EPSRC Durham Symposium

Geometric and Algebraic Aspects of Integrability

What is Darboux Integrability?

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Overview

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$$u_{xy} = e^u, \quad I = u_{xx} - \frac{1}{2}u_x^2, \quad D_y(I) = 0$$

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With this classical literature (including the tricks), one can solve "explicitly" these so-called Darboux integrable equations.

But many structural questions about these equations remain;

In the context of this conference: Equivalence, Symmetries, IVP, Bäcklund, Zero Curvature.

AND one would really like a SIMPLE organizing principle for all these classical integration methods and examples.

Goals:

1. Motivate a new Lie group theoretic definition of DI.

2. Show how this new definition can be used to effective study all these questions and organize the subject.

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Lie Symmetry Reduction For ODE.

$$u'' - \frac{u'}{u} = 0 \quad \text{(with symmetry } (x,u) \to (\lambda x, \lambda u))$$

$$s = \frac{u}{x}, \; v = y' \quad \text{(symmetry invariants)}$$

$$v' = \frac{v}{v(v-s)} \quad \text{(symmetry reduction)}$$

In this talk we shall deal exclusively with the second type of reduction.

The General Mathematical Setting

Let $\mathcal{I}$ be a differential system on M (encoding some differential equations).

Let G be a Lie group acting on M and define $\Phi_g : M \to M$ by $\Phi_g(x) = g \cdot x$.

Then G is a symmetry group of $\mathcal{I}$ if $\Phi_g^*(\mathcal{I}) = \mathcal{I}$.

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Assume that G acts regularly on M so that $\pi : M \to M/G$ is a smooth submersion.

Definition. The symmetry reduction of $(\mathcal{I}, M)$ by G is the differential system $(\mathcal{I}/G, M/G)$ defined by

$$\mathcal{I}/G = \{\text{forms } \omega \text{ on the reduced space } M/G \mid \pi^*(\omega) \in \mathcal{I}\}.$$

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Symmetry Reduction as a Black Box

$$\begin{bmatrix} \text{a manifold } M \\ \text{vectors or forms } \mathcal{I} \text{ on } M \\ \text{a group } G \text{ preserving } \mathcal{I} \end{bmatrix} \longrightarrow \begin{bmatrix} \text{a smaller space } M/G \\ \text{vectors or forms } \mathcal{I}/G \text{ on } M/G \end{bmatrix}$$

complicated equations $\longrightarrow$ simple equations   ODE reduction

simple equations $\longrightarrow$ complicated equations   Darboux

Superposition Formula For Linear ODE

To help set the stage for what is coming, consider the differential system for 2 copies of a linear second order ODE.

$$y'' + a(x)y' + b(x)y = 0$$

The manifold coordinates are (x, u, p, v, q).

The Pfaffian system is

$$I = \{\, du - p\,dx, dp - (ap + bu)\,dx, dv - q\,dx, dq - (aq + bv)\,dx \,\}.$$

The general linear group is a symmetry group.

Example 1.

Reduce the Pfaffian system I by the special linear group:

$$\Gamma = \{u\partial_u + p\partial_p - v\partial_v - q\partial_q, v\partial_u + q\partial_p, u\partial_v + p\partial_q\}.$$

Calculate the differential invariants for this group.

$$\mathrm{Inv} = \{x, W = uq - vp\}.$$

The reduced differential equation is the differential syzygy

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which is Abel's Identity for the Wronksian.

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Now there are 4 invariants which we write as:

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The reduced differential equation is the differential syzygy

$$U_{xxx} + 3aU_{xx} + (a' + 2a^2 + 4b)U_x + (2b' + 4ab)U = 0.$$

This is the symmetric power of the original 2nd order ODE.

Exercise. Reduce using some other groups.

Liouville Equation

The Liouville equation $u_{xy} = exp(u)$ is the most famous example of a Darboux integrable equation.

The general solution is

$$u = ln(2\frac{f'(x)g'(y)}{(f(x)+g(y))^2})$$

I want to show how this equation and its solution can be obtained by symmetric reduction –

in exactly the same spirit as we derived Abel's identity and the symmetric power equation.

The manifold is the product of jet spaces $J^3(\mathbb{R},\mathbb{R}) \times J^3(\mathbb{R},\mathbb{R})$ with coordinates

$$\{x, u, u_1, u_2, u_3, y, v, v_1, v_2, v_3\}$$

The differential system is the contact system

$$\begin{aligned} \mathcal{C}_1 + \mathcal{C}_2 = \{& du - u_1 dx, du_1 - u_2 dx, du_2 - u_3 dx \\ & dv - v_1 dy, dv_1 - v_2 dy, dv_2 - v_3 dx\} \end{aligned}$$

The symmetry group to be used for the reduction is the simultaneous standard projective action of sl_2 on the dependent variables.

$$\begin{aligned} \Gamma = \{& \partial_u - \partial_v, \quad u\partial_u + v\partial_v + u_1\partial_{u_1} + v_1\partial_{v_1} + \cdots, \\ & u^2\partial_u + 2uu_1\partial_{u_1} - v^2/2\partial_v - v * v_1\partial_{v_1} + \cdots\} \end{aligned}$$

Overview

Symmetry Reduction

Linear ODE

Liouville

Milestones

What is DI?

Properties of DI

Application 1

Application 2

Application 3

Application 4

Application 5

Application 6

Application 7

Application 8

Conclusions

Integrable Systems

The differential invariants are Γ are

$$\text{Inv} = \{x, y, U = log\frac{2u_1 v_1}{(u+v)^2},$$

$$U_x = \frac{u_2}{u_1} - 2\frac{u_1}{(u+v)}$$

$$U_y = \frac{v_2}{v_1} - 2\frac{v_1}{(u+v)}$$

$$U_{xx} = \frac{u_3}{u_1} + \cdots\}$$

The syzygy for the sl_2 differential invariants is:

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> The Liouville equation is the symmetry reduction of a pair of contact systems by the diagonal action of sl_2.

The symmetry group used to make the reduction is called the internal symmetry group.

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Vessiot: 1939, 1941. Here Vessiot gave symmetry group representations of all equations

$$u_{xy} = f(x, y, u, u_x, u_y)$$

which are Daboux integrable at the 2-jet level.

$$u_{xy} = 0 \qquad u_{xy} = \frac{u_x}{u - x}$$

$$u_{xy} = u_x u \qquad u_{xy} = 2\frac{\sqrt{u_x u_y}}{x + y}$$

$$u_{xy} = e^u \qquad u_{xy} = e^u\sqrt{u_x^2 - 1}$$

In so doing he explicitly solved one such equation which Goursat could not solve using intermediate integrals

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The internal symmetry groups all arise as tranformation groups in the plane, as classified many years earlier by S. Lie.

Leznov and Saveliev, 1980 ... 1999. The Toda lattice equations

$$u^i_{xy} = exp(a_{ij} u^j) \quad [a_{ij}] = \text{Cartan matrix}$$

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The representation of the Toda lattice equations by symmtry reduction is found in

– *Representation Theory and Integration of Nonlinear Spherically Symmetric Equations to Gauge Theories*

See also

– *Two-Dimensional Exactly and Completely Integrable Dynamical Systems: Monopoles, Instantons, Dual Models, Relativistic Strings, Lund-Regge Model, Generalized Toda Lattice, etc. it*

. . . all enumerated dynamical systems are joined together due to the presence of non-trivial internal symmetry groups. Just this fact allows one to find explicit expressions for the solutions of the corresponding equations in terms of Lie algebra and group representation theory.

What is Darboux Integrability ?

With the above remarks as motivation we make the following new definition.

Definition: A differential system is called Darboux integrable if it is the differential syzgies of a diagonal group action for the common symmetry of a pair of auxiliary differential equations.

More precisely: A differential system $(\mathcal{I}, M)$ is called Darboux integrable if

$$\mathcal{I}, = (\mathcal{K}_1 + \mathcal{K}_2)/G, \quad M = (M_1 \times M_2)/G$$

where

– $(\mathcal{K}_1, M_1)$ and $(\mathcal{K}_2, M_2)$ are two Pfaffian systems.

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and the following technical requirements holds:

– G acts regularly on M_1 and M_2

– G acts freely on M_1 and M_2

–G acts transversely to $\mathcal{K}_1$ and $\mathcal{K}_2$

We call $\mathcal{K}_1$ and $\mathcal{K}_2$ the defining differential systems and G the internal symmetry group (or Vessiot group).

Meta Principle of DI Systems

Every question you have about a DI system should be answered in terms of the defining differential systems and the internal group G.

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	PDE Property	Symmetry Reduction Property
1.	Intermediate Integrals	Differential Invariants for G
2.	Closed Form Solutions	Defining systems are contact
3.	IVP by quadrature	Solvable G
4.	Equivalence Problem	E.P. for the defining systems
5.	Symmetries	Normalizers of G
6.	Bäcklund Transformations	Subgroups of G
7.	Zero Curvature	Representation theory of G
8.	Leznov and Saveliev	Parabolic Geometry
9.	Classification of DI	Class. of Group Actions

Application 1: Intermediate Integrals

Let us recall the general definition (in terms of a distribution of vector fields (dual to a Pfaffian system).

A distribution $\mathcal{H}$ is called hyperbolic if

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \quad \text{with} \quad [\mathcal{H}_1, \mathcal{H}_2] \subset \mathcal{H}$$

An intermediate integral is a function f such that

$$X(f) = 0 \quad \text{for all } X \text{ in } \mathcal{H}_1.$$

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> Theorems on the existence and number of differential invariants immediately translate to theorems on differential invariants.

Example. The differential invariant for the projective action of sl_2 is the Schwarzian derivative which projects to the intermediate integral for Liouville equation.

$$\frac{u'''}{u'} - \frac{3}{2}\Big(\frac{u''}{u}\Big)^2 \longrightarrow U_{xx} - \frac{1}{2}U_x^2$$

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This 1-1 correspondence between intermediate integrals and differential invariants is very important.

Application 2: Equations with closed form solutions

On November 8, 1908 Forysth gave the Presidential Address to the Cambridge Mathematical Society. This address contains an nice summary of classical geometric integration methods and concludes with a number of open problems.

One of these is to classify all 2nd order scalar PDE in the plane whose general integral is

$$x = V_1(\alpha, \beta, \phi(\alpha), \psi(\beta), \phi', \psi' ...)$$
$$y = V_2(\alpha, \beta, \phi(\alpha), \psi(\beta), \phi', \psi' ...),$$
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Here I would simply state a similar result – if the defining systems are jet spaces, then the general solution to DI system is of the above form.

Application 3: Generalizations of d'Alebert formula

The solution to the Cauchy problem

$$u_{tt} - u_{xx} = 0 \quad u(0,x) = a(x) \quad u_t(0,x) = b(x)$$

is given by the well-known d'Alembert formula

$$u = \frac{1}{2}(a(x-t) + a(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} b(\zeta)\, d\zeta$$

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Theorem The Cauchy problem for a DI integrable system (in 2 independent variables) can be solved by quadratures if the internal symmetry group is solvable.

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Theorem The Cauchy problem for a DI integrable system (in 2 independent variables) can be solved by quadratures if the internal symmetry group is solvable.

This goes back to the basic theorem of Lie on solving ODE by quadratures but now in the context of lifting integral curves.

Example 1.

The solution to the non-linear Cauchy problem

$$u_{xy} = \frac{u_x u_y}{u - x}, \quad u(x,x) = f(x), u_x(x,x) = g(x)$$

is

$$u = x + (f(y) - x)exp(A(x,y)) + exp(-A(0,x)) \int_{t=x}^{t=y} A(0,t)dt,$$

$$A(s,t) = \int_s^t g(\zeta)/(\zeta - f(\zeta))\, d\,\zeta$$

Example 2. The Cauchy problem for $u_{xy} = e^u$ requires the solving a pair of Riccati equations.

Application 3: Equivalence of Darboux Integrable Systems

Theorem. Two DI integrable systems are equivalent if their internal symmetry groups are isomorphism, the actions are equivalent, and their defining differential systems are equivariantly equivalent.

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Project. Within the framework of the DifferentialGeometry software construct a database of known DI systems and their realizations as symmetry reductions.

Example 1. The relativistic string equation (Barbashov, Nesterenko, Chervakov)

$$\theta_{xx} - \theta_{tt} + \frac{\cos(\theta)}{\sin(\theta)^3}(\varphi_x^2 - \varphi_t^2) = 0, \quad (\cot(\theta)^2\varphi_x)_x = (\cot(\theta)^2\varphi_t)_t$$

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$$ds^2 = \frac{1}{1 - e^u}(du^2 + dv^2).$$

$$x' = x + t, \quad t' = x - t, \quad u = \sqrt{e^{\arctan(\theta)} - 1}, \quad v = \frac{1}{2}\varphi.$$

Example 2.

In his classic treatise, Goursat gives two very different 2 examples of DI systems.

$$U_{xy} = 2\frac{\sqrt{U_x U_y}}{x+y}, \quad \text{and}$$

$$u_{xx} + u^2 u_{yy} + 2uu_y^2 = 0$$

It would seem that he was unaware that these systems are equivalent under the transformation

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Remark This transformation is algebraically invertible once we enlarge the underlying 7 manifolds to 8 dimensions.

Application 5: Symmetries of Darboux Integrable Systems

Let $(\mathcal{I}, M)$ be a differential system with full symmetry algebra Σ.

Let Γ be a sub-algebra of Σ.

Then the algebra $\mathrm{nor}_{\Sigma}(\Gamma)/\Gamma$ always determines a sub-algebra of the full symmetry algebra of the reduced system. $(\mathcal{I}/G, M/G)$.

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Generically one expects the reduced system to have *more* symmetries.

But for DI we have the remarkable:

Theorem. Let $(\mathcal{I}, M)$ be a DI system with defining systems $(\mathcal{K}_a, M_a)$ with symmetry algebras Σ_a. Let be $\Gamma \subset \Sigma_a$ be the internal symmetry algebra.

Then the full symmetry algebra of $(\mathcal{I}, M)$ is determined by the normalizer Γ_{diag} in $\Sigma_1 \oplus \Sigma_2$.

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Let Γ be a sub-algebra of Σ.

Then the algebra $\mathrm{nor}_\Sigma(\Gamma)/\Gamma$ always determines a sub-algebra of the full symmetry algebra of the reduced system. $(\mathcal{I}/G, M/G)$.

Generically one expects the reduced system to have *more* symmetries.

But for DI we have the remarkable:

Theorem. Let $(\mathcal{I}, M)$ be a DI system with defining systems $(\mathcal{K}_a, M_a)$ with symmetry algebras Σ_a. Let be $\Gamma \subset \Sigma_a$ be the internal symmetry algebra.

Then the full symmetry algebra of $(\mathcal{I}, M)$ is determined by the normalizer Γ_{diag} in $\Sigma_1 \oplus \Sigma_2$.

Corollary. If $\Sigma_1 = \Sigma_2 = \Sigma$ then

$$\dim \Gamma = \dim \mathrm{nor}_\Sigma(\Gamma) + \dim \mathrm{cent}_\Sigma(\Gamma) - \dim(\Gamma)$$

If the internal symmetry algebra is a MAS then

$$\dim \Gamma = \dim \mathrm{nor}_\Sigma(\Gamma).$$

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Example 1. Thanks to D. The, B. Doubrov, F. Stazzullo

For the defining differential systems take

$$\psi' = (\psi'')^2$$

The symmetry algebra is the exceptional algebra g_2.

There are 2 MAS.

Roots	Normalizer in g_2	Equation
$\alpha_4, \alpha_5, \alpha_6$	9	$rt - s^2 = 3t^4$
$\alpha_3, \alpha_5, \alpha_6$	7	Messy

Application 6: Bäcklund Transformations for Darboux Integrable Systems

Bäcklund transformations for a Darboux integrable system can be constructed from different subgroups of the symmetry groups of the defining differential systems.

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Bäcklund transformations for a Darboux integrable system can be constructed from different subgroups of the symmetry groups of the defining differential systems.

Step 1. Start with a Darboux integrable system.

$$(\mathcal{K}_1 \times \mathcal{K}_2) \xrightarrow{\mathbf{q}_{G_{\mathrm{diag}}}} \mathcal{I} = (\mathcal{K}_1 \times \mathcal{K}_2)/G_{\mathrm{diag}}$$

Step 2. Reduce by another symmetry group L.

$$(\mathcal{K}_1 \times \mathcal{K}_2)$$

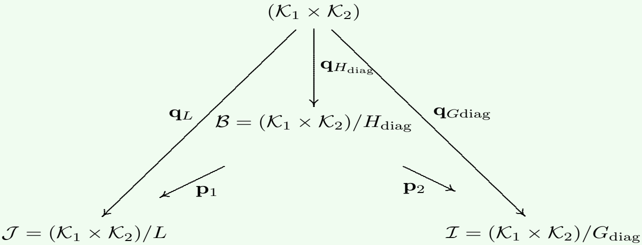
$\mathbf{q}_L$, $\mathbf{q}_{G\mathrm{diag}}$

$$\mathcal{J} = (\mathcal{K}_1 \times \mathcal{K}_2)/L \qquad \mathcal{I} = (\mathcal{K}_1 \times \mathcal{K}_2)/G_{\mathrm{diag}}$$

Step 3. Symmetry reduce by the intersection $H = L \cap G$.

$$
\begin{array}{ccccc}
 & & (\mathcal{K}_1 \times \mathcal{K}_2) & & \\
 & \swarrow \mathbf{q}_L & \downarrow \mathbf{q}_{H_{\mathrm{diag}}} & \mathbf{q}_{G\mathrm{diag}} \searrow & \\
 & & \mathcal{B} = (\mathcal{K}_1 \times \mathcal{K}_2)/H_{\mathrm{diag}} & & \\
\mathcal{J} = (\mathcal{K}_1 \times \mathcal{K}_2)/L & & & & \mathcal{I} = (\mathcal{K}_1 \times \mathcal{K}_2)/G_{\mathrm{diag}}
\end{array}
$$

Step 4. Calculate the orbit projection maps $\mathbf{p_1}$ and $\mathbf{p_2}$.

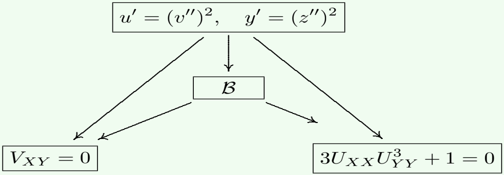


Step 5. Remove the scaffolding to arrive at a Bäcklund transformation.

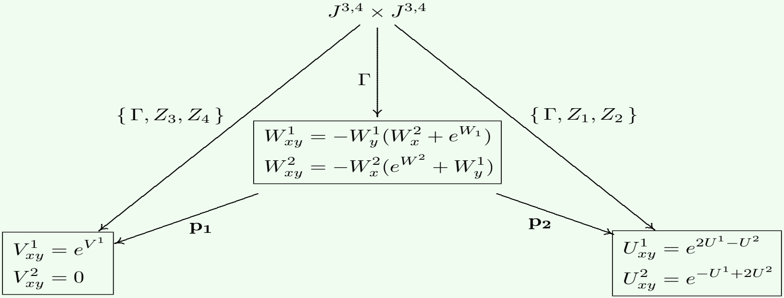
$$\begin{array}{ccc} & \mathcal{B} = (\mathcal{K}_1 \times \mathcal{K}_2)/H_{\mathrm{diag}} & \\ \swarrow \mathbf{p}_1 & & \mathbf{p}_2 \searrow \\ \mathcal{J} = (\mathcal{K}_1 \times \mathcal{K}_2)/L & & \mathcal{I} = (\mathcal{K}_1 \times \mathcal{K}_2)/G_{\mathrm{diag}} \end{array}$$

ALL known examples of Bäcklund transformations for Darboux integrable systems can be constructed by symmetry reduction.

Example 1. A Bäcklund transformation for a fully non-linear equation



Example 2. A de-coupling Bäcklund transformation for the A_2 Toda lattice.



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Application 7: Zero Curvature Formulations for Darboux Integrable Systems

Zero curvature formulations for Darboux integrable systems can be constructed from linear representations of the internal symmetry group.

We illustrate with Liouville's equation

Step 1. The coordinates for $J^3(R, R) \times J^3(R, R)$ are

$$(x, z, z_1, z_2, z_3, y, w, w_1, w_2, w_3)$$

Here is diagonal action used in the symmetry reduction to Liouville's equation.

$$\Gamma_1 = \partial_z - \partial_w,$$
$$\Gamma_2 = z\partial z + z_1\partial_{z_1} + w\partial_w + w_1\partial_{w_1} + \ldots \text{(prolonged to order 3)}$$
$$\Gamma_3 = \frac{z^2}{2}\partial_z + zz_1\partial_{z_1} + \frac{w^2}{2}\partial_w + ww_1\partial_{w_1} + \ldots$$

Step 2. Create an extension by adding (vector space coordinates t_1, t_2 for the adjoint representation)

$$\tilde{\Gamma}_1 = t_2 \partial_{t_1} + \Gamma_1$$
$$\tilde{\Gamma}_2 = t_1 \partial_{t_1} - t_2 \partial_{t_2} + \Gamma_2$$
$$\tilde{\Gamma}_3 = t_1 \partial_{t_2} + \Gamma_3$$

Step 3. Calculate the Pfaffian system which is $\tilde{\Gamma}$ invariant and linear in the new variables

$$\vartheta^1 = dt_1 - \lambda(\frac{z}{z_1} t_1 - \frac{z_2}{z_1} t_2)\, dx$$
$$\vartheta^2 = dt_2 - \lambda(\frac{1}{z_1} t_1 - \frac{z}{z2} t_2)\, dx$$

(to which the contact forms on $J^3(R,R) \times J^3(R,R)$ are added)

Step 4. Calculate the reduced differential system in terms of the differential invariants

$$\sigma_1 = \frac{\sqrt{2}\sqrt{w_1}}{z+w}(t_1 - z t_2), \quad \sigma_2 = \frac{1}{\sqrt{w_1}}(t_1 + w t_2), \quad u = \log\left(\frac{2 z_1 w_1}{(z+w)^2}\right)$$

to be the zero curvature formulation

$$\frac{d}{dx}\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2}e^{u} \\ \sqrt{2}\lambda e^{-u} & 0 \end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix}$$

$$\frac{d}{dy}\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix} = \begin{bmatrix} \frac{1}{2}u_y & 0 \\ \frac{\sqrt{2}}{2} & -\frac{1}{2}u_y \end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix}$$

of $u_{xy} = e^{u}$.

Application 8: Classification of Darboux Integrable f-Gordon Equations

$$u_{xy} = f(x, y, u, u_x, u_y) \qquad (*)$$

In 1899 Goursat give a classification of equations (*) which are DI integrable at order 2.

In 1939, 1941 Vessiot re-produced Goursat's result using the symmetry reduction approach discussed today - in effect one simply calculates the DI systems determined by Lie's classification of vector field systems in the plane.

In 2001 Ziber and Sokolov classified f-Gordon equations (*) which are DI integrable all order. From our perspective, the equations are

[1] reduction of the contact systems on $J^k(R, R)$ Eq 2; Eq 3; Eq 4; Eq 5; Eq 6; Eq 7.

[2] reduction of $z' = y^{(n)}$ by the 2-step nilpotent algebras (and simple variations)

[3] reduction of the Hilbert-Cartan equation $z' = y''$ by 5 dimensional sub-algebras of the exceptional algebra g_2. Eq 8; Eq 9.

Conclusions

1. The definition of Darboux integrability in terms of symmetry reduction by an internal symmetry group is (locally) equivalent to the classical definition in terms of the existence of intermediate integrals/ Darboux invariants/ ... A-Fels-Vaasiliou

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WIP

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THANK-YOU

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