Cartan's C-class equations: solving differential equations by differentiation

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Outline

C-class equations

E. Cartan, Les esapces généralisés et l'intégration de certaines classes d'équations différentielles (1938):

Definition. A given class of differential equations of order n

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

will be said to be a class (C) if there exists an infinite group (in the sense of Lie) $\mathcal{G}$ transforming equations of the class into equations of the class and such that the differential invariants with respect to $\mathcal{G}$ of an equation of the class be the first integrals of the equation.

Example 1: 2nd order ODEs

- Equations $y'' = F(x, y, y')$ satisfying

$$\frac{d^2 F_{y'}}{dx^2} - 4\frac{d^2 F_{yy'}}{dx} - F_{y'}\frac{dF_{y'}}{dx} + 4F_{y'}F_{yy'} - 3F_yF_{y'} + 6F_y = 0.$$

Here $F_y = \partial_y F$, $F_{y'} = \partial_{y'} F$ and $\frac{d}{dx} = \partial_x + y'\partial_y + F\partial_{y'}$.

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$$(x, y) \mapsto (A(x, y), B(x, y)).$$

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- It can be shown that any equation from this class, except for thouse equivalent to $y'' = 0$, can be integrated by means the operation of differentiation and at most two quadratures.

Projective connections on $\mathbb{R}^2$ and cubic 2nd order ODEs

- Any affine connection $\{\Gamma^i_{jk}\}$ on the plane (x, y) defines an equation on *unparametrized* geodesics:

$$y'' = -\Gamma^2_{11}(y')^3 + (\Gamma^2_{22} - 2\Gamma^1_{12})(y')^2 + (2\Gamma^2_{12} - \Gamma^1_{11})y' + \Gamma^2_{11}.$$

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- Two affine connections are said to be projectively equivalent if their unparametrized geodesics coincide. A class of projectively equivalent affine connections is called *a projective connection*.

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- There is a 1-1 correspondence between projective connections and second order ODEs:

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- In particular, the class of such equations is invariant under any (local) diffeomorphisms on the plane (= changes of variables, or point transformations). This means that the condition $\frac{\partial^4 F}{(\partial y')^4} = 0$ is invariant under point transformations, or that $\frac{\partial^4 F}{(\partial y')^4}$ is a *relative invariant* of second order ODEs $y'' = F(x, y, y')$.

Duality and the second relative invariant

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- Define these curves as, for example, graphs of functions $b = b(a)$, differentiate the above relation two times by a and exclude "parameters" x and y. We get so-called *dual* 2nd order ODE:

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$$b'' = G(a, b, b').$$
It is defined modulo point transofrmations in the (a, b) space.
- It can be shown that the dual equation is cubic with respect to b' if and only if the initial equation $y'' = F(x, y, y')$ satisfies:
$$\frac{d^2 F_{y'}}{dx^2} - 4\frac{d^2 F_{yy'}}{dx} - F_{y'}\frac{dF_{y'}}{dx} + 4F_{y'}F_{yy'} - 3F_yF_{y'} + 6F_y = 0.$$

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- (Weyl) These invariants can be found as functions of the curvature of the projective connection and its covariant derivatives. This is tedious, but very explicit and constructive process.
- However, one needs to integrate the equation first to find its dual equations and the corresponding projective connection. Can we do better and construct this projective connection without integrating the equation?

Cartan connections for general 2nd order ODEs

- Given an arbitrary 2nd order ODE $y'' = F(x, y, y')$, Élie Cartan constructs a so-called *Cartan connection* on the space $(x, y, z = y')$ of contact elements on on the plane ($=$ jet space $J^1(\mathbb{R}, \mathbb{R})$).

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- Cartan connection is modelled by a homogeneous space $PSL(3, \mathbb{R})/B$ and consists of the following data:
  1. principal B-bundle $\pi \colon \mathcal{G} \to J^1(\mathbb{R}, \mathbb{R})$;
  2. 1-form $\omega \colon T\mathcal{G} \to \mathfrak{sl}(3, \mathbb{R})$ that satisfies properties similar to the Maurer-Cartan form on the Lie group $PSL(3, \mathbb{R})$.
  3. the form $d\omega + [\omega, \omega]$ is zero only on vertical vector fields and forms the curvature tensor Ω of the Cartan connection.

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  3. the form $d\omega + [\omega, \omega]$ is zero only on vertical vector fields and forms the curvature tensor Ω of the Cartan connection.
- The construction of this Cartan connection for a given 2nd order ODE is very explicit. All components $\omega_{ij}, i, j = 1, \ldots, 3$ of the connection form ω are expressed explicitly in terms of the function $F(x, y, z)$ and its partial derivatives.

Integration of C-class 2nd order ODEs

- The connection descends to the parameter space (=space of solutions) if and only if the curvature of this connection vanishes identically on the direction $\partial_x + z\partial_y + F\partial_z$ along the solutions of the equation. This turns to be equivalent to the above "complicated" condition on $F(x, y, z)$.

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- In this case we end up with a Cartan connection with vanishing curvature. This happens if and only if the initial 2nd order ODE is *trivializable* (=equivalent to the trivial equation $y'' = 0$).
- Integrating the equation is then equivalent to finding this trivialization transformation. And this is equivalent to integrating the form ω itself, i.e. constructing the map $f\colon \mathcal{G} \to PSL(3, \mathbb{R})$ such that $f^{-1}df = \omega$. Such problems are known are *Lie type equations*.

Example 2: 3rd order ODEs

- Equations $y'^{(3)} = F(x, y, y', y'')$ satisfying

$$\frac{d^2 F_{y''}}{dx^2} - 2F_{y''}\frac{dF_{y''}}{dx} - 3\frac{dF_{y'}}{dx} + \frac{4}{9}\left(F_{y''}\right)^3 + 3F_{y'}F_{y''} + 6F_y = 0.$$

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- The pseudogroup $\mathcal{G}$ consists of all contact transformations:

$$(x, y, y') \mapsto (A, B, C), \quad dB - C\,dA = \lambda(dy - y'\,dx).$$

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- There is a natural conformal structure on the (3-dimensional) solution space of ODEs of a given class.

More recent examples

- Scalar ODEs of 4th order (R. Bryant, 1991): under certain (explicit) conditions on the right hand side of the equation $y^{IV} = F(x, y, y', y'', y''')$ the solution space carries a natural torsion-free affine connection with $GL(2, \mathbb{R})$ holonomy.

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- Systems of m ODEs of 2nd order (D. Grossman, 2000): if the associated Cartan connection has vanishing torsion, the solution space carries a natural Segre (or Grassmanian) structure defined as a decomposition of the tangent space as a tensor product of $\mathbb{R}^2 \otimes \mathbb{R}^m$. If $m = 2$, this is equivalent to the conformal structure of split signature $(2, 2)$.

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- In both cases it is shown that the natural Cartan connections for these classes of equations descend to the solution spaces and can be used to construct first integrals and integrate the equation.

Questions

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- What are the conditions on the equation which guarantee that such structures exist?
- Is there a way to construct connections for these structures without integrating an equation?

Generalized Wilczynski invariants

- Consider a linear system on $y(x) \in \mathbb{R}^m$:

$$y^{(k)} + P_{k-1}(x)y^{(k-1)} + \cdots + P_0(x)y(x) = 0$$

up to transformations $(x, y) \mapsto (\lambda(x), \mu(x)y)$, $\mu(x) \in GL(m)$.

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- Then the following expressions become fundamental invariants for the class of linear equations:

$$\Theta_r = \sum_{j=1}^{r-1} (-1)^{j+1} \frac{(2r-j-1)!(k-r+j-1)!}{(r-j)!(j-1)!} P_{k-r+j-1}^{(j-1)},$$

for $r = 2, \ldots, k$.

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- *Generalized Wilczynski invariants* W_r, $r = 2, \ldots, k$ for a non-linear system are defined as invariants Θ_r evaluated at the linearization of the system.

Structures on the solution spaces for scalar ODEs

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- It is no longer true for non-linear ODEs, as they possess other invariants independent of the generalized Wilczynski ones. However, ODEs with vanishing generalized Wilczynski invariants carry a very special geometric structure on the solution space $\mathcal{S}$.
- In case of a scalar ODE we get a GL_2 structure on the solution space iff $W_3 = \cdots = W_k = 0$. The structure can be defined as a rational curve P^1 embedded into each (projectivized) tangent space:

  $$[1 : t : \cdots : t^{k-1}].$$

  In other words, the tangent space $T\gamma\mathcal{S}$ of the solution space at a "point" γ is identified with an irreducible representation V_{k-1} of the $\mathfrak{sl}_2$.

Structures on the solution spaces for systems of ODEs

- In case of systems of n ODEs we get a $GL_m \otimes SL_2$ structure iff $W_2 = W_3 = \cdots = W_k = 0$. The structure can be defined as a projective variety $P^1 \times P^{m-1}$ embedded into each (projectivized) tangent space:

$$[z_1 : t\, z_1 : \cdots : t^{k-1} z_1 : \quad z_2 : t\, z_2 : \cdots : t^{k-1} z_2 : \ \cdots \qquad z_m : t\, z_n : \cdots : t^{k-1} z_m].$$

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- For scalar 3rd order ODEs we get conformal structures on 3-dimensional manifolds equipped with an Einstein-Weyl structure.
- For systems of two equations of 2nd order we get an ASD conformal structure on a 4-dimensional manifold.

Equations as double fibration

- Generic system of m equations of order k:

$$y_i(x)^{(k)} = F_i(x, y_j^{(l)}),\ 1 \leq i, j \leq m;\ 0 \leq l \leq k-1.$$

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- Geometrically it is a section $\sigma\colon J^{k-1}(\mathbb{R}, \mathbb{R}^m) \to J^k(\mathbb{R}, \mathbb{R}^m)$ or just a submanifold $\mathcal{E} \subset J^k$ with locally diffeomorphic projection to J^{k-1}.

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  - point transformations for systems of ODEs ($m \geq 2, k \geq 2$).

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  - contact transformations for scalar ODEs ($m = 1, k \geq 3$)
  - point transformations for systems of ODEs ($m \geq 2, k \geq 2$).
- There are two natural foliations on $\mathcal{E}$:
  - the foliation on solutions lifted to J^k. Its tangent direction is given by total derivative.
  - "vertical" foliation of fibers of projection $\pi\colon \mathcal{E} \to J^{k-2}$.

Equations as double fibration

- Generic system of m equations of order k:

$$y_i(x)^{(k)} = F_i(x, y_j^{(l)}),\ 1 \leq i, j \leq m;\ 0 \leq l \leq k-1.$$

- Geometrically it is a section $\sigma\colon J^{k-1}(\mathbb{R}, \mathbb{R}^m) \to J^k(\mathbb{R}, \mathbb{R}^m)$ or just a submanifold $\mathcal{E} \subset J^k$ with locally diffeomorphic projection to J^{k-1}.
- As a pseudogroup $\mathcal{G}$ we take
  - contact transformations for scalar ODEs ($m = 1, k \geq 3$)
  - point transformations for systems of ODEs ($m \geq 2, k \geq 2$).
- There are two natural foliations on $\mathcal{E}$:
  - the foliation on solutions lifted to J^k. Its tangent direction is given by total derivative.
  - "vertical" foliation of fibers of projection $\pi\colon \mathcal{E} \to J^{k-2}$.
- This double fibration completely determines the extrinsic geometry of ODEs. Hence, all invariants of ODEs are exactly the invariants of this double fibration.

Cartan-Tanaka approach

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- For $m = 1, k \geq 4$, or $m \geq 2, k \geq 3$ this Cartan connection is no longer modelled by parabolic homogeneous spaces.
- In these cases the symbol algebra $\mathfrak{g}$ (=symmetry algebra of the trivial system) is a semi-direct product of $\mathfrak{sl}(2) \times \mathfrak{gl}(m)$ and an irreducible representation $V = \mathcal{V}_{k-1} \otimes \mathbb{R}^m$.

Understanding the curvature

- All invariants of the ODEs under the action of the pseudogroup $\mathcal{G}$ $\Leftrightarrow$ invariants of the corresponding double fibration $\Leftrightarrow$ invariants of the associated Cartan connection

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- This part of the curvature can be identified via a pure algebraic object: $H^2_+(\mathfrak{g}_-, \mathfrak{g})$. We call it the *fundamental invariants*.
- Generalized Wilczynski invariants form a part of the fundamental invariants. But there are others!

C-class equations of any order

Theorem

The following classes of equations:

- *scalar ODEs of order* ≥ 3 *viewed up to contact transformations;*
- *systems of ODEs of order* ≥ 2 *viewed up to point transformations*

with vanishing generalized Wilczynski invariants form (C) classes.

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- Idea of the proof. Generalized Wilczynski invariants form part of the curvature of the Cartan connection associated with a given ODE. If they vanish, we can use Bianchi identity to prove that the curvature tensor vanishes on the direction tangent to the solutions and that the connection descends to a natural connection on a solution space.

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- However, direct use of Bianchi identities is very messy. Smart algebraic techniques coming splitting operators in parabolic geometries are required to sort this out.

Examples

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- Complete deformation family of a non-degenerate conic in P^2 is the space of all conics and is given by the following 5th order ODE:

$$9(y'')^2y^{(5)} - 45y''y'''y^{(4)} + 40(y''')^3 = 0.$$