Geometric Cauchy problems for surfaces associated to harmonic maps

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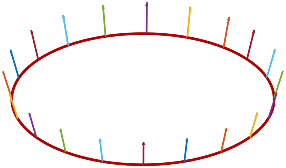
Credits: parts of this work are collaborations with Josef Dorfmeister, Martin Svensson and Peng Wang

3 August 2016

Geometric Cauchy problems

A classical problem

Björling's problem for minimal surfaces:



Prescribed normal field on curve $\rightarrow$ Unique minimal surface

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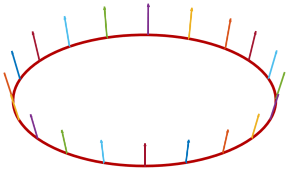

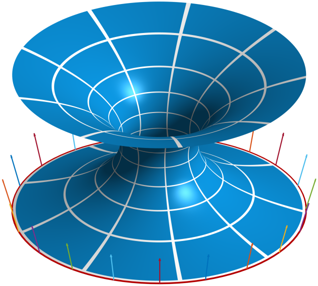

Prescribed normal field on curve $\rightarrow$ Unique minimal surface

Schwarz formula:

$$f(z) = \Re\left\{\alpha(z) - i\int_{x_0}^{z} N(w) \times \alpha'(w)\mathrm{d}w\right\},$$

$\alpha(z)$ and $N(z)$ are the *holomorphic extensions*.

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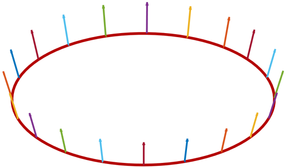

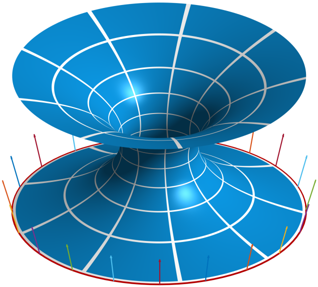

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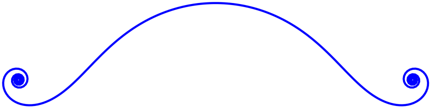
Curve $+\ldots$ generates surface of type X

e.g. space curve given by:

$$\kappa(s) = 1 - s^4, \qquad \tau(s) = 0.$$



Find the (unique?) surface of (e.g.) constant Gauss curvature $K = 1$ containing this curve as:

1. a geodesic
2. a cuspidal edge singularity
3. or with some arbitrary prescribed surface normal

These are called **geometric Cauchy problems**

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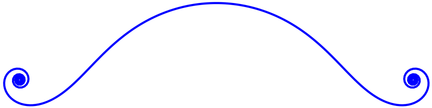

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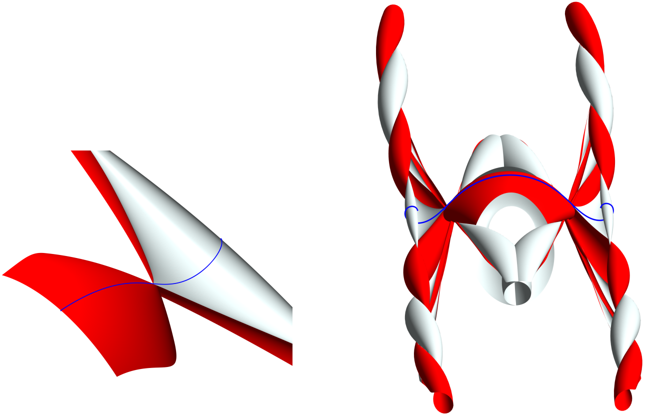

As a geodesic curve (the CGC $K = 1$ solution)

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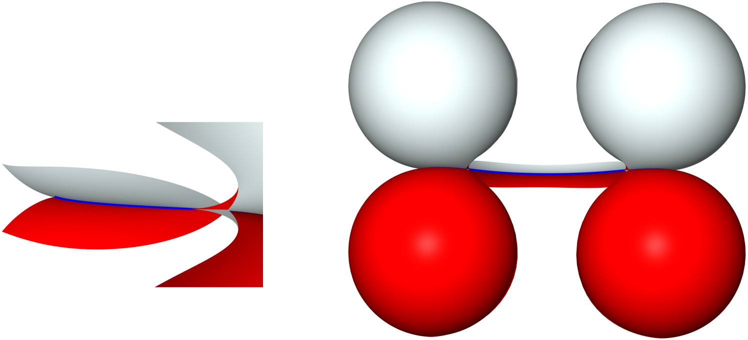

As a cuspidal edge singular curve

Curve $+ \ldots$ generates surface of type X

e.g. Find the unique CGC $K = 1$ surface containing the curve with surface geometry given by:

$$\kappa_g(s) = 1, \quad \kappa_n(s) = 1, \quad \tau_g(s) = \sin(s)$$

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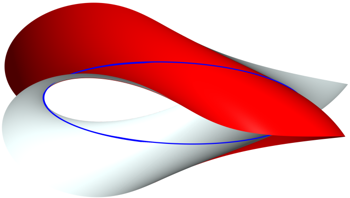

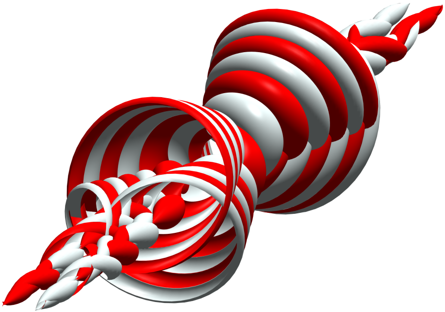

Special surfaces and harmonic maps

Special surfaces and harmonic maps

Many important classical surfaces correspond to *harmonic maps* from either $\mathbb{R}^2$ or $\mathbb{R}^{1,1}$ into G/K.

Examples:

- Constant mean curvature (CMC) surfaces in space forms
- Constant Gauss curvature (CGC) surfaces in space forms
- Willmore surfaces

Example: Constant Gauss Curvature Surfaces

1. $N : \mathbb{C} \to \mathbb{S}^2$ is harmonic iff

$$N \times N_{z\bar{z}} = 0,$$

iff

$$f_z = iN \times N_z,$$

is *integrable* i.e. $(f_z)_{\bar{z}} = (f_{\bar{z}})_z$.

Moreover: $f : \mathbb{C} \to \mathbb{R}^3$ (with induced metric) is CGC, with $K = 1$.

2. $N : \mathbb{R}^{1,1} \to \mathbb{S}^2$ is (Lorentzian)-harmonic iff

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Loop group lift of a harmonic map into G/K

$$G = G^{\mathbb{C}}_{\rho}, \qquad K = G_{\sigma}$$

Loop group $\Lambda G^{\mathbb{C}} := \{\gamma : \mathbb{S}^1 \to G^{\mathbb{C}}\}$. Twisted subgroup is the fixed point subgroup

$$\Lambda G^{\mathbb{C}}_{\hat{\sigma}}, \quad \text{for} \quad \hat{\sigma} x(\lambda) := \sigma(x(-\lambda)).$$

Real forms determined by the involutions:

$$\hat{\rho}_1 x(\lambda) := \rho(x(1/\bar{\lambda})), \qquad \hat{\rho}_2 x(\lambda) := \rho(x(\bar{\lambda})).$$

Note:

$$\Lambda G = \Lambda G^{\mathbb{C}}_{\hat{\rho}_1}.$$

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Riemannian case: Harmonic maps $\mathbb{C} \supset U \to G/K$,

Characterized by $F : U \to \Lambda G^{\mathbb{C}}_{\hat{\rho}_1 \hat{\sigma}} = \Lambda G_{\hat{\sigma}}$

$$F^{-1}\mathbf{d}F = A_{-1}\lambda^{-1}\mathbf{d}z + \alpha_0 + \overline{A_{-1}}\lambda\mathbf{d}\bar{z},$$

For any $\lambda_0 \in \mathbb{S}^1$ the map

$$F\big|_{\lambda_0} : U \to G$$

projects to a harmonic map $f : U \to G/K$.

Call such F an *admissible frame*.

Lorentzian case: $\mathbb{R}^{1,1} \supset V \to G/K$,

Characterized by $F : V \to \Lambda G^{\mathbb{C}}_{\hat{\rho}_2 \hat{\sigma}}$,

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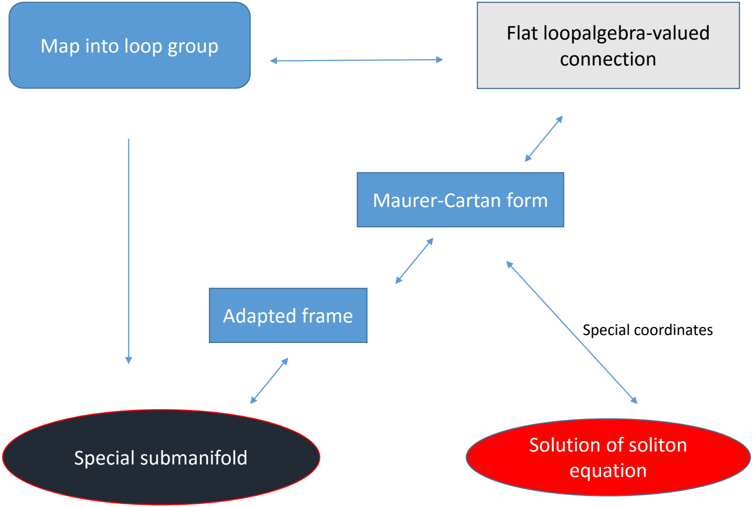
Link with Soliton equations



Important loop group decompositions

Set $\Lambda^{\pm} G^{\mathbb{C}} = \{\gamma \in \Lambda G^{\mathbb{C}} \mid \gamma = \sum_{n=0}^{\infty} a_n \lambda^{\pm n}\}$.

We need:

1. The **Birkhoff decomposition**
   1.1
   $$\Lambda^{-} G^{\mathbb{C}} \cdot \Lambda^{+} G^{\mathbb{C}}$$
   is open an dense in the identity component of $\Lambda G^{\mathbb{C}}$.
   1.2 For compact G:
   $$\Lambda G^{\mathbb{C}}_{\hat{\rho}_2} = \Lambda^{-} G^{\mathbb{C}}_{\hat{\rho}_2} \cdot \Lambda^{+} G^{\mathbb{C}}_{\hat{\rho}_2}$$

   Analogue: $A = LU$ matrix factorization.

2. The **Iwasawa decomposition** (for compact G):
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Riemannian-harmonic maps

Riemannian case (Dorfmeister/Pedit/Wu)

$\Rightarrow$

Given admissible frame $F : U \to \Lambda G^{\mathbb{C}}_{\hat{\rho}_1 \hat{\sigma}} = \Lambda G_{\hat{\sigma}}$

$$F^{-1}\mathrm{d}F = A_{-1}\lambda^{-1}\mathrm{d}z + \alpha_0 + \overline{A_{-1}}\lambda\mathrm{d}\bar{z},$$

Birkhoff decompose: $F(z) = F_-(z)F_+(z)$ (with normalization), then

$$F_-^{-1}\mathrm{d}F_- = B_{-1}\lambda^{-1}\mathrm{d}z, \qquad B_{-1} \text{ holo., } B_{-1}(z) \in \mathfrak{g}^{\mathbb{C}}.$$

$\Leftarrow$

Conversely: given a holomorphic 1-form with values in Lie($\Lambda G^{\mathbb{C}}\hat{\sigma}$),

$$\eta = \sum_{n=-1}^{\infty} B_n(z)\lambda^n\mathrm{d}z,$$

1. solve $\Phi^{-1}\mathrm{d}\Phi = \eta$, with $\Phi(z_0) = I$,
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$$\Phi(z) = F(z)G_+(z)$$

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Applications: e.g. CGC $K = 1$ (spherical) surfaces

$F : U \to \Lambda G$ admissible frame for the harmonic Gauss map.

The CGC surface can be obtained from F by the *Sym formula:*

$$f = i\lambda \frac{\partial F}{\partial \lambda} F^{-1} \Big|_{\lambda=1} \quad =: \quad \mathcal{S}(F).$$

Numerical Implementation

e.g. DPW for spherical surfaces:

"holomorphic potential": $\eta = \sum_{i=-1}^{\infty} A_i \lambda^i \mathrm{d}z$
integrate: $\Phi^{-1}\mathrm{d}\Phi = \eta$.
Iwasawa: $\Phi = FH_+$.
Sym: $f = \mathcal{S}(F)$.

Implementation: Can represent $\sum_{i=-n}^{n} A_i \lambda^i$ as a matrix:

$$\begin{pmatrix} A_0 & \dots & A_n & 0 & & \dots & & 0 \\ A_{-1} & A_0 & \dots & A_n & 0 & \dots & & 0 \\ \vdots & & & & & & & \\ 0 & \dots & A_{-n} & \dots & A_0 & \dots & A_n & \dots & 0 \\ \vdots & & & & & & & \\ 0 & & \dots & & & 0 & A_{-n} & \dots & A_0 \end{pmatrix}$$

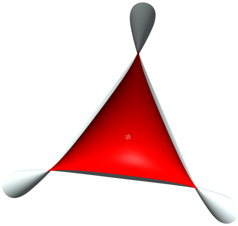
Loop group decompositions $\leftrightarrow$ matrix decompositions

Examples

Simplest potentials:

$$\eta = \begin{pmatrix} 0 & a(z) \\ b(z) & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z.$$

$a = 1$
$b = 0$



$a = z$
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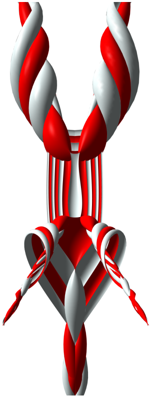

$a = 1 + z$
$b = 0.5 + 0.5z - z^2$

Summary of DPW for spherical surfaces:

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All spherical surfaces can be constructed this way.

Limitation: Geometric information lost in the Iwasawa splitting, can *not* read off geometric infomation from η.

To exploit: many choices of potential for a given surface.

Somewhat analogous method and statements hold for surfaces associated to *Lorentzian* harmonic maps (such as CGC $K = -1$).

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Using DPW for Geometry

Problem: Find the potential η that produces the solution with some desired geometric properties.

One approach Use known potentials (e.g. rotational) to define more complicated solutions, e.g. potentials on n-punctured sphere with prescribed *end behaviour*.

Drawback: there are not that many known potentials.

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Another idea: prescribed geometry along a curve

The **geometric Cauchy problem:**

- Specify *sufficient geometric data* along a curve for a unique solution
- Find formulas for DPW-type potentials in terms of this data.

Solving the GCP for harmonic maps

Recall:
Riemannian harmonic:

$$F \leftarrow \Phi \quad \text{via} \quad \Phi = FH_+ \quad \text{Iwasawa}$$

Many choices of potentials, hence of Φ.

Essential idea: Find potentials such that the Iwasawa/Birkhoff decomposition is *trivial* along the curve, i.e. such that

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Main point: F contains the *geometric information*, while Φ are the "*Weierstrass data*".

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Solving the GCP

Outline:

- Choose coordinates $z = x + iy$ so that the curve is $y = 0$.
- Prescribe sufficient information to construct the loop group frame $F_0(x)$ along $y = 0$, from γ and N.
- Write $\alpha = F^{-1}\mathrm{d}F = \left(A_{-1}\lambda^{-1} + \alpha_0 + \overline{A_{-1}}\lambda\right)\mathrm{d}x$.
- Let η be the holomorphic extension of α.
- Apply DPW to η: solve $\Phi^{-1}\mathrm{d}\Phi = \eta$, Iwasawa split $\Phi = FH_+$, then F is an admissible frame.
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Solving the GCP

Outline:

- Choose coordinates $z = x + iy$ so that the curve is $y = 0$.
- Prescribe sufficient information to construct the loop group frame $F_0(x)$ along $y = 0$, from γ and N.
- Write $\alpha = F^{-1}\mathrm{d}F = \left(A_{-1}\lambda^{-1} + \alpha_0 + \overline{A_{-1}}\lambda\right)\mathrm{d}x$.
- Let η be the holomorphic extension of α.
- Apply DPW to η: solve $\Phi^{-1}\mathrm{d}\Phi = \eta$, Iwasawa split $\Phi = FH_+$, then F is an admissible frame.
- Along $y = 0$ we have $F(x, 0) = \Phi(x, 0) = F_0(x)$ by construction.

Theorem

Give real analytic functions

$$\kappa_g(s), \quad \kappa_n(s), \quad \tau_g(s),$$

The unique spherical surface containing a curve along $\{y = 0\}$ with the prescribed geodesic and normal curvature and geodesic torsion is obtained from the DPW potential

$$\eta = \left[\left[\frac{\tau_g(z) - i}{2} e_1 - \frac{\kappa_n(z)}{2} e_2 \right] \frac{1}{\lambda} + \kappa_g(z) e_3 + \left[\frac{\tau_g(z) + i}{2} e_1 - \frac{\kappa_n(z)}{2} e_2 \right] \lambda \right] \mathrm{d}z.$$

(All functions extended holomorphically, Here e_i are an o.n. basis for $\mathfrak{g}$.)

Similarly, given real analytic

$$\kappa(s), \qquad \tau(s),$$

with $\kappa \not\equiv 0$, holomorphically extend and then:

$$\hat{\eta} = \left(\frac{\tau(z) - i}{2} \lambda^{-1} e_1 + \kappa(z) e_3 + \frac{\tau(z) + i}{2} \lambda e_1 \right) \mathrm{d}z,$$

generates the *singular* curve solution.

Lorentzian-harmonic maps

"DPW"' for Lorentzian harmonic maps (Krichever, M. Toda)

$\Rightarrow$: Given $F : V \to \Lambda G^{\mathbb{C}}_{\hat{\rho}_2 \hat{\sigma}}$,

$$F^{-1}\mathrm{d}F = A_1 \lambda \mathrm{d}x + \alpha_0 + A_{-1}\lambda^{-1}\mathrm{d}y,$$

Birkhoff: $F(x,y) = X_+(x,y)G_-(x,y) = Y_-(x,y)G_+(x,y)$ (with normalizations), then

$$X_+^{-1}\mathrm{d}X_+ = B_1(x)\lambda \mathrm{d}x,$$
$$Y_-^{-1}\mathrm{d}Y_- = C_{-1}(y)\lambda^{-1}\mathrm{d}y.$$

$\Leftarrow$
Conversely: given 1-forms (χ, ψ) on $\mathbb{R}$ with values in Lie$(\Lambda G^{\mathbb{C}}\hat{\sigma}\hat{\rho}_2)$,

$$\chi = \sum_{n=-\infty}^{1} B_n(x)\lambda^n \mathrm{d}x, \qquad \psi = \sum_{n=-1}^{\infty} C_n(y)\lambda^n \mathrm{d}y,$$

1. Solve $X^{-1}\mathrm{d}X = \chi$, and $Y^{-1}\mathrm{d}Y = \psi$,
2. Birkhoff decompose

$$X^{-1}(x)Y(y) = H_-(x,y)H_+(x,y)$$

Then $F := XH_-$ is an admissible frame.

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The GCP for Lorentz-harmonic maps

"DPW" construction:

$$F = XH_- \leftarrow (X, Y) \quad \text{via} \quad X^{-1}Y = H_-H_+ \quad \text{Birkhoff}$$

Many choices of potentials, hence of (X, Y).

Analogous to Riemannian case: Find potentials such that the Birkhoff decomposition is trivial along the curve, i.e. such that

$$F\big|_\gamma = X\big|_\gamma = Y\big|_\gamma.$$

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Solving the GCP (non-characteristic curve)

Required admissible frame:

$$F^{-1}\mathrm{d}F = A_1\lambda \mathrm{d}x + \alpha_0 + A_{-1}\lambda^{-1}\mathrm{d}y,$$

Potential pairs of form:

$$\chi = X^{-1}\mathrm{d}X = \sum_{n=-\infty}^{1} B_n(x)\lambda^n \mathrm{d}x,$$

$$\psi = Y^{-1}\mathrm{d}Y = \sum_{n=-1}^{\infty} C_n(y)\lambda^n \mathrm{d}y,$$

Related by $F := XH_-$, where

$$X^{-1}(x)Y(y) = H_-(x,y)H_+(x,y)$$

- Choose null coordinates s.t. initial curve given by $y = x$.
- Set $u = (x+y)/2$, $v = (x-y)/2$, then initial curve is $v = 0$, and $\mathbf{d}y = \mathbf{d}x = \mathbf{d}u$ along the curve.
- Construct $F_0(u) = F(u, 0)$, so

$$\alpha_0 = F_0^{-1}\mathrm{d}F_0 = A_1\lambda \mathrm{d}u + \alpha_0 + A_{-1}\lambda^{-1}\mathrm{d}u.$$

- Set $\chi = \psi = \alpha_0$.

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$$\alpha_0 = F_0^{-1}\mathrm{d}F_0 = A_1\lambda \mathrm{d}u + \alpha_0 + A_{-1}\lambda^{-1}\mathrm{d}u.$$

- Set $\chi = \psi = \alpha_0$.

Pseudospherical surfaces (Lorentzian harmonic)

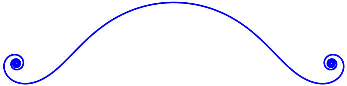
- Analogous results to spherical surfaces
- Main difference: solution not unique for *characteristic curves*

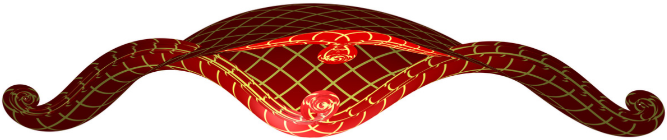
Convenient way to generate examples:

Given curvature functions κ and τ there is a unique CGC $K = -1$ surface containing this curve as a cuspidal edge

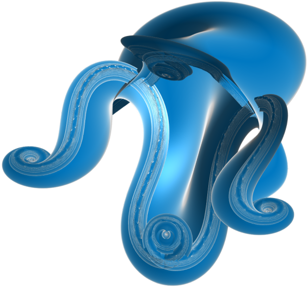
(degenerate where $\kappa = 0$ or $\tau = \pm 1$).

$$\kappa(s) = 1 - s^4, \qquad \tau(s) = 0$$

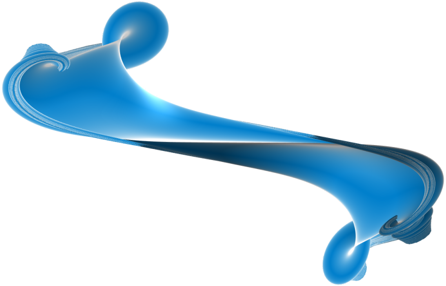




Examples



$\kappa(s) = 2 - s^2$
$\tau = 0$

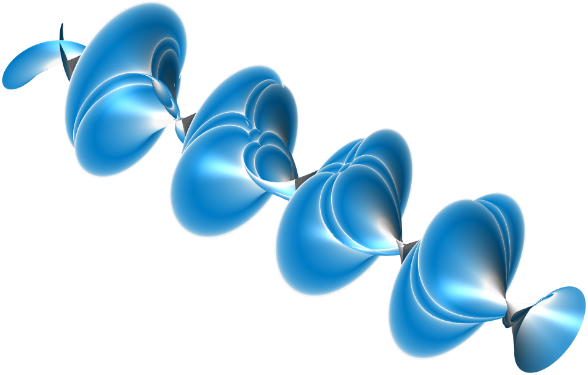


$\kappa(s) = s^2$
$\tau = 1/2$

Examples that are not weakly regular

Viviani figure 8 space curve $\gamma(t) = 0.3\left(1 + \cos t, \sin t, 2\sin\frac{t}{2}\right)$.

- $\tau = \pm 1$ twice each on the curve.
- Solution to SG-equation not defined at these points
- The Lorentzian harmonic map *is* defined



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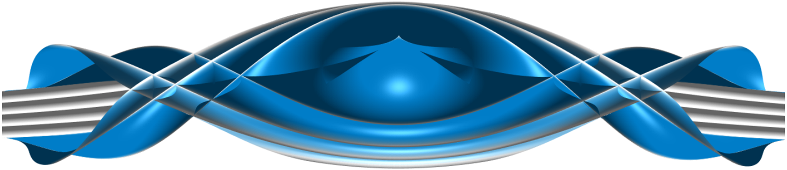
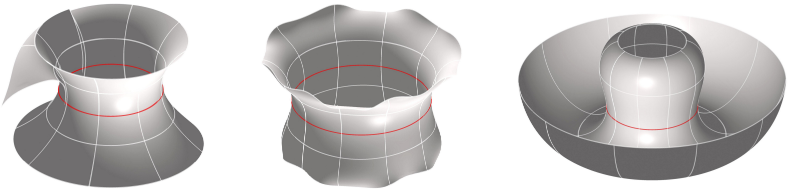


Willmore surfaces

Willmore Surfaces

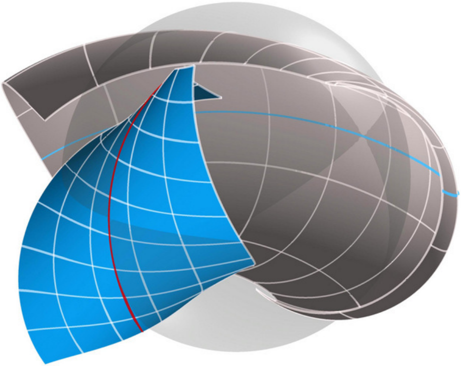
Elliptic PDE

- Gauss map *Riemannian-harmonic* (like spherical surfaces)
- Uniqueness: need more than just the surface normal.

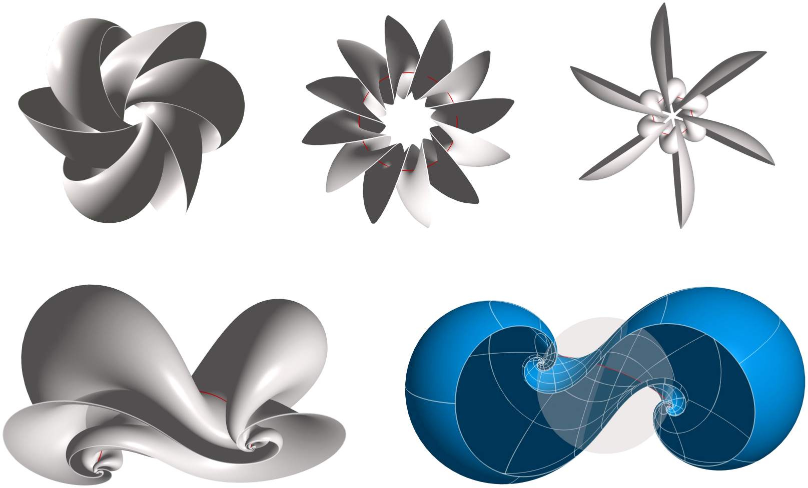


Willmore Surfaces

It is sufficient to prescribe the *dual surface* $\hat{Y}$ in addition to Y and the conformal Gauss map along the curve.



Equivariant Willmore Surfaces



Summary

- We discussed surface classes with harmonic Gauss maps
- All solutions can be constructed from holomorphic Weierstrass-type data (Riemannian) or d'Alembert-type data (Lorentzian) called potentials.
- The challenge is to explicitly write down the potential for a given geometric problem
- We can solve this given geometric Cauchy data along a curve.

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