 - \frac{\varepsilon_2}{4} z^4} \, \underline{e^{(\varepsilon_1 x_1 + \varepsilon_2 x_2) z - \varepsilon_2 x_1 z^2}};$$

So, four types of confluent Lauricella functions for $G(2,5)$

1. Top cell

$$\eta_0(x;z) =\cdot (1-x_1 z)^{-\varepsilon_1}(1-x_2 z)^{-\varepsilon_2}$$

generic Lauricella function at $n=2$.

2. Co-dimen. one cell

$$\eta_1(y;z) = \cdot (1-y_1 z)^{-\varepsilon_1} e^{y_2 z}$$

3. $5=\{3,2\}$

$$\eta_2(y;z) = \cdot \ e^{(y_1+y_2)z}$$

4. most degenerate case $5=\{5\}$

$$\eta_3(y;z) = \cdot \ e^{y_1 z + y_2 z^2}$$

22.VII.

Confluent Lauricella functions for $Gr(2,6)$ $(n=3)$

1. Top cell

$$\eta_0(x;z) = \cdot (1-x_1 z)^{-\varepsilon_1}(1-x_2 z)^{-\varepsilon_2}(1-x_3 z)^{-\varepsilon_3}$$

2. Co-dimen. one cell

$$\eta_1(y;z) = \cdot (1-y_1 z)^{-\varepsilon_1}(1-y_2 z)^{-\varepsilon_2} e^{y_3 z}$$

3.

$$\eta_2(y;z) = \cdot (1-y_1 z)^{-\varepsilon_1} e^{(y_2+y_3)z}$$

4. $$\eta_3(y;z) = \cdot (1-y_1 z)^{-\varepsilon_1} e^{y_2 z + y_3 z^2}.$$

5. $$\eta_4(y;z) = \cdot \; e^{(y_1+y_2+y_3)z}$$

6. $$\eta_5(y;z) = \cdot \; e^{y_1 z + y_2 z} e^{y_3 z^2}$$

7. most degenerate cell co-dim-5

$$\eta_6(y;z) = \cdot \; e^{y_1 z + y_2 z^2 + y_3 z^3}.$$

More cases for $Gr(2, n+3)$ $n \geq 4$

Most degenerate case for arbitr. n

$$\eta(y, z) = \cdot\ e^{y_1 z + y_2 z^2 + \dots + y_n z^n}.$$

Confluence limits.

$G(2,5)$

1. $\eta_0 \to \eta_1$:

$x_1 = y_1$, ε_1 - constant, $x_2 = \delta y_2$, $\varepsilon_2 = \frac{1}{\delta}$, $\delta \to 0$

classical formula $\lim\limits_{\delta \to 0} (1 - \delta z y_2)^{-\frac{1}{\delta}} = e^{z y_2}$.

2. $\eta_0 \to \eta_2$:

$x_1 = \delta y_1$, $x_2 = \delta y_2$, $\varepsilon_1 = \varepsilon_2 = \frac{1}{\delta}$, $\delta \to 0$

3. $\eta_0 \to \eta_3$:

$$x_1 = \delta y_2^{1/2} + \frac{1}{2}\delta^2 y_1,$$
$$x_2 = -\delta y_2^{1/2} + \frac{1}{2}\delta^2 y_1.$$

$\varepsilon_1 = \varepsilon_2 = \frac{1}{\delta^2}$, $\delta \to 0$

$G(2,6)$ $\qquad n = 3.$

1. $\eta_0 \to \eta_1$, $\eta_0 \to \eta_2$, $\eta_0 \to \eta_3$ as for $G(2,5)$

2. $\eta_0 \to \eta_4$

$$x_i = \delta y_i, \quad i = 1,2,3, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{\delta}, \quad \delta \to 0$$

3. $\eta_0 \to \eta_6$:

$$x_1 = \delta y_3^{1/3},$$

$$x_2 = \delta q y_3^{1/3} + \delta^2 \hat{q} y_2 y_3^{-1/3}$$

$$x_3 = \delta q^2 y_3^{1/3} - \delta^2 \hat{q} y_2 y_3^{-1/3} + \delta^3 y_1.$$

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{\delta^3}, \quad \delta \to 0$$

$$q = e^{\frac{2\pi i}{3}}$$

$$\hat{q} = \frac{q}{q^2 - 1}.$$

For $Gr(2, n+3)$

$$x_i = a_{1i}(y)\delta + a_{2i}\cdot\delta^2 + \ldots + a_{ni}\cdot\delta^n.$$

$i = 1, \ldots, n.$

– family of transformations such that

$$(-1)^{k+1}\sigma_k(x) = \delta^n y_k, \qquad \sigma_k(x) = \sum_{1 \le j_1 < \ldots < j_n} x_{j_1} x_{j_2} \cdots x_{j_n}$$

Confluent EPD equations.

Gr(2,5)

1. Top cell

$$(x_1 - x_2)\frac{\partial^2 \eta_0}{\partial x_1 \partial x_2} = \varepsilon_2 \frac{\partial \eta_0}{\partial x_1} - \varepsilon_1 \frac{\partial \eta_0}{\partial x_2}$$

– EPD equation

2. $\eta_0 \to \eta_1$

$$y_1 \frac{\partial^2 \eta_1}{\partial y_1 \partial y_2} = \frac{\partial \eta_1}{\partial y_1} - \varepsilon_1 \frac{\partial \eta_1}{\partial y_2}.$$

3. $\eta_0 \to \eta_2$

$$\frac{\partial \eta_2}{\partial y_1} - \frac{\partial \eta_2}{\partial y_2} = 0$$

4. $\eta_0 \to \eta_3$

$$\frac{\partial \eta_3}{\partial y_2} = \frac{\partial^2 \eta_3}{\partial y_1^2} \quad !$$

Gr(2, 6). $n=3$

1. Top cell

$$(x_i - x_k)\frac{\partial^2 \eta_0}{\partial x_i \partial x_k} = \varepsilon_k \frac{\partial \eta_0}{\partial x_i} - \varepsilon_i \frac{\partial \eta_0}{\partial x_k} \qquad i \neq k,\ i,k = 1,2,3$$

– EPD system.

2. $\eta_0 \to \eta_1$

$$(y_1 - y_2)\frac{\partial^2 \eta_1}{\partial y_1 \partial y_2} = \varepsilon_2 \frac{\partial \eta_1}{\partial y_1} - \varepsilon_1 \frac{\partial \eta_1}{\partial y_2},$$

$$y_1 \frac{\partial^2 h_1}{\partial y_1 \partial y_3} = \frac{\partial \eta_1}{\partial y_1} - \varepsilon_1 \frac{\partial \eta_1}{\partial y_3}.$$

. . .

4. $\eta_0 \to \eta_3$

$$y_1 \frac{\partial^2 h_3}{\partial y_1 \partial y_2} = \frac{\partial h_3}{\partial y_1} - \varepsilon_1 \frac{\partial h_3}{\partial y_2},$$

$$\frac{\partial h_3}{\partial y_3} = \frac{\partial^2 h_3}{\partial y_2^2}.$$

. .

7. $\eta_0 \to \eta_6$ most degenerate case

$$\frac{\partial \eta_6}{\partial y_2} = \frac{\partial^2 \eta_6}{\partial y_1^2}, \qquad \frac{\partial \eta_6}{\partial y_3} = \frac{\partial^2 \eta_6}{\partial y_1 \partial y_2}.$$

or $$\frac{\partial \eta_6}{\partial y_3} = \frac{\partial^3 \eta_6}{\partial y_1^3}.$$

$Gr(2, n+3)$ arbitrary n

Most degenerate cell : $n+3 = \{n+3\}$

$$\frac{\partial \eta}{\partial y_r} = \frac{\partial^2 \eta}{\partial y_i \partial y_j}$$

any $i+j=r$
$i, j, r = 1, \dots, n$

or

$$\frac{\partial \eta}{\partial y_r} = \frac{\partial^r \eta}{\partial y_1^r}$$

$i = 1, 2, \dots, n$

Equations for multidimensional Airy function.
Gelfand, Retakh, Serganova (1988)
degenerate GKZ system

Simplest solution

$$\eta = e^{y_1 z + y_2 z^2 + \dots + y_n z^n}.$$

Dynamics of critical points for confluent Lauricella type functions

Kodama, Konopelchenko (2015)

General scheme:

Let $\eta_k(u)$ - solution of confluent EPD system

Consider

$$W(y;t) = \sum_{k=1}^{N} t_k \eta_k(y) + \hat{\eta}(y)$$

$t_1, \dots t_N$ - parameters

Critical points

$$\left.\frac{\partial W}{\partial y_i}\right|_{y=u} = 0 \qquad i=1,\dots,n$$

$\Rightarrow$ Hodograph type equation

$$\sum_{k=1}^{N} t_k \frac{\partial \eta_k}{\partial u_i} + \frac{\partial \tilde{\eta}}{\partial u_i} = 0 \qquad i=1\dots n$$

$\Rightarrow$ Systems of quasi-linear PDEs

$$\frac{\partial \bar{u}}{\partial t_k} = A_k(\bar{u}) \frac{\partial \bar{u}}{\partial t_1} \qquad \bar{u} = (u_1, \dots, u_n)^T$$

Infinite family of commuting equations.

Simple choice

$$\eta(y;z) = \sum_{k=0}^{\infty} \eta_k(y) z^k \quad \text{as } z \to 0$$

and

$$W(y;t) = \sum_{k=1}^{N} t_k \eta_k(y) + \hat{\eta}(y)$$

Examples.

$Gr(2,5)$

Function $\quad \eta_{(1)} = (1-y_1 z)^{-\varepsilon} e^{y_2 z}$

$\Rightarrow$

$$W = t_1(\varepsilon y_1 + y_2) + t_2\left(\frac{\varepsilon(\varepsilon+1)}{2} y_1^2 + \varepsilon y_1 y_2 + \frac{1}{2} y_2^2\right) + \dots$$

Critical point (u_1, u_2) and $\dfrac{\partial^2 \eta_{(1)}}{\partial u_1^2} = 0$

$\Rightarrow$ the system

$$\frac{\partial u_1}{\partial t_2} = (u_1 + \varepsilon u_1 + u_2)\frac{\partial u_1}{\partial t_1},$$

$$\frac{\partial u_2}{\partial t_2} = (\varepsilon u_1 + u_2)\frac{\partial u_2}{\partial t_1}$$

\+ infinite hierarchy of commuting equations.

Function $\eta_{(3)}(y;z) = e^{y_1 z + y_2 z^2}$

So

$$\eta_{(3)k}(y) = p_k(y_1, y_2) -$$

– elementary Schur polynomials.

$$p_1 = y_1, \quad p_2 = \frac{1}{2}y_1^2 + y_2, \quad p_3 = \frac{1}{6}y_1^3 + y_1 y_2, \ldots$$

and

$$W = t_1 y_1 + t_2\left(\frac{1}{2}y_1^2 + y_2\right) + t_3 \cdots + \hat{W}$$

Critical points

$$\frac{\partial W}{\partial u_1} = t_1 + t_2 u_1 + \ldots \quad \frac{\partial \tilde{\hat{W}}}{\partial u_1} = 0$$

$$\frac{\partial W}{\partial u_2} = t_2 + \cdots \quad \frac{\partial \tilde{\hat{W}}}{\partial u_2} = 0.$$

also $\dfrac{\partial^2 W}{\partial u_1^2} = 0$ but $\dfrac{\partial^2 W}{\partial u_1 \partial u_2} \neq 0$ in general

since $\dfrac{\partial W}{\partial u_2} + \dfrac{\partial^2 W}{\partial u_1^2} = 0.$ also for $\tilde{\hat{W}}$.

Differentiating w.r.t. t_1 and t_2 – and using $\dfrac{\partial^2 \tilde{\hat{W}}}{\partial u_1^2} = 0$, $\dfrac{\partial^2 \tilde{\hat{W}}}{\partial u_1 \partial u_2} = \dfrac{\partial^2 W}{\partial u_1 \partial u_2}$, $\dfrac{\partial^2 \tilde{\hat{W}}}{\partial u_2^2} = \dfrac{\partial^2 W}{\partial u_2^2}$?

one gets

$$1 + \frac{\partial^2 W}{\partial u_1 \partial u_2} \cdot \frac{\partial u_2}{\partial t_1} = 0,$$

$$u_1 + \frac{\partial^2 W}{\partial u_1 \partial u_2} \cdot \frac{\partial u_2}{\partial t_2} = 0,$$

$$\frac{\partial^2 W}{\partial u_1 \partial u_2} \cdot \frac{\partial u_1}{\partial t_1} + \frac{\partial^2 W}{\partial u_2^2} \cdot \frac{\partial u_2}{\partial t_1} = 0,$$

$$1 + \frac{\partial^2 W}{\partial u_1 \partial u_2} \cdot \frac{\partial u_1}{\partial t_2} + \frac{\partial^2 W}{\partial u_2^2} \cdot \frac{\partial u_2}{\partial t_2} = 0$$

Eliminating $\frac{\partial^2 W}{\partial u_1 \partial u_2}, \frac{\partial^2 W}{\partial u_2^2} \Rightarrow$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{t_2} = \begin{pmatrix} u_1 & 1 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{t_1}.$$

Hierarchy of commuting flows

$$\frac{\partial \bar{u}}{\partial t_k} = \begin{pmatrix} p_k & p_{k-1} \\ 0 & p_k \end{pmatrix} \frac{\partial \bar{u}}{\partial t_1}. \qquad k = 2, 3, 4, \ldots$$

Second flow

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{t_3} = \begin{pmatrix} \frac{1}{2} u_1^2 + u_2, & u_1 \\ 0 & \frac{1}{2} u_1^2 + u_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{t_1}.$$

$Gr(2,6)$

Function

$$\eta_{(3)}(y;z) = (1-y_1 z)^{-\varepsilon} e^{y_2 z + y_3 z^2} = \sum_{k=0}^{\infty} \eta_{(3)k}(y) z^k$$

$$W = \sum_{k=1}^{n} t_k \eta_{(3)k}(y) + \hat{W}.$$

Critical points and equation:

$$\frac{\partial \bar{u}}{\partial t_k} = A_k \frac{\partial \bar{u}}{\partial t_1}. \qquad k = 2,3,4\dots$$

with

$$A_k = \begin{pmatrix} G_k & 0 & 0 \\ 0 & F_k & F_{k-1} \\ 0 & 0 & F_k \end{pmatrix}$$

- commuting hierarchy

First flow

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{t_2} = \begin{pmatrix} 2u_1 + u_2 & 0 & 0 \\ 0 & u_1 + u_2 & 1 \\ 0 & 0 & u_1 + u_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{t_1}.$$

Function $\eta_{(6)}(y;z) = e^{y_1 z + y_2 z^2 + y_3 z^3}$.

So $\eta_{(6)k}(y) = p_k(y_1, y_2, y_3)$

– elementary Schur polynomials.

$\Rightarrow$ equations

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{t_2} = \begin{pmatrix} u_1 & 1 & 0 \\ 0 & u_1 & 1 \\ 0 & 0 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{t_1}.$$

and higher commuting flows.

Most degenerate case for $Gr(2, n+3)$

$$\eta(y;z) = e^{y_1 z + y_2 z^2 + \dots + y_n z^n}.$$

$\Rightarrow$ $\eta_k(y) = p_k(y_1, \dots, y_n)$

and

$$W(y,t) = \sum_{k=1}^{N} t_k p_k(y) + \tilde{W}$$

$$\frac{\partial W}{\partial y_k} = \frac{\partial^k W}{\partial y_1^k}.$$

Dynamics of critical points

$$\frac{\partial \bar{u}}{\partial t_k} = A_k \frac{\partial \bar{u}}{\partial t_1} \qquad k = 1, 2, 3, \ldots$$

with

$$A_k = \begin{pmatrix} p_k & p_{k-1} & \cdots & p_{k-(n-1)} \\ 0 & p_k & p_{k-1} \cdots & p_{k-(n-2)} \\ \vdots & & p_k & \\ & & \ddots & p_{k-1} \\ 0 & \cdots & 0 & p_k \end{pmatrix}$$

First flows

$$A_1 = \begin{pmatrix} u_1 & 1 & 0 & \cdots & 0 \\ 0 & u_1 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \cdots & & & \ddots & 1 \\ 0 & \cdots & & & u_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \frac{1}{2}u_1^2 + u_2, & u_1, & 1, & \cdots & 0 \\ 0 & \frac{1}{2}u_1^2 + u_2, & u_1 & 1 \cdots & 0 \\ \vdots & & & & \vdots \\ & & & & u_1 \\ 0 & & \cdots & & \frac{1}{2}u_1^2 + u_2 \end{pmatrix}$$

commuting hierarchy.

Variables u_i - no Riemann invariant
Jordan blocks

Hodograph equations

$$\sum_{k=1}^{N} t_k A_k(u) + B(u) = 0$$

with $B = \sum_{k=0}^{\infty} c_k A_k(u)$

Matrix hodograph equation
Kodama (1989)

but only $\underline{n}$ independent equation
due to the Jordan form of A_k.

Most degenerate cases-

\- pure parabolic systems!

Properties are <u>different</u> from those
in diagonal form.

Linearization

$$u_t = u u_x + v_x$$
$$v_t = u v_x$$

Reciprocal transformation

$$(t,x) \to (y,z)$$

$$dy = dt$$
$$dz = u\,dt + dx$$

$$\Rightarrow \quad u_y = v_z$$
$$v_y = 0.$$

Pavlov (2003)

n-component system

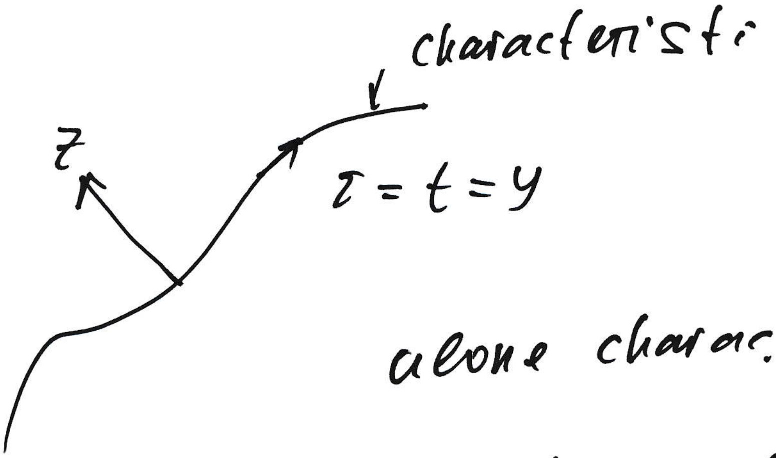
$$u_{i\,t} = u_1 u_{i\,x} + u_{i+1\,x} \qquad i = 1, \dots, N-1$$
$$u_{N\,t} = u_1 u_{N\,x}.$$

Single characteristic

$$\frac{dx}{dt} = -u_1$$

along the characteristic $\dfrac{\partial u_i}{\partial \tau} = \dfrac{\partial u_{i+1}}{\partial x}$ $\quad i = 1, \dots, N-1$

$$\frac{\partial u_n}{\partial \tau} = 0$$



along charac. $u\,dt + dx = 0$

variable z: $dz = u_1 dt + dx$.

Reciprocal transformation $(t,x) \to (y,z)$

$$dy = dt$$
$$dz = u_1\, dt + dx$$

$$\Rightarrow \begin{cases} \dfrac{\partial u_i}{\partial y} = \dfrac{\partial u_{i+1}}{\partial z}, & i = 1, \dots, n-1 \\[2mm] \dfrac{\partial u_N}{\partial y} = 0 \end{cases}$$

Limit $n \to \infty$

First flow

$$\frac{\partial u_i}{\partial t} = u_1 \frac{\partial u_i}{\partial x} + \frac{\partial u_{i+1}}{\partial x} \qquad i = 1, 2, \dots$$

- infinite chain

Formally positive part of Pavlov's hydrodynamic chain (2003)

but with all u_i - functionally independe

infinite hierarchy

Gradient catastrophe

System

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x .$$

Critical points of function W

$\Rightarrow$

$$v_x = -\frac{1}{W_{uv}}, \qquad v_t = -\frac{u}{W_{uv}},$$

$$u_x = \frac{W_{vv}}{W_{uv}^2}, \qquad u_t = -\frac{1}{W_{uv}} + u\frac{W_{vv}}{W_{uv}^2} .$$

Gradient catastrophe ($u_x, u_t, v_x, v_t \to \infty$)
at

$$W_{uv}\big|_{u_0 v_0} = 0.$$

v_x – first order singularities

u_x, u_t – second (and first) order singularities !

First singular sector

$$W_{u_0v_0} = W_{u_0u_0} = 0 , \quad W_{u_0u_0v_0} \neq 0.$$

Multi-scaling expansion

$$x = x_0 + \varepsilon^{\alpha} x^*$$
$$t = t_0 + \varepsilon^{\beta} t^*$$
$$u = u_0 + \varepsilon^{\gamma} u^*$$
$$v = v_0 + \varepsilon^{\delta} v^*.$$

balance – $\alpha = \beta = 3\gamma,\ \delta = 2\gamma$

$\gamma = 1. \quad \Rightarrow$

$$W = W_0 + \varepsilon^3 \left(u_0 x^* + \left(\tfrac{1}{2} u_0^2 + v_0\right) t^* \right)$$
$$+ \quad \varepsilon^4 W^* \quad + o(\varepsilon^5)$$

with

$$W^* = u^* x^* + u_0 u^* t^* +$$
$$+ \tfrac{1}{2} a v^{*2} + \tfrac{1}{2} a u^{*2} v^* + \tfrac{1}{24} a u^{*4}$$

$$a = W_{v_0 v_0} = W_{u_0 u_0 v_0} = W_{u_0 u_0 u_0 u_0}.$$

Critical points

$$\frac{\partial W^*}{\partial u^*} = x^* + u_0 t^* + a u^* v^* + \tfrac{1}{6} u^{*3} = 0$$

$$\frac{\partial W^*}{\partial v^*} = a v^* + \tfrac{1}{2} A u^{*2} = 0.$$

$$\Rightarrow \quad u^* \sim \xi^{1/3}, \quad v^* \sim \xi^{2/3}$$

$$\Rightarrow \quad \frac{\partial u^*}{\partial \xi} \sim \xi^{-2/3}, \quad \frac{\partial v^*}{\partial \xi} \sim \xi^{-1/3} \qquad \xi = x^* + \alpha t^*$$

Different degrees of singularity!

Regularization

Adding derivatives (standard!?)

$$\bar{W}^* \to \mathcal{L} = v^* \frac{\partial u^*}{\partial \xi} + W^*(u^*, v^*).$$

$\Rightarrow$ Euler-Lagrange equation

$$\frac{\partial u^*}{\partial \xi} + a\left(\frac{1}{2}u^{*2} + v^*\right) = 0$$

$$\frac{\partial v^*}{\partial \xi} - \xi - a\left(\frac{1}{6}u^{*3} + u^* v^*\right) = 0$$

$$\Rightarrow \quad \left\| \frac{\partial^2 u^*}{\partial \xi^2} + a\left(\xi - \frac{a}{3}u^{*3}\right) = 0 \right\|$$

not a Painlevé-II.

Suleimanov (2002) $\quad \frac{\partial^2 v}{\partial x^2} = v^3 - tv + x$

ZhETP. Singularities in diffusion eq.

Higher gradient catastrophes

Higher strata of singular sector (Kodama, Kon 2002)

$$\frac{\partial W}{\partial u} = \frac{\partial^2 W}{\partial u^2} = \frac{\partial^3 W}{\partial u^3} = \dots = \frac{\partial^{k+2} W}{\partial u^{k+2}} = 0, \quad \frac{\partial^{k+3} W}{\partial u^{k+3}} \neq 0, \dots$$

$k = 1, 2, \dots$

Necessity of sufficient number of independent variables:

1. Hierarchy of commuting flows - times $t_3, t_4, \dots$
2. Families of initial data - parameters $t_3, t_4, \dots$

Expansion near catast point

$$W = W(u_0, v_0) + \varepsilon^{\alpha}\left(u_0 x^* + \left(\tfrac{1}{2}u_0^2 + v_0\right)t^* + \sum_{k=3} P_k(u_0, v_0)\, t_k^*\right) +$$
$$+ \varepsilon^{\alpha+\gamma} \cdot \xi u^* + \varepsilon^{(k+3)\gamma} A_k P_{k+3}(u^*, v^*) + \dots$$

where $\xi = x^* + u_0 t^* + \sum_{k=3} P_{k-1}(u_0, v_0)\, t_k^*$

and $\delta = 2\gamma$, $\alpha = \beta$

Balance $\alpha = (k+2)\gamma = ?$

$$W = W(u_0, v_0) + \varepsilon^{k+3} W^*(u^*, v^*), \qquad \gamma = 1$$

$$W^* = \xi u^* + A_k P_{k+3}(u^*, v^*). \qquad A_k = \left.\frac{\partial^{k+3} W}{\partial u^{k+3}}\right|_0$$

Critical points

$$\frac{\partial W^*}{\partial u^*} = \xi + A_k \frac{\partial P_{k+3}}{\partial u^*} = \xi + A_k P_{k+2}(u^*, v^*) = 0$$

$$\frac{\partial W^*}{\partial v^*} = A_k \frac{\partial P_{k+3}}{\partial v^*} = A_k P_{k+1}(u^*, v^*) = 0$$

So

$$\left.\begin{aligned} &\xi + A_k P_{k+2}(u^*, v^*) = 0 \\ &P_{k+1}(u^*, v^*) = 0 \end{aligned}\right| \qquad k = 1, 2, 3 \ldots$$

$\Rightarrow$

$$u^* \sim \xi^{\frac{1}{k+2}}, \quad v^* \sim \xi^{\frac{2}{k+2}}$$

$\Rightarrow$ $\dfrac{\partial u^*}{\partial \xi} \sim \xi^{-\left(\frac{k+1}{k+2}\right)}, \quad \dfrac{\partial v^*}{\partial \xi} \sim \xi^{-\frac{k}{k+2}}.$

At $\underline{k=2}$. $\qquad P_3(u^*, v^*) = \frac{1}{3} u^{*3} + u^* v^*.$

$\Rightarrow$

$$\xi - \frac{1}{16} A_2 u^{*4} = 0$$

Regularization: $W^* \to \mathcal{L} = v^* \frac{\partial u^*}{\partial \xi} + W^* \Rightarrow$

$$\left.\begin{aligned} &\frac{\partial u^*}{\partial \xi} + A_k P_{k+1}(u^*, v^*) = 0, \\ &\frac{\partial v^*}{\partial \xi} - \xi - A_k P_{k+2}(u^*, v^*) = 0. \end{aligned}\right|$$

Gradient catastrophe for n-component Jordan system

$$\frac{\partial W}{\partial x_k} = \frac{\partial^k W}{\partial x_1^k} \qquad k = 1, \dots, n.$$

Superposition

$$W = x \cdot u_1 + t\left(\tfrac{1}{2} u_1^2 + u_2\right) + \tilde{W}(u_1, \dots, u_n)$$

Critical points $(u_1, \dots, u_n)$

$$\frac{\partial W}{\partial u_k} = 0 \qquad k = 1, \dots, n$$

$$\Rightarrow \quad \frac{\partial W}{\partial u_1} = \frac{\partial^2 W}{\partial u_1^2} = \dots = \frac{\partial^n W}{\partial u_1^n} = 0,$$

regular sector $\dfrac{\partial^{n+1} W}{\partial u_1^{n+1}} \neq 0, \dots$

Relation $\left(W_k \equiv \dfrac{\partial^k W}{\partial u_1^k} \right)$ $x = t_0$, $t = t_1$

$$\begin{pmatrix} W_{n+1} & W_{n+2} & \dots & & W_{2n} \\ 0 & W_{n+1} & W_{n+2} & \cdots & W_{2n-1} \\ \cdot & \cdot & \cdot & \cdot & \\ & & & W_{n+1} & W_{n+2} \\ 0 & 0 & & 0 & W_{n+1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}_{t_k} = - \begin{pmatrix} P_{k-n+1} \\ P_{k-n+2} \\ \vdots \\ \vdots \\ P_k \end{pmatrix}$$

$\Rightarrow$

$$\frac{\partial u_n}{\partial t_k} = -\frac{p_k}{W_{n+1}},$$

$$\frac{\partial u_{n-1}}{\partial t_k} = p_k \frac{W_{n+2}}{W_{n+1}^2} - \frac{p_{k-1}}{W_{n+1}}$$

$\dots$

$$\frac{\partial u_l}{\partial t_k} = C_{n-l+1}\frac{1}{(W_{n+1})^{n-l+1}} + C_{n-l}\frac{1}{(W_{n+1})^{n-l}} + \dots + C_1 \frac{1}{W_{n+1}}. \qquad l = 1,\dots, n \quad k = 0,1$$

Gradient catastrophe (first)

$$\frac{\partial^{n+1} W}{\partial u^{n+1}} \equiv W_{n+1} = 0.$$

So the behaviour of $\frac{\partial u_l}{\partial t_k}$ near the point of g.c. $W_{n+1} \sim \varepsilon \to 0$

$$\frac{\partial u_l}{\partial t_k} \sim \frac{\cdot}{\varepsilon^{n-l+1}} + \frac{\cdot}{\varepsilon^{n-l}} + \dots + \frac{\cdot}{\varepsilon}.$$

Expansion near cat. point $(x_0, t_0, \bar{u}_0)$

$$x = x_0 + \varepsilon^\alpha x^*$$
$$t = t_0 + \varepsilon^\alpha t^*$$
$$u_k = u_{k0} + \varepsilon^k u_k^*$$

Balance $\alpha = n+1$ and

$$W = W_0 + \varepsilon^{n+2} W^*(u_1^*, \dots, u_n^*)$$

with

$$W^* = \xi u_1^* + A_{n+2} \cdot P_{n+2}(u_1^*, \dots, u_n^*).$$

where $A_{n+2} = \left.\frac{\partial^{n+2} W}{\partial u_1^{n+2}}\right|_0$ and $P_k(x_1 \dots x_n)$ are elementary Schur polynomial of n- variables

$$e^{\sum_{k=1}^{n} z^k x_k} = \sum_{l=1}^{\infty} z^l P_l(x_1, \dots, x_n) \qquad z \to 0.$$

$\Rightarrow$ Critical point of W^*

$$\frac{\partial W^*}{\partial u_1^*} = \xi + A_{n+2} P_{n+1}(u_1^*, \dots, u_n^*) = 0$$

$$\frac{\partial W^*}{\partial u_k^*} = A_{n+2} P_{n+2-k}(u_1^*, \dots, u_n^*) = 0 \qquad k = 2, \dots, n.$$

Regularization.

$n=3$

Equations

$$\frac{\partial W^*}{\partial u_1^*} = \xi + A_5\left(\frac{1}{24}u_1^{*4} + \frac{1}{2}u_1^{*2}u_2^* + \frac{1}{2}u_2^{*2} + u_1^* u_3^*\right) = 0$$

$$\frac{\partial W^*}{\partial u_2^*}: \qquad \frac{1}{6}u_1^{*3} + u_1^* u_2^* + u_3^* = 0$$

$$\frac{\partial W^*}{\partial u_3^*}: \qquad \frac{1}{2}u_1^{*2} + u_2^* = 0$$

$\Rightarrow$

$$\xi + \frac{1}{4}A_5 u_1^{*4} = 0,$$

$$u_2^* = -\frac{1}{2}u_1^{*2},$$

$$u_3^* = \frac{1}{3}u_1^{*3}.$$

$$\Rightarrow \frac{\partial u_1^*}{\partial \xi} \sim \xi^{-3/4}, \quad \frac{\partial u_2^*}{\partial \xi} \sim \xi^{-1/2}, \quad \frac{\partial u_3^*}{\partial \xi} \sim \xi^{-1/4}$$

small ξ

Similarity with second stratum at $n=2$. !?