From the Bow Integrable System to the Kähler Potential on the Moduli Spaces of G-monopoles and instantons

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Outline

- Motivation for studying bow integrable systems:
  - Moduli spaces of Yang-Mill instantons on ALF spaces.
  - Solutions of supersymmetric 3D quantum field theories.
  - Geometric Langlands for complex surfaces.
  - String theory brane dynamics.
  - Bow’s internal beauty.
- Analytic results: curvature decay, asymptotic form, and Index.
- Up Transform: Bow -> Instanton.
- Down Transform: Instanton -> Bow (via scattering or via index bundle).
- Spectral Curves and Dynkin diagrams.
- Kähler potential on the space of solutions.

Motivation

Physics

a) Yang-Mills Theory (Strong, Week, and Electromagnetic interactions) studies connection one-form A
on a Hermitian bundle $E \longrightarrow M^4$.

with curvature of A is $F=dA+A\wedge A$ and action $S[A]=\int \mathrm{tr}\, F\wedge {*F}$

Its Euler-Lagrange equation is the Yang-Mills Equation $d_A{*F}=0$. *Extremum*

b) Euclidean Feynman Path Integral

$$< \mathcal{O}(A) >= \int \mathcal{O}(A) e^{-\frac{1}{\hbar}S[A]} \mathcal{D}A$$

$$\int tr(F \pm *F) \wedge (F \pm *F) > 0 \qquad \Rightarrow \qquad S[A] \geq \int \pm \mathrm{tr} F \wedge F$$

second Chern character

dominant contributions are delivered by the minima of the Yang-Mills action:

$$F = - * F$$ Anti-Self-Duality Equation

Minimum

Def: An **Instanton** is a connection A, with square integrable curvature and $F=-{*F}$.

Geometry

A_0 ALE: Flat metric in "radial coordinates"

$\mathbb{R}^4 \simeq \mathbb{H} \ni q = a e^{I \frac{\tau}{2}}$ with a pure imaginary

(x,τ) are "polar coordinates" on $\mathbb{R}^4$

$$x = x_1 I + x_2 J + x_3 K := q I \bar{q} = a I \bar{a}$$

$$ds^2 = dq d\bar{q} = \frac{1}{4}\left(\frac{1}{|x|} d\vec{x}^2 + \frac{(d\tau + \omega)^2}{\frac{1}{|x|}}\right) \qquad d\omega = *_3 dV(x)$$

A_0 ALF: Taub-NUT space (TN)

$$ds^2 = dq d\bar{q} = \frac{1}{4}\left(V(x) d\vec{x}^2 + \frac{(d\tau + \omega)^2}{V(x)}\right)$$

$$V(x) = l + \frac{1}{|x|}$$

A_{k-1} ALF: multi-Taub-NUT space TN_k

$$V(x) = l + \sum_{\sigma=1}^{k} \frac{1}{|x - \nu_\sigma|}$$

Analytic questions about Instantons on Taub-NUT:

- Does L^2 Yang-Mills solution have to be L^∞?
  Does F_A necessarily decay at infinity?

- How fast does Instanton curvature decay?
  (e.g. on $\mathbb{R}^4$ it decays as $1/r^4$, what is it for ALF?)

- What is its asymptotic form?

- What is the behavior of Harmonic Forms?

- How many of them? Index theorem?

Analytic Results:

with Andres Larrain-Hubach and Mark Stern

Theorem (Decay)**:** For (M,g) a complete Riemannian manifold of bounded geometry and a connection A on E $\longrightarrow$ M

1) $F \in L^2$ and $d_A * F = 0$ $\implies$ $\lim_{|x|\to\infty} |F(x)| = 0.$

2) If F=*F and F∈L^2 and M is the multi-Taub-NUT space, then there is C>0, such that |r^2 F|<C.

Theorem (Asymptotic)**:**

An instanton on TN_k with generic holonomy can be put (by a choice of trivialization) in the form

$$A = -i\mathrm{diag}(a_1, a_2, \ldots, a_n) + O\left(\frac{1}{r^2}\right),$$

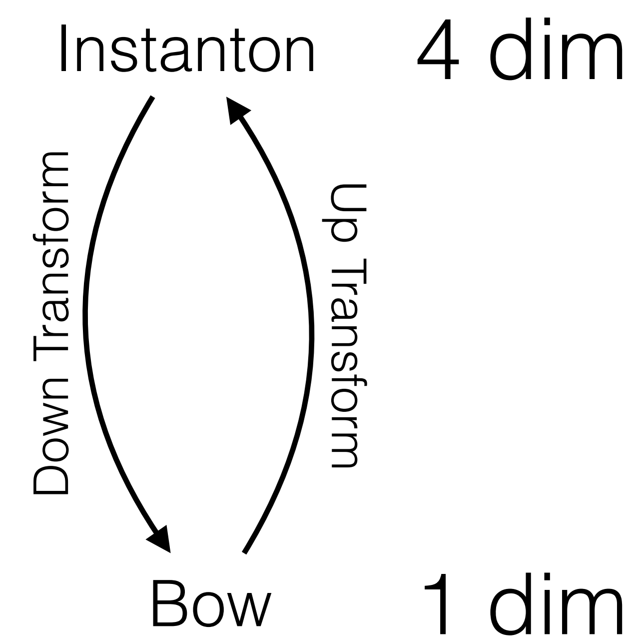
with $$a_j = \left(\lambda_j + \frac{m_j}{r}\right)\frac{d\tau + \omega}{V} + \frac{m_j}{k}\omega.$$

Monopole charges
=Chern number of
holonomy eigenbundles

Def: For a point $r\hat{n}$ write eigenvalues of the holonomy of A around $S^1_{r\hat{n}}$ as $e^{4\pi i \mu_j / l}$.
An instanton A has generic holonomy, if there is a direction $\hat{n}$ such that the limits $\lim_{r\to\infty} \mu_j(r\hat{n}) = \lambda_j$ exist with all λ_j distinct.

Theorem (Up)**:**

The connection resulting from the Up Transform is an instanton on TN_k, with the instanton asymp. holonomy and topological class determined by the bow representation.



Theorem (Bijection)**:**

Up Transform is a bijection of Bow and Instanton moduli spaces.

$$\mathrm{Up}\circ\mathrm{Down}=I_{TN} \qquad \mathrm{Down}\circ\mathrm{Up}=I_{Bow}$$

Theorem (Isometry)**:**

Up Transform is an isometry of the hyperkähler moduli space of a Bow representation and the corresponding moduli space of an Instanton on TN_k.

Index of the Dirac Operator

Theorem (Index)**:**

$$\mathrm{ind}_{L^2} D_A^+ = \sum_j \left((\frac{1}{2} - \{\lambda_j/l\})(k\lfloor \lambda_j/l \rfloor - m_j) - \frac{k}{2}\{\lambda_j/l\}^2 \right) + \frac{1}{8\pi^2} \int F \wedge F,$$

Here $\quad A = -i\mathrm{diag}(a_1, a_2, \ldots, a_n) + O\left(\frac{1}{r^2}\right),$

with $\quad a_j = \left(\lambda_j + \frac{m_j}{r}\right) \frac{d\tau + \omega}{V}$ - $\frac{m_j}{k}\omega.$

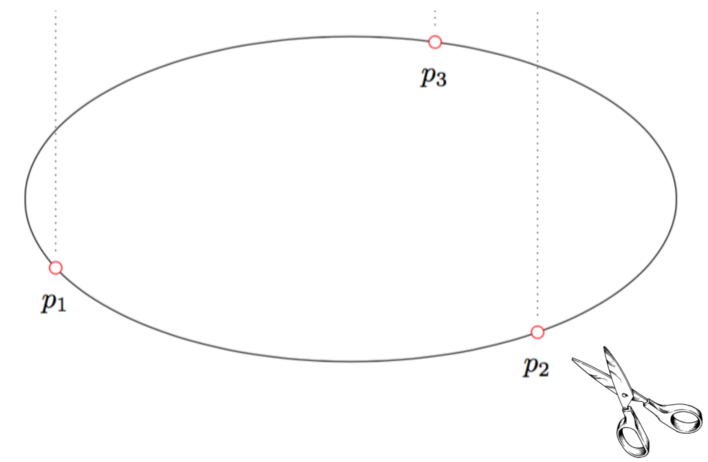
Up Transform

TN$_k$ corresponds to A$_{k-1}$ Bow
(example for k=3):

Bow:

Representation *R* of the bow:

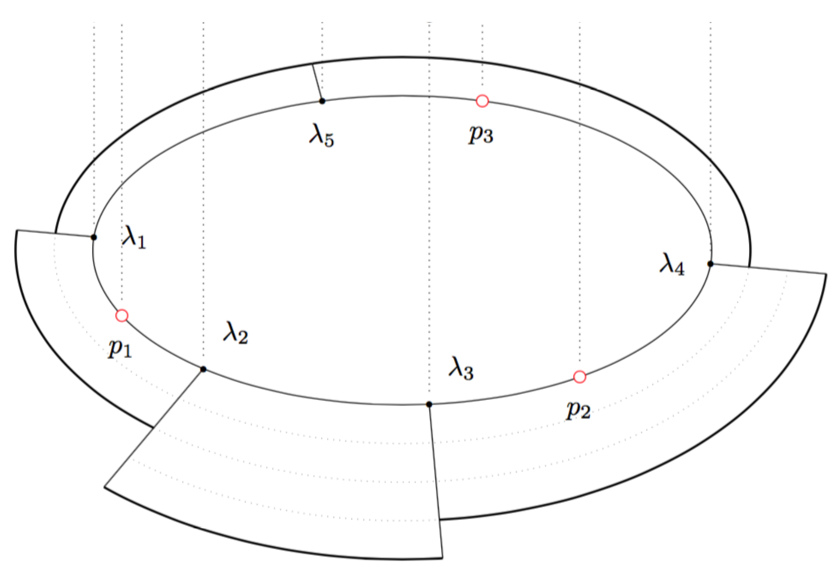
Circle diagram:



$\{\lambda_j\}, R(s)$ constant on each subinterval

R(s) determines the rank of the Hermitian bundle E over each subinterval.

If R is continuous at λ, introduce W$_\lambda$=$\mathbb{C}$



Let e_1, e_2, and e_3 denote quaternionic units representation and S be a 2-dim representation space.

- Affine space: Dat(R)=B⊕F⊕N is hyperkähler

B: $$B_\sigma^+ = \begin{pmatrix} B_{\sigma,\sigma+1}^\dagger \\ B_{\sigma+1,\sigma} \end{pmatrix} \in \mathrm{Hom}(E_{p_\sigma-}, S \otimes E_{p_\sigma+})$$

F: $$Q_\lambda = \begin{pmatrix} J_\lambda^\dagger \\ I_\lambda \end{pmatrix} \in \mathrm{Hom}(W_\lambda, S \otimes E_\lambda)$$

N: $$D = \frac{d}{ds} + T_0 + e_j T_j \in \mathrm{Con}(S \otimes E)$$

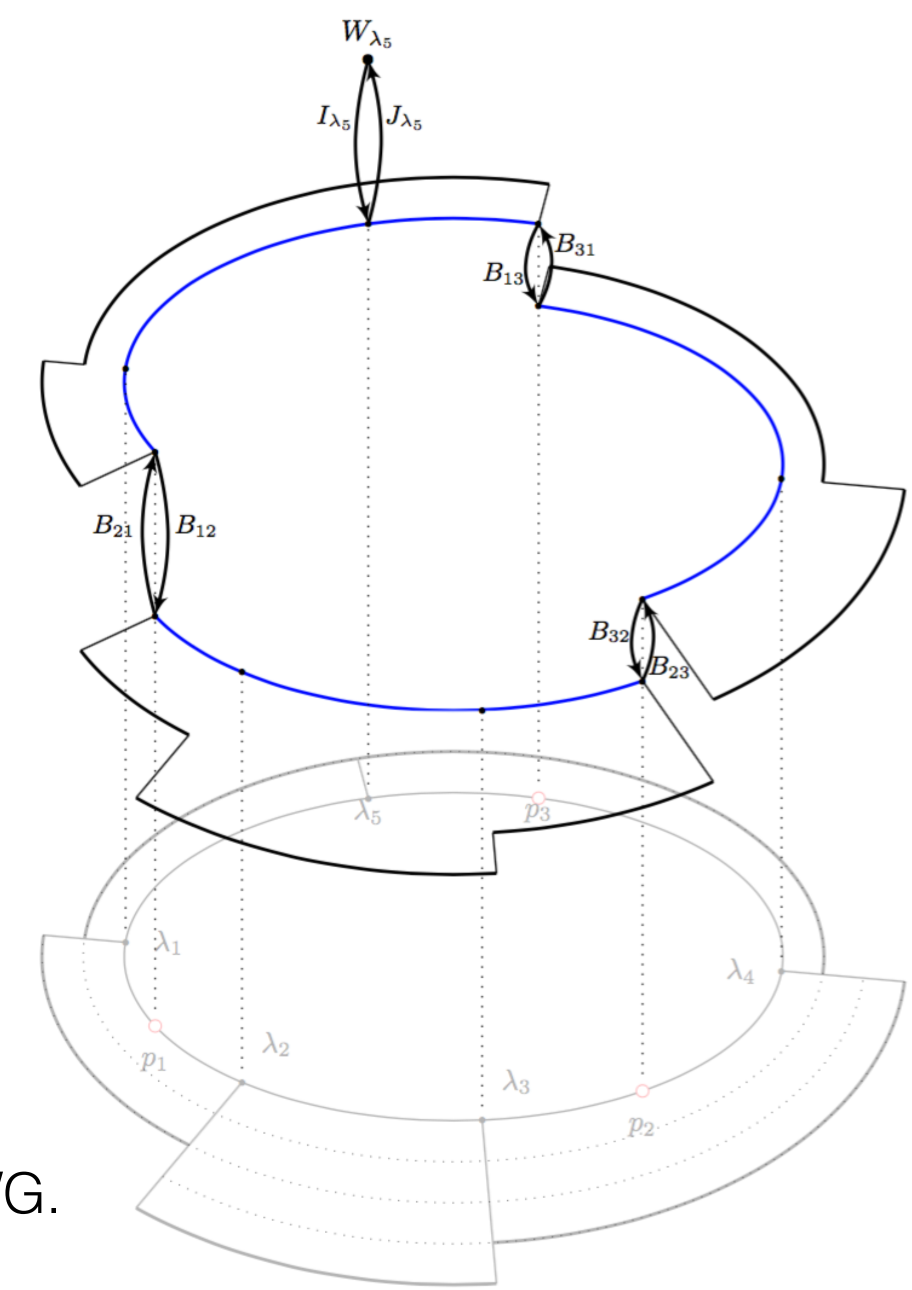
- The group G of gauge transformations on E act triholomorphically on Dat(R)!

$$B_\sigma^+ \mapsto g(p_\sigma-) B_\sigma^+ g(p_\sigma+),$$

$$Q_\lambda \mapsto g(\lambda) Q_\lambda,$$

$$T_0(s) \mapsto g^{-1}(s) T_0 g(s) + g^{-1}(s) \frac{d}{ds} g(s),$$

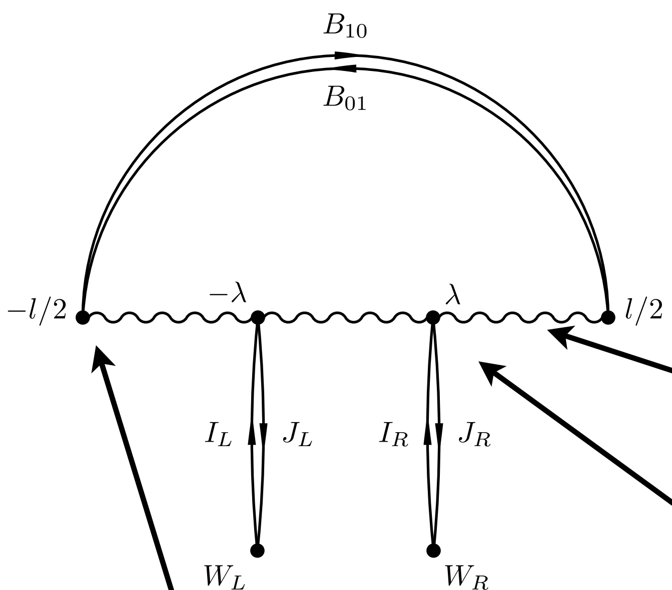
$$T_j(s) \mapsto g^{-1}(s) T_j g(s).$$



- Bow moduli space is the hyperkähler reduction M_R=Dat(R)///G.

Moment Map Conditions

For a bow representation its moment map conditions are:



Real form:

$$\begin{cases} \frac{d}{ds}T_1 + [T_0, T_1] = [T_2, T_3] \\ \frac{d}{ds}T_2 + [T_0, T_2] = [T_3, T_1] \\ \frac{d}{ds}T_3 + [T_0, T_3] = [T_1, T_2] \end{cases}$$

Nahm's Eqs

$$T_1(\lambda+) - T_1(\lambda-) = \frac{1}{2}\left(I_\lambda J_\lambda + J_\lambda^\dagger I_\lambda^\dagger\right),$$
$$T_2(\lambda+) - T_2(\lambda-) = \frac{i}{2}\left(J_\lambda^\dagger I_\lambda^\dagger - I_\lambda J_\lambda\right),$$
$$T_3(\lambda+) - T_3(\lambda-) = \frac{1}{2}\left(J_\lambda^\dagger J_\lambda - I_\lambda I_\lambda^\dagger\right),$$

$$T_1(p_{\sigma L}) = \sum_{\substack{e\in\mathcal{E}\\ h(e)=p_{\sigma L}}} \left(\frac{1}{2}\left(B_e^{LR}B_e^{RL} + (B_e^{RL})^\dagger(B_e^{LR})^\dagger\right) - \nu_{e1}\right), \qquad T_1(p_{\sigma R}) = \sum_{\substack{e\in\mathcal{E}\\ t(e)=p_{\sigma R}}} \left(\frac{1}{2}\left(B_e^{RL}B_e^{LR} + (B_e^{LR})^\dagger(B_e^{RL})^\dagger\right) - \nu_{e1}\right),$$
$$T_2(p_{\sigma L}) = \sum_{\substack{e\in\mathcal{E}\\ t(e)=p_{\sigma L}}} \left(\frac{i}{2}\left((B_e^{RL})^\dagger(B_e^{LR})^\dagger - B_e^{LR}B_e^{RL}\right) - \nu_{e2}\right), \qquad T_2(p_{\sigma R}) = \sum_{\substack{e\in\mathcal{E}\\ t(e)=p_{\sigma R}}} \left(\frac{i}{2}\left((B_e^{LR})^\dagger(B_e^{RL})^\dagger - B_e^{RL}B_e^{LR}\right) - \nu_{e2}\right),$$
$$T_3(p_{\sigma L}) = \sum_{\substack{e\in\mathcal{E}\\ t(e)=p_{\sigma L}}} \left(\frac{1}{2}\left((B_e^{RL})^\dagger B_e^{RL} - B_e^{LR}(B_e^{LR})^\dagger\right) - \nu_{e3}\right), \qquad T_3(p_{\sigma R}) = \sum_{\substack{e\in\mathcal{E}\\ t(e)=p_{\sigma R}}} \left(\frac{1}{2}\left(B_e^{RL}(B_e^{RL})^\dagger - (B_e^{LR})^\dagger B_e^{LR}\right) - \nu_{e3}\right).$$

This is the Integrable System associated with a Bow Rep.

Complex form:

$$D = \frac{d}{ds} - iT_0 - T_3 \text{ and } T = T_1 + iT_2,$$

$$[D,T] - \delta_{(s+\frac{l}{2})} B_{01} B_{10} + \delta_{(s-\frac{l}{2})} B_{10} B_{01} + \sum_{\alpha \in \{L,R\}} \delta_{(s-\lambda_\alpha)} I_\alpha J_\alpha = 0$$

$$[D^\dagger, D] + [T^\dagger, T] + \delta_{(s+\frac{l}{2})}(B_{10}^\dagger B_{10} - B_{01} B_{01}^\dagger) + \delta_{(s-\frac{l}{2})}(B_{01}^\dagger B_{01} - B_{10} B_{10}^\dagger) + \\ + \sum_{\alpha \in \{L,R\}} \delta_{(s-\lambda_\alpha)}(J_\alpha^\dagger J_\alpha - I_\alpha I_\alpha^\dagger) = 0.$$

Quaternionic form:

$$\mathfrak{D}^\dagger = \left(\frac{d}{ds} - iT_0 - \mathbf{T}\right) \oplus \bigoplus_\alpha \delta(s-\lambda_\alpha) Q_\alpha \oplus \bigoplus_{a \in \{\text{Arrows}\}} \left(\delta(s-t(a))\, \reflectbox{B}^a,\ \delta(s-h(a)) B^a\right),$$

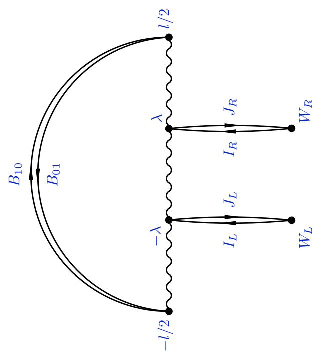
$$\mathrm{Im}_{\mathbb{H}}\, \mathfrak{D}^\dagger \mathfrak{D} = 0$$

||

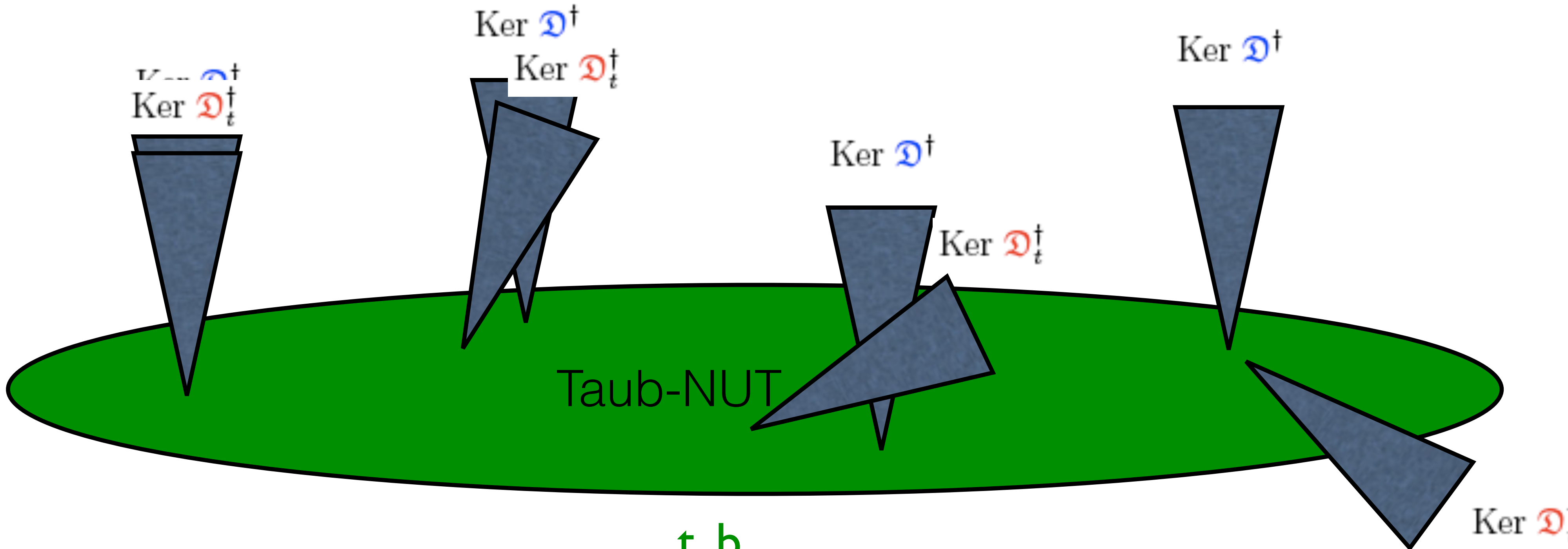
Moment map

w/ SC '11

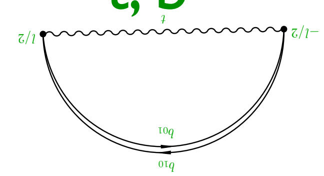
$\mathfrak{M}_0$



T, B, Q



t, b



$$\boxed{\mathfrak{D}_t^\dagger = \mathfrak{D}^\dagger \otimes 1 + 1 \otimes \mathfrak{d}^\dagger.}$$

$$\boxed{\mathfrak{D}_t^\dagger \Upsilon = 0}$$

the self-dual connection on TN is

$$A = (\Upsilon, (d + a_s)\Upsilon)$$

$$A = \left(\Upsilon, \left(\frac{\partial}{\partial \tau} + \frac{s}{2V}\right)\Upsilon\right) d\tau + \left(\Upsilon, \left(\frac{\partial}{\partial t_j} + \omega_j \frac{s}{2V}\right)\Upsilon\right) dt_j.$$

1) Find solutions of the LARGE bow rep. and a small bow rep.

2) Form Dirac operators

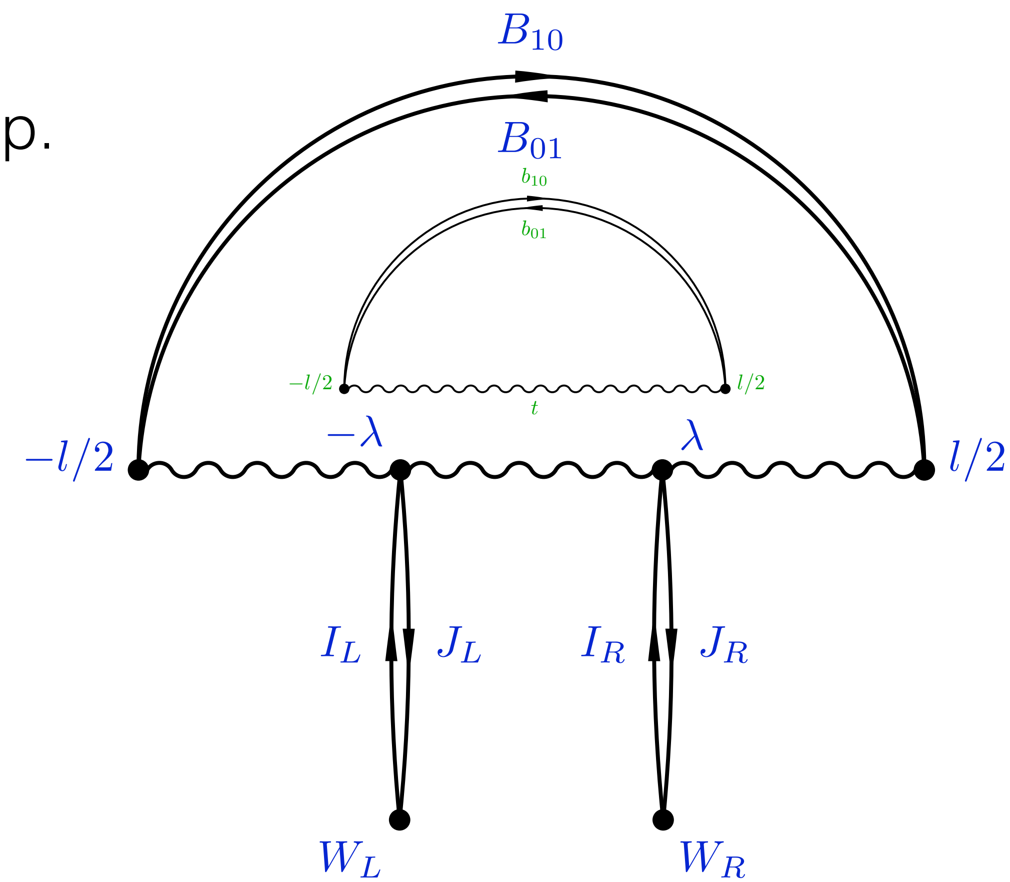
$$\mathfrak{D}^\dagger = \left(\frac{d}{ds} - iT_0 - \mathcal{T}\right) \oplus \bigoplus_\alpha \delta(s-\lambda_\alpha) Q_\alpha \oplus \bigoplus_{a \in \{\mathrm{Arrows}\}} \left(\delta(s-t(a))\, \reflectbox{B}^a,\ \delta(s-h(a)) B^a\right),$$

$$\mathfrak{d}^\dagger = \left(\frac{d}{ds} - it_0 - \mathfrak{t}\right) \oplus 0 \oplus \bigoplus_{a \in \{\mathrm{Arrows}\}} \left(\delta(s-h(a)) b^a \oplus \delta(s-t(a)) \mathrm{d}^a\right).$$

with opposite moment map values

and the twisted operator

$$\boxed{\mathfrak{D}_t^\dagger = \mathfrak{D}^\dagger \otimes 1 + 1 \otimes \mathfrak{d}^\dagger.}$$



3) Form the orthonormal basis of solutions $\Upsilon = (\chi(s), f_\lambda, \nu_+, \nu_-)$

to the Bow Dirac equation $\mathfrak{D}_t^\dagger \Upsilon = 0$

4) Find the Higgs field and the gauge field of the singular monopole from

$$A = (\Upsilon, (d + a_s)\Upsilon)$$

Down Transform

Since the Taub-NUT space is a moduli space of a small bow representation **s**

it comes not only equipped with a metric,

but also equipped with a series of instantons. It is assembled of

$$a^0 = \frac{d\tau + \omega}{V} = \frac{\theta^0}{\sqrt{V}}$$, which can be multiplied by ay real factor, and

and, also for each NUT

$$a_\sigma = \frac{1}{|x - \nu_\sigma|} \frac{d\tau + \omega}{V} - \eta_\sigma$$

These assemble into the connection on the *tautological bundle*:

$$a(s) = s a^0 + \sum_{\sigma | p_\sigma < s} a_\sigma .$$

Dirac Operator

Clifford Algebra: $\{c^p, c^q\} = -2\delta^{pq}$, $c^p = Cl(\theta^p)$, $\gamma^5 = c^0 c^1 c^2 c^3$

Spin bundle splits into $S = S^+ \oplus S^-$,

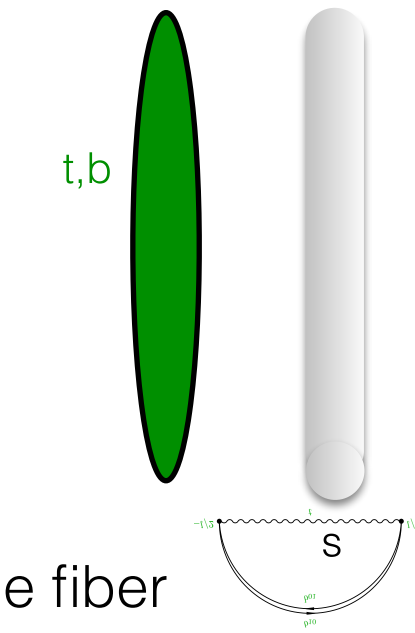
Not trivial

Trivial

S^- carries a representation of quaternions $I^j = c^0 c^j$!

The fiber of S^-, after being trivialized, is identified with S on the bow of the Up Transform.

$$\begin{array}{ccc} & & G_s \\ & & \downarrow \\ TN & \leftarrow & L_s \end{array}$$



- TN_k space can be viewed as a HK quotient,
  TN_k=(Level Set)/G where G is the group of bow gauge transformations.

- Let $H_s \subset G$ be its subgroup of gauge transformations acting trivially on the fiber at s. We have $H_s \to G \to G_s$, where G_s is the group acting on the fiber at s.
  Thus TN_k space comes with a family of G_s tautological principal line bundles (Level Set)/H_s=$L_s \to$ALF,
  moreover, since the Level Set inherits a metric, L_s comes with ASD connection a_s.

- Associated family of Dirac operators: d_s acting on $S \otimes L_s$.

- Given an instanton on $\mathscr{E} \to$ALF, there is an associated Dirac operator D acting on $S \otimes \mathscr{E}$.

- Form a family of Twisted Dirac operators $D_s := D \otimes 1_{L_s} + 1_{\mathscr{E}} \otimes d_s$.

- Due to anti-self-duality it satisfies $D_s D_s = -\nabla^* \nabla$ (covariant Laplace-Beltrami) on S^-.

- We obtain the index bundle (Ker D_s)=E$\to$Bow. This is the bow representation.

$$Es = \mathrm{Ker}\, D_s,\ R(s) = \mathrm{Ind}\, D_s,\ W_\lambda = \mathrm{Ker}_{\mathrm{Bounded}}\, \nabla^* \nabla$$

- Family of Dirac operators parameterized by the point s on the Bow: $D_s\Psi = 0$

L[2] solutions span E→Bow

- Whenever s matches a holonomy eigenvalue λ there is a covariantly constant at infinity solution to the Laplace equation: $\nabla^*\nabla f_\lambda = 0$

- Small bow rep. data (t,b) induces similar data on the LARGE rep. E→Bow:

$$T_0(s) = \int_{TN} \Psi^\dagger i\frac{d}{ds}\Psi d^4\mathrm{Vol} \qquad T_j(s) = \int_{TN} \Psi^\dagger t_j \Psi d^4\mathrm{Vol}$$

$$B_{th}(s) = \int_{TN} \Psi_t^\dagger b_{th} \Psi_h d^4\mathrm{Vol} \qquad B_{ht}(s) = \int_{TN} \Psi_h^\dagger b_{ht} \Psi_t d^4\mathrm{Vol}$$

$$Q_\lambda = \int_{TN} \Psi_\lambda^\dagger D_\lambda f_\lambda d^4\mathrm{Vol}$$

- (T,B,Q) satisfy the moment map conditions on the Bow for the LARGE bow rep.

(T,B,Q) is a solution of the integrable system associated with the bow representation R.

Completeness

Question: Are these the same as the initial bow data, i.e. is Down$\circ$Up$=1_{\mathrm{Bow}}$?

To answer this question we express solutions Ψ of the Bow Dirac equation in terms of solutions Υ of the TN Dirac equation.

Quaternionic units rep. on S-:

$$I^j = \frac{1-\gamma^5}{2} c^j c^0$$

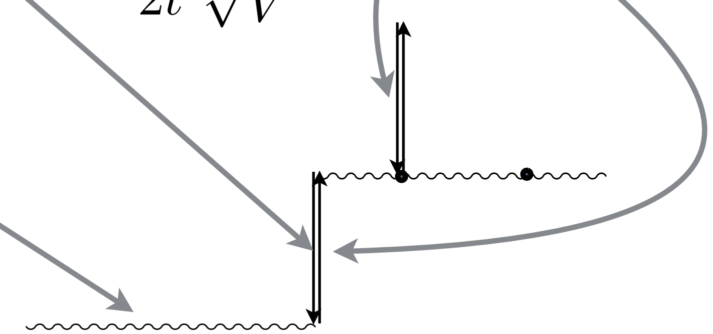
Clifford(vierbein)

Key technical relations:

$$[D_s, i\partial_s] = Cl(a_s) = \frac{c^0}{\sqrt{V}} \quad \Big| \quad [D_s, t^j] = I^j\, Cl(a_s) = I^j \frac{c^0}{\sqrt{V}}$$

- To relate Down and Up transforms express solution (χ,ν,f) of Bow Dirac through solutions ψ of TN Dirac:

$$\nabla^*\nabla\chi = \frac{c^0}{\sqrt{V}}\Psi \qquad \nabla^*\nabla\nu_+ = i\frac{b_+^\dagger}{2t}\frac{c^0}{\sqrt{V}}\Psi_t \qquad \nabla^*\nabla\nu_- = i\frac{b_+^\dagger}{2t}\frac{c^0}{\sqrt{V}}\Psi_h \qquad \nabla^*\nabla f_\lambda = 0$$



These form an orthonormal basis of solutions of the Bow Dirac equation!

$$(i\partial_s + I^j(t^j - T^j))\chi + (\delta(s-t)b_+ - \delta(s-h)B_+)\nu_+ + (\delta(s-t)B_- - \delta(s-h)b_-)\nu_- - \delta(s-\lambda)Q = 0$$

in short

$\mathfrak{D}_S(\chi,\nu_+,\nu_-,f_\lambda)=0$

Moreover, just as ($\chi,\nu_+,\nu_-,f_\lambda$) satisfy the Poisson equation on ALF, $\boldsymbol{\Psi}$ satisfy the Poisson equation on the bow:

$$\left((i\partial_s - T^0)^2 + (t^j - T^j)^2\right)\Psi = 4\frac{c^0}{\sqrt{V}}\chi$$

These Dirac and Poisson relations

$$D_s \Psi = 0 \qquad \mathfrak{D}_s(\chi, \nu_+, \nu_-, f_\lambda) = 0$$

$$\nabla^* \nabla \chi = \frac{c^0}{\sqrt{V}} \Psi \qquad \left((i\partial_s - T^0)^2 + (t^j - T^j)^2 \right) \Psi = 4 \frac{c^0}{\sqrt{V}} \chi$$

together with the appropriate index theorem

$$\mathrm{ind}_{L^2} D^+ = \mathrm{tr} \left(\frac{k}{2}\{\Lambda\}^2 - \frac{k}{2}\{\Lambda\} - \{\Lambda\}(k\Lambda - M) + \frac{1}{2}(k\Lambda - M) \right) + \frac{1}{8\pi^2} \int \mathrm{tr} F \wedge F.$$

prove that

Up∘Down=1 and Down∘Up=1

Isometry

Infinitesimal deformation $A \mapsto A + a$ of an instanton satisfies $(d_A a)^+ = 0, \quad d_A^* a = 0.$

Leading to a change in Ker D $\quad D\delta\Psi + Cl(a)\Psi = 0$

$$D_A \Psi = 0$$

$$\delta\Psi = -D(\nabla^*\nabla)^{-1} Cl(a)\Psi$$

- Key observations: - Dirac operator on the Bow is $\mathfrak{D} = \Psi^\dagger \mathfrak{d} \Psi$

  - and the commutator has a particularly simple form

$$[D, \mathfrak{d}] = (1, \frac{b_+^\dagger}{2t}, \frac{b_-^\dagger}{2t}, 0) =: R$$

- The resulting deformation of the Bow data is $\hat{a} := \delta(T, B) = (R\Psi, (\nabla^*\nabla)^{-1} Cl(a)\Psi)$

$$< \hat{a}, \hat{a}' > = < (\nabla^*\nabla)^{-1} Cl(a)\Psi, R\Psi\hat{a}' > = < Cl(a)\Psi, \underbrace{R\Psi\hat{a}'(\nabla^*\nabla)^{-1}}_{Cl(\hat{a}')\Upsilon} = (Cl(a)\Psi, Cl(\hat{a}')\Upsilon)$$

- Identical calculation on the bow side (using analogous Up relations) gives

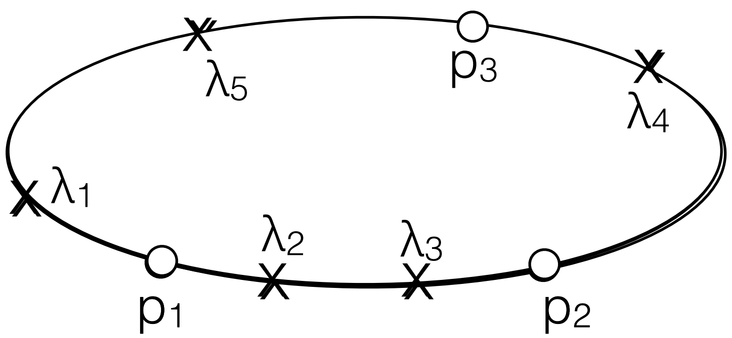
$$(a, a') = (a \underbrace{((-i\partial_s - T^0)^2 + (t^j - T^j)^2)^{-1}\Upsilon}_{\Psi}, \hat{a}'\Upsilon).$$

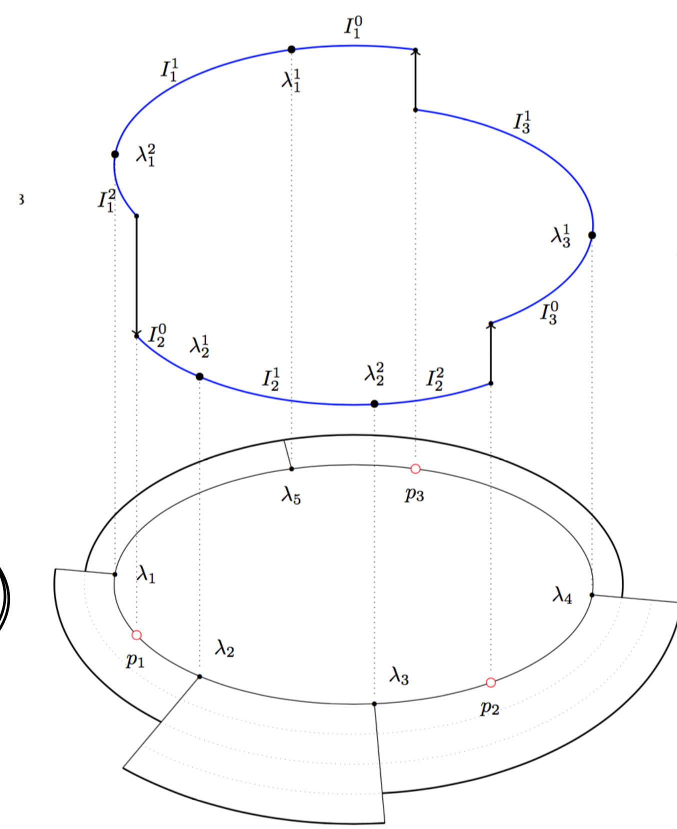
- Thus $(a, a') = < \hat{a}, \hat{a}' >$ and the Up and Down transforms are isometries.

Bow Integrable System

U(n) YM instanton on A_k ALF space => two Dynkin diagrams: $\tilde{A}_{k-1}$ and $\tilde{A}_{n-1}$

$\tilde{A}_{k-1}$





In a balanced bow representation, each λ interval carries the Nahm system:

$$\begin{cases} \frac{d}{ds}T_1 + [T_0, T_1] = [T_2, T_3] \\ \frac{d}{ds}T_2 + [T_0, T_2] = [T_3, T_1] \\ \frac{d}{ds}T_3 + [T_0, T_3] = [T_1, T_2] \end{cases}$$

with Lax pair:

$$M = \frac{d}{ds} + T_0 - iT_3 - \zeta(T_1 - iT_2),$$

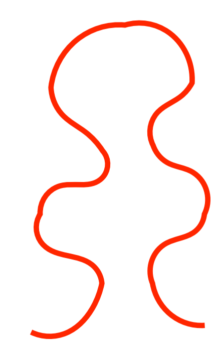
$$L = T_1 + iT_2 + 2\zeta i T_3 - \zeta^2(T_1 - iT_2),$$

$$\left[\frac{d}{ds} + M, L\right] = 0$$

and spectral curve

$$T\mathbb{P}^1 \ni S_{I_j} := \left\{\eta\frac{d}{d\zeta} \in T\mathbb{P}^1 \middle| \det(L - \eta)\right\} = 0$$

$\tilde{A}_{n-1}$

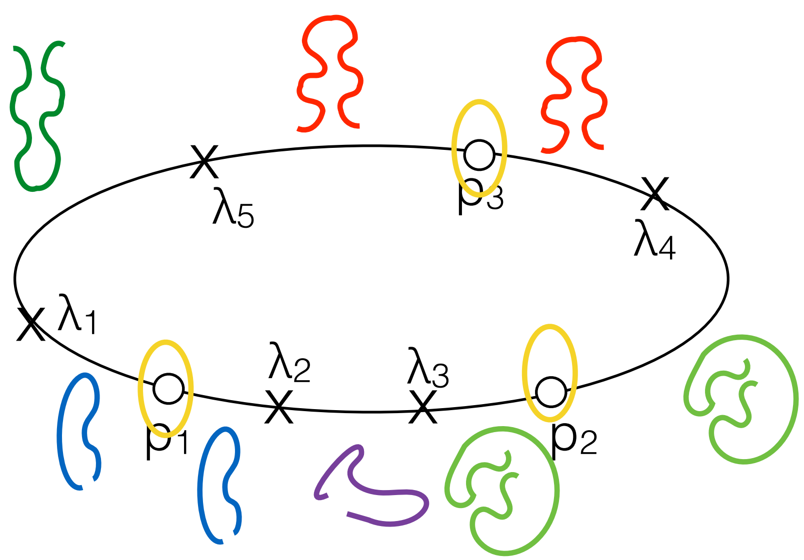


Spectral Curve changes across a λ-point, but remains the same across a p-point:

moment map conditions at a p-point read

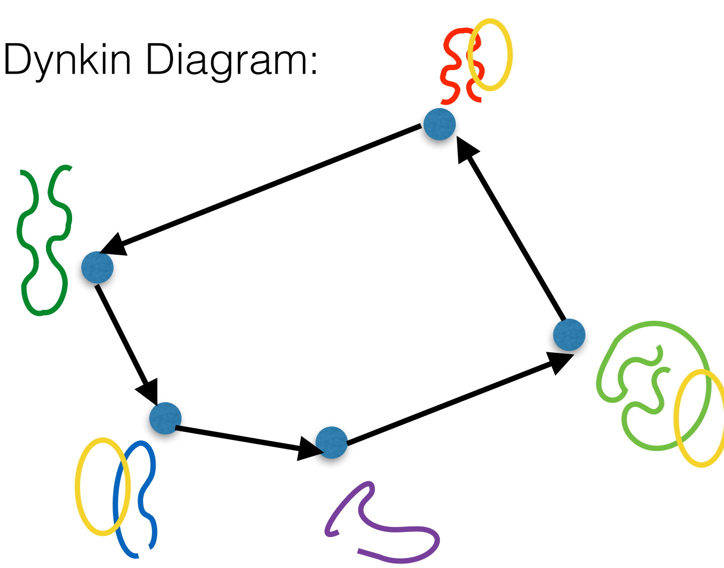
on the right: $L(p_\sigma +) = (B_{\sigma,RL} + \zeta B^\dagger_{\sigma,LR})(B_{\sigma,LR} - \zeta B^\dagger_{\sigma,RL}) - \big((\nu_1 + i\nu_2) + 2\zeta i \nu_3 - \zeta^2(\nu_1 - i\nu_2)\big)$

on the left: $L(p_\sigma -) = (B_{\sigma,LR} - \zeta B^\dagger_{\sigma,RL})(B_{\sigma,RL} + \zeta B^\dagger_{\sigma,LR}) - \big((\nu_1 + i\nu_2) + 2\zeta i \nu_3 - \zeta^2(\nu_1 - i\nu_2)\big)$



Reciprocal bow (cutting at λ-points instead) is of A_n type i.e. it is determined by the gauge group.

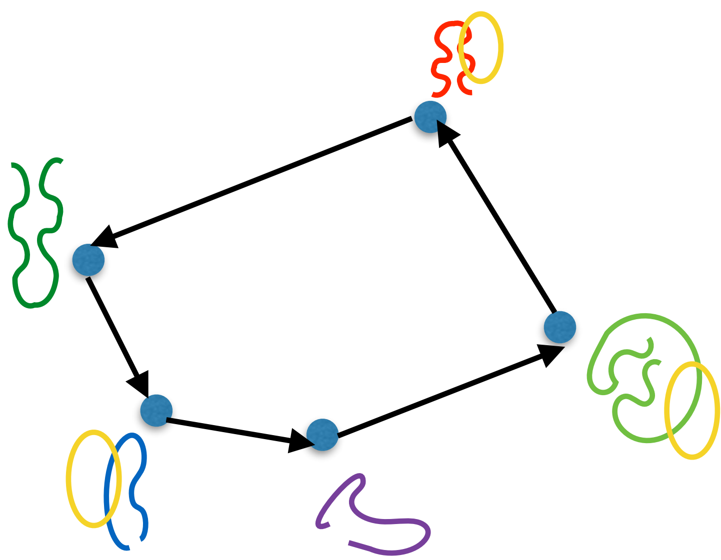
$U^{aff}(n)$ Dynkin Diagram:



What is the significance of p-points?

Each p-point has an assigned moment map level: each determines a section of TP^1.

$$\eta = \big((\nu_1 + i\nu_2) + 2\zeta i\nu_3 - \zeta^2(\nu_1 - i\nu_2)\big)$$



- Each vertex of the affine Dynkin diagram carries a spectral curve.
- p-points assign P^1 curves (moment P^1) to some vertices.
- All of these curves are in TP^1.
- Connected vertices => respective curve intersection divisor.
- A curve at a vertex intersection with the moment P^1's of that vertex.
- Ignore all other intersections.

Alternatively, one can view this as a single multi-component curve

Exact Metric via the Generalized Legendre Transform

w/ Roger Bielawski

In terms of finite HK quotient ingredients the symplectic structure on each interval is

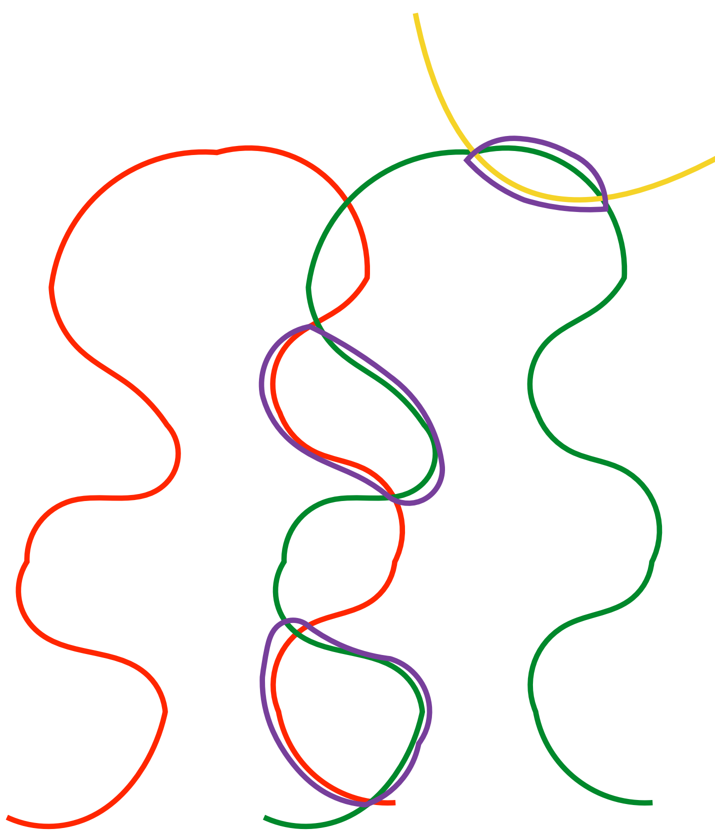
$$\omega = \mathrm{Tr}\left(H^{-1}dH \wedge dL + LH^{-1}dH \wedge H^{-1}dH\right)$$

Spectral curve on i$^{\text{th}}$ interval is

$$\eta_i^{r_i} + a_1^i(\zeta)\eta^{r_i-1} + \ldots + a_{r_i-1}^i(\zeta)\eta + a_{r_i-1}^i(\zeta) = 0$$

with polynomial coefficients

$$a_\alpha^i(\zeta) = z_i + v_i\zeta + w_{2,i}\zeta^2 + \ldots + w_{2r_i-2,i}\zeta^{2r_i-2} + (-1)^{r_i-1}\bar{v}_i\zeta^{2r_i-1} + (-1)^{r_i}\bar{z}_i\zeta^{2r_i}$$



Form Legendre potential, which is a function of coefficients of these polynomials

$$F = \sum_{i\in\text{Intervals}} l_i \frac{1}{2\pi i}\oint_0 \frac{\eta_i^2}{\zeta^3}d\zeta + \sum_{e\in\text{Edges}} \frac{1}{2\pi i}\oint_{\Gamma_e} (\eta_{h(e)} - \eta_{t(e)})\log(\eta_{h(e)} - \eta_{t(e)})\frac{d\zeta}{\zeta^2}$$

$$+ \sum_{\substack{\lambda\in\Lambda \\ \text{weights}}} \frac{1}{2\pi i}\oint_{\Gamma_e} (\eta_{i(\lambda)} - \nu_\lambda)\log(\eta_{i(\lambda)} - \nu_\lambda)\frac{d\zeta}{\zeta^2}$$

more succinctly

$$F = -\frac{1}{2\pi i}\oint_0 \frac{\eta^2}{\zeta^3}d\zeta + \oint_C \frac{\eta}{\zeta^2}d\zeta$$

performing Legendre transform

Constraints on the spectral curves.

$$u_i = \frac{\partial F}{\partial v_i}, \qquad \frac{\partial F}{\partial w_{\alpha,i}} = 0,$$

gives the Kähler potential: $K(z,\bar{z},u,\bar{u}) = F(z,\bar{z},v,\bar{v}) - uv - \bar{u}\bar{v}.$

using GLT of
Hitchin, Karlhede, Lindstrom, Rocek '87

Conclusion

- Yang-Mills instantons on multi-Taub-NUT (with generic asymptotic holonomy) have abelian instanton asymptotic.
- Solutions of Bow integrable system are in 1-to-1 isometric correspondence with the instantons.
- A bow combines two Dynkin diagrams:
  - one of the gauge group,
  - another of the underlying base space.
- Kähler potential on the moduli space of U(n) instantons on TN_k (and a conjecture for G instantons).