Geometry of the movable poles of real solutions of Painlevé III(0,0,4,-4)

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My starting points, my motivation

Griffiths in the 60ies: variations of (polarized) Hodge structures as tools within complex algebraic geometry.

Cecotti-Vafa 1991 (and Dubrovin 1991): tt^* equations,
– big generalization of variations of Hodge structures.

Simpson 1989: harmonic bundles: related, but weaker, less rigid.

Hertling 2002: frame for tt^*, variations of TERP structures (Twistor Extension Real Pairing) [definition not in this talk].

Geometry of TERP structures in the semisimple rank 2 case (first interesting case far from variations of Hodge structures)

$$\updownarrow$$

real solutions (with/without singularities) on $\mathbb{R}_{>0}$ of the Painlevé III equation of type $(0, 0, 4, -4)$: $P_{III}(0, 0, 4, -4)$

Related work on $P_{III}(0,0,4,-4)$

Work on real solutions on $\mathbb{R}_{>0}$ of the special Painlevé equation:

McCoy-Tracy-Wu 1977: other methods.

Its-Novokshenov 1986: with isomonodromic connections,
positive: rich, hard results, precise formulas,
critic: formulas not transparent, only matrices, no vector bundles,
only open charts of the moduli spaces.

Niles (PhD with Its) 2009: extending Its-Novokshenov.

Claus Hertling + Martin Guest: joint paper (also on multivalued complex solutions on $\mathbb{C}^*$ and their vector bundles). There and

Today: complete picture of real solutions on $\mathbb{R}_{>0}$ of $P_{III}(0,0,4,-4)$.

Painlevé III equations

For any 4 parameters $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$,

$$P_{III}(\alpha, \beta, \gamma, \delta): \qquad f'' = \frac{(f')^2}{f} - \frac{1}{x} f' + \frac{1}{x}(\alpha f^2 + \beta) + \gamma f^3 + \delta \frac{1}{f}$$

is a complex differential equation of 2nd order on $\mathbb{C}^*$.

Basic fact: $\forall\ x_0 \in \mathbb{C}^* \quad \forall$ (regular) initial value $(f_0, \widetilde{f}_0) \in \mathbb{C}^* \times \mathbb{C}$, locally a unique holomorphic solution f with $f(x_0) = f_0, f'(x_0) = \widetilde{f}_0$ exists.

Big old theorem (Painlevé property): This solution f extends to a global multivalued meromorphic function f on $\mathbb{C}^*$.

$P_{III}(0, 0, 4, -4)$: f solution $\Rightarrow\ -f, f^{-1}, -f^{-1}$ solutions.

Poles and zeros on $\mathbb{C}^*$ of solutions of $P_{III}(0,0,4,-4)$

Lemma: Let f be a solution near $x_0 \in \mathbb{C}^*$ of the $P_{III}(0,0,4,-4)$ equation with $f(x_0) = 0$, $f(x) = \sum_{k\geq 1} f_k \cdot (x-x_0)^k$. Then

$$f_1 = \pm 2, \quad f_2 = \frac{8}{x_0 \cdot f_1}, \quad f_3 \text{ is free,}$$
$$f_k \text{ for } k \geq 4 \text{ are determined by } f_1 \text{ and } f_3.$$

$\longrightarrow$ Any solution f has on $\mathbb{C}^*$ only simple zeros and simple poles, with leading part $f_1 = \pm 2$ resp. $f_{-1} = \pm\frac{1}{2}$, and it is determined by the following data at a simple zero or simple pole x_0 :

$$(f_1, f_3) \in \{\pm 2\} \times \mathbb{C} \text{ resp. } (f_{-1}, f_1) \in \{\pm\frac{1}{2}\} \times \mathbb{C}.$$

$\longrightarrow$ 4 1-par families of *singular* initial values for any $x_0 \in \mathbb{C}^*$.

Spaces of initial values

For any $x_0 \in \mathbb{C}^*$ the local solutions f of the $P_{III}(0,0,4,-4)$ equation are determined
by a *regular* initial value $(f(x_0), f'(x_0)) \in \mathbb{C}^* \times \mathbb{C}$
or by a *singular* initial value (f_1, f_3) resp. (f_{-1}, f_1).

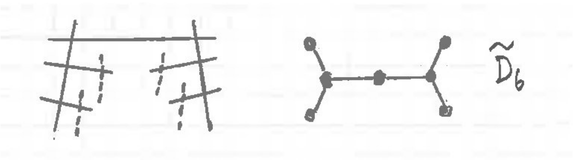
$\longrightarrow$ The *space of initial values* at x_0 is $M_{ini}(x_0)$,

as a set $M_{ini}(x_0) \cong \mathbb{C}^* \times \mathbb{C} \,\dot{\cup}\, (4 \text{ copies of } \mathbb{C})$,

as a 2 dim complex algebraic manifold, it is given by 4 affine charts, each $\cong \mathbb{C} \times \mathbb{C}$, each consists of $\mathbb{C}^* \times \mathbb{C}$ and 1 copy of $\mathbb{C}$.

Okamoto 1979: first, different approach to $M_{ini}(x_0)$:

$$M_{ini}(x_0) \cong S - Y, \quad S \text{ a projective surface,}$$
$$Y \text{ an anticanonical divisor of type } \widetilde{D}_6,$$



Painlevé property: A local solution near $x_0 \in \mathbb{C}^*$ extends to a global multivalued solution on $\mathbb{C}^*$.
$\longrightarrow$ $M_{ini}(x_0)$ represents all global multivalued solutions on $\mathbb{C}^*$, and any homotopy class of a path from $x_0 \in \mathbb{C}^*$ to $x_1 \in \mathbb{C}^*$ induces an *analytic* isomorphism

$$M_{ini}(x_0) \overset{anal}{\cong} M_{ini}(x_1).$$

Space of monodromy data

Big fact from the relation of $P_{III}(0,0,4,-4)$ to isomonodromic connections:

There is a canonical 2-dim algebraic manifold M_{mon} which represents all global multivalued solutions on $\mathbb{C}^*$ of $P_{III}(0,0,4,-4)$.

And in a presentation of it by equations, the parameters are meaningful and are related to the monodromy data of the isomonodromic connections, and for any $x_0 \in \mathbb{C}^*$

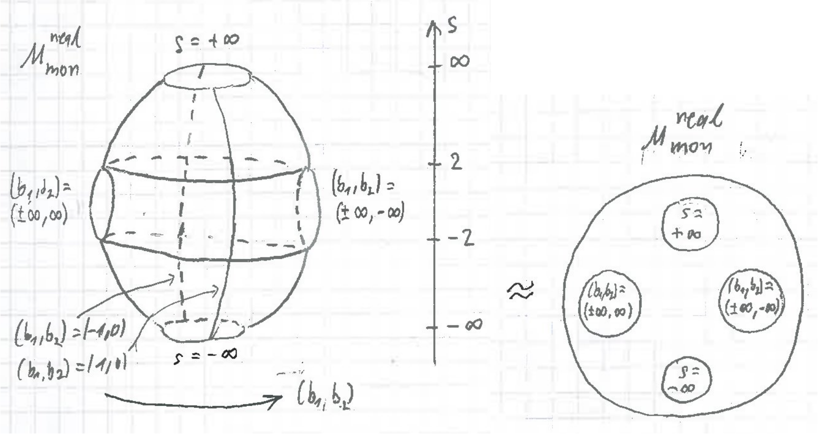
$$M_{ini}(x_0) \overset{anal}{\cong} M_{mon}.$$

How M_{mon} arises: no time in this talk, black box.

Now, a main point of the talk: how well the parameters of M_{mon} help to understand the real solutions on $\mathbb{R}_{>0}$ of $P_{III}(0,0,4,-4)$.

Monodromy data for real solutions on $\mathbb{R}_{>0}$ of $P_{III}(0, 0, 4, -4)$

$$M^{real}_{mon} = \{(s, b_1, b_2) \in \mathbb{R}^3 \mid b_1^2 + (\frac{s^2}{4} - 1)b_2^2 - 1 = 0\}$$



Initial data for real solutions on $\mathbb{R}_{>0}$ of $P_{III}(0,0,4,-4)$

$M^{real}_{ini}(x_0)$ is given by 4 charts, each $\cong \mathbb{R}\times\mathbb{R}$, each consists of $\mathbb{R}^*\times\mathbb{R} \cong \{\text{regular initial values}\}$ (2 components!!) and $\mathbb{R} \cong$ (1 stratum of singular initial values),

$$M^{real}_{mon} \overset{anal}{\cong} M^{real}_{ini}(x_0) \quad \text{for any } x_0 \in \mathbb{R}_{>0}.$$

For all $x_0 \in \mathbb{R}_{>0}$ together,

$$\begin{aligned} M^{real}_{ini} &:= \bigcup_{x_0\in\mathbb{R}_{>0}} M^{real}_{ini}(x_0) \\ &\cong (\text{2 open components, each } \cong \mathbb{R}^2_{>0}\times\mathbb{R}) \\ &\quad \text{separated by (4 walls, each } \cong \mathbb{R}_{>0}\times\mathbb{R}). \end{aligned}$$

On 1 component $f>0$, on the other component $f<0$,
2 walls of simple zeros, 2 walls of simple poles.

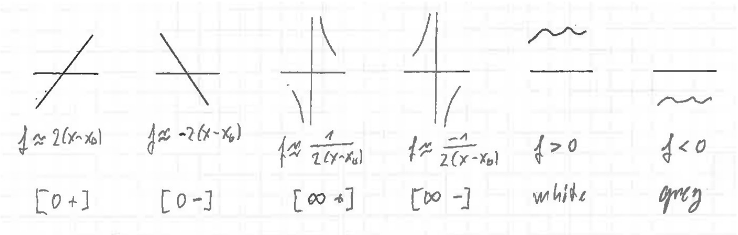
Real 3-dim space showing all solutions

The isomorphism

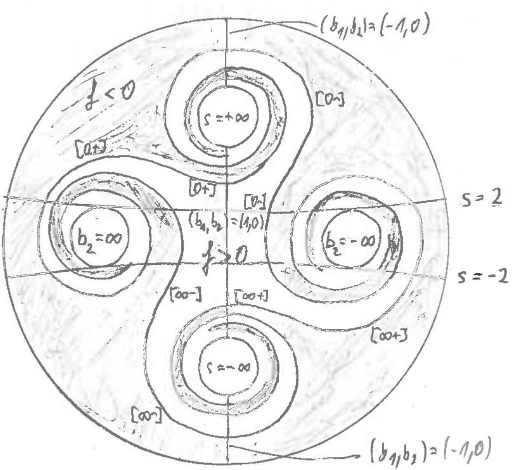
$$\begin{aligned} \mathbb{R}_{>0} \times M^{real}_{mon} &\cong M^{real}_{ini} \quad \text{with} \\ \{x_0\} \times M^{real}_{mon} &\cong M^{real}_{ini}(x_0) \end{aligned}$$

is analytic & complicated & beautiful and tells (together with the 6 strata in M^{real}_{ini}) all about the zeros and poles of the solutions.

The 6 strata, 4 walls of zeros and poles, 2 open components:

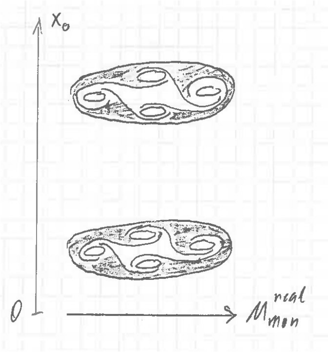


M_{mon}^{real} with the 6 strata of one $M_{ini}^{real}(x_0)$



(Picture partially conjectural)

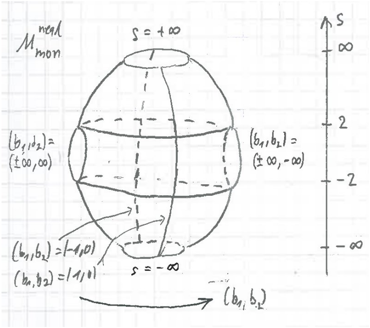
The vertical lines give the solutions of $P_{III}(0, 0, 4, -4)$



x_0 large:
small spirals around $s = \pm\infty$
large spirals around $(b_1, b_2) = \pm(\pm\infty, \infty)$

x_0 small:
large spirals around $s = \pm\infty$
small spirals around $(b_1, b_2) = \pm(\pm\infty, \infty)$

3 theorems on the real solutions on $\mathbb{R}_{>0}$ of $P_{III}(0, 0, 4, -4)$



On the next slides

Theorem 1: asymptotics near 0.
Only the parameter s is used.

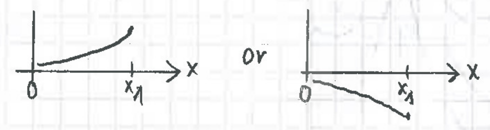
Theorem 2: asymptotics near ∞.
Only the parameters (b_1, b_2) are used.

Theorem 3: global results on the sequences of zeros and poles.

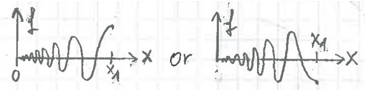
Theorem 1: asymptotics near 0.

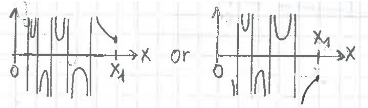
Fix a real solution f on $\mathbb{R}_{>0}$ of $P_{III}(0,0,4,-4)$ and its data $(s, b_1, b_2) \in M^{real}_{mon}$.

(a) Then f has no zeros or poles near 0 $\iff$ $|s| \leq 2$.
$\exists\ x_1 > 0$ s.t. for $b_1 \geq 1$ resp. $b_1 \leq -1$:



(b) If $|s| > 2$ then $\exists\ x_1 > 0$ s.t.

for $s > 2$:  ...[0+][0−]...

and for $s < -2$:  ...[∞+][∞−]...

About the asymptotics near 0

McCoy-Tracy-Wu 1977 studied only the solutions which are smooth near ∞ [Thm 2(a)]; for them, they knew the asymptotics near 0 [Thm 1(a)+(b)]; they had the globally smooth solutions [an implication of Thm 3].

Its-Novokshenov 1986 studied only the solutions which are smooth near 0 and got for their asymptotics precise formulas [Thm 1(a)].

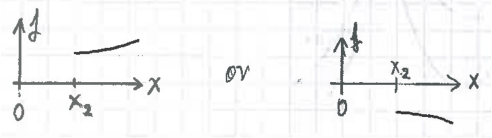
Niles 2009 obtained formulas for the asymptotics near 0 for all complex multivalued solutions on $\mathbb{C}^*$: Thm 1(a)+(b). He and Its-Novokshenov used isomonodromic connections and Bessel functions/Hankel functions.

A review and different derivation of his formulas is in Guest-Hertling 2015.

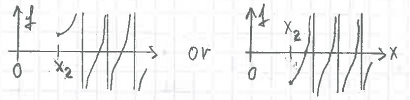
Theorem 2: asymptotics near ∞.

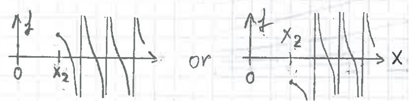
Fix a real solution f on $\mathbb{R}_{>0}$ of $P_{III}(0,0,4,-4)$ and its data $(s, b_1, b_2) \in M^{real}_{mon}$.

(a) f has no zeros or poles near $\infty \iff (b_1, b_2) = (\pm 1, 0)$.
$\exists\ x_2 > 0$ s.t. for $b_1 = 1$ resp. $b_1 = -1$:



(b) If $(b_1, b_2) \neq (\pm 1, 0)$ then $\exists\ x_2 > 0$ s.t.

for $b_2 > 0$:  ...[0+][∞−]...

and for $b_2 > 0$:  ...[0−][∞+]...

About the asymptotics near ∞

McCoy-Tracy-Wu studied only the solutions smooth near ∞ [Thm 2(a)]. Their techniques: hard analysis, not attractive, some proofs probably incomplete.

Its-Novokshenov obtained for solutions not smooth near ∞ [Thm 2(b)] rather precise formulas, derived with Mathieu functions and the WKB method. But they did not cover all cases (only solutions smooth near 0).

The theory of variations of TERP structures (limit theorems: nilpotent orbits and mixed TERP structures) gives sufficient control on the asymptotics near ∞ for all solutions: work of Hertling, Mochizuki, Sabbah.

It applies also to the asymptotics near 0. But there Niles' results are complete.

Theorems 1+2+3: asymptotics again + the global result

The theorems 1 [asymptotics near 0] and 2 [asymptotics near ∞] in a table:

	on $]0, x_1]$		on $[x_2, \infty[$
$\|s\| \leq 2, b_1 \geq 1$	$f(x) > 0$	$(b_1, b_2) = (1, 0)$	$f(x) > 0$
$\|s\| \leq 2, b_1 \leq -1$	$f(x) < 0$	$(b_1, b_2) = (-1, 0)$	$f(x) < 0$
$s > 2$	...[0+][0−]...	$b_2 > 0$	...[0+][∞−]...
$s < -2$	...[∞+][∞−]...	$b_2 < 0$	...[0−][∞+]...

Theorem 3:

$\boxed{\exists\ x_1, x_2 \text{ s.t. } x_1 = x_2. \text{ So no } \textit{intermediate mixed zone} \text{ exists.}}$

It implies (known): f has at most finitely many zeros or poles $\iff$ f has nowhere zeros or poles $\iff$ $|s| \leq 2$ and $(b_1, b_2) = (\pm 1, 0)$.

About the global result

Theorem 3 is completely new: the asymptotics near 0 and ∞ glue without *intermediate mixed zones*.

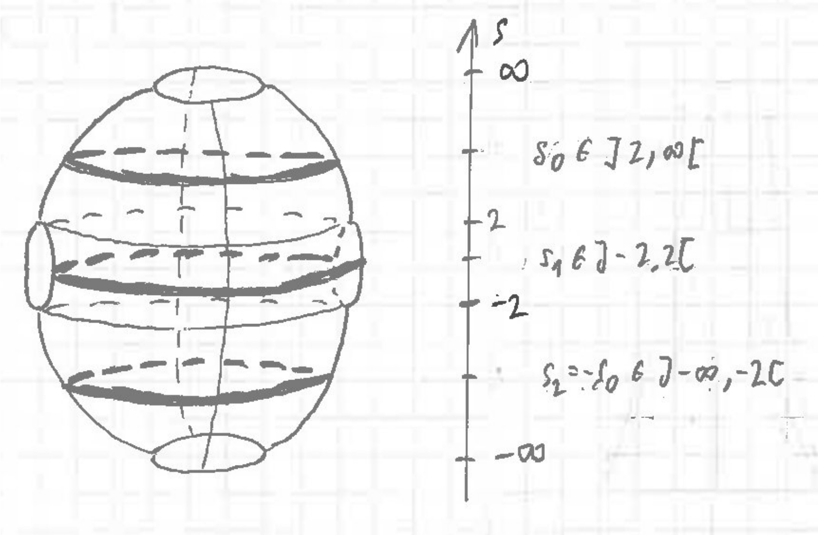
The strata in s for the asymptotics near 0 and the strata in (b_1, b_2) for the asymptotics near ∞ intersect in 14 strata.

$\longrightarrow$ 14 cases of sequences of zeros and poles exist.

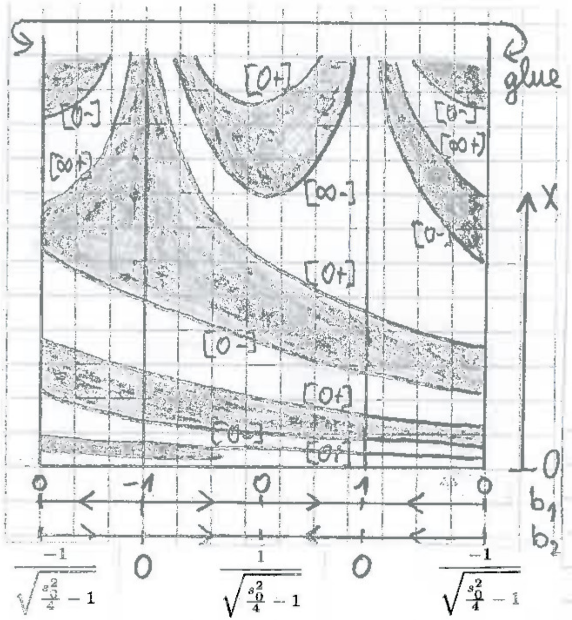
$$
\text{Theorem 3} \quad \Longleftarrow \quad \left\{ \begin{array}{l} \text{the following pictures of three sections for} \\ \mathbb{R}_{>0} \times M_{mon}^{real}|_{s=s_i},\, i = 0, 1, 2 \end{array} \right.
$$

$$
\Longleftarrow \quad \left\{ \begin{array}{l} \text{from the theorems 1+2 on the asymptotics} \\ \\ \text{and from the global geometry of the moduli} \\ \text{spaces: the isomorphism } \mathbb{R}_{>0} \times M_{mon}^{real} \cong M_{ini}^{real} \\ \text{and the 6 strata on } M_{ini}^{real}. \end{array} \right.
$$

Three sections $\mathbb{R}_{>0} \times M_{real}^{mon}|_{s=s_k}, \;\; k = 0, 1, 2$

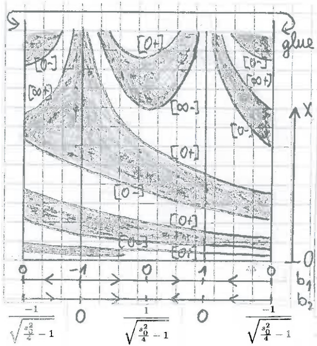


The section for $s = s_0 > 2$

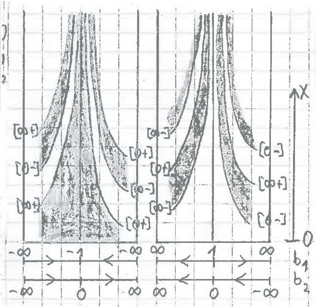


All three sections

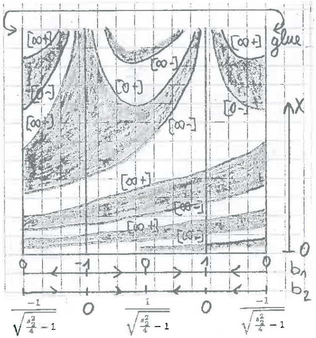
$\mathbb{R}_{>0} \times M_{mon}^{real}|_{s=s_0}$
for $s_0 > 2$



$\mathbb{R}_{>0} \times M_{mon}^{real}|_{s=s_1}$
for $-2 < s_1 < 2$



$\mathbb{R}_{>0} \times M_{mon}^{real}|_{s=s_2}$
for $s_2 = -s_0 < 0$



Isomonodromic connections

The Painlevé I-VI equations are related to isomonodromic families of 2nd order linear differential equations: this is old, Fuchs + Garnier early 20th century, taken up again by Okamoto 1986.

Painlevé I-VI equations are related to isomonodromic families of rank 2 vector bundles on $\mathbb{P}^1$ with meromorphic connections: Flaschka-Newell 1980 for some cases including $P_{III}(0,0,4,-4)$, Jimbo-Miwa-Ueno 1981 for other cases.

Its-Novokshenov 1986 and Niles 2009 and Guest-Hertling 2015 take up the connections of Flaschka-Newell 1980.

Its-Novokshenov and Niles work with matrices with some symmetries, Guest-Hertling with vector bundles with rich additional structure: variations of TERP structures.

Isomonodromic connections II

For $(x_0, f_0, \widetilde{f}_0) \in$ (2 open components of M_{ini}^{real}),
one has a rank 2 complex trivial vector bundle on $\mathbb{P}^1$
with global basis $\underline{v} = (v_1, v_2)$
and (flat) meromorphic connection ∇ with poles of order 2 at 0
and at ∞,

$$\nabla_{z\partial_z} \underline{v}(z) = \underline{v}(z) \cdot \left[\frac{x_0}{z} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{x_0}{2} \frac{\widetilde{f}_0}{f_0} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - z \cdot x_0 \begin{pmatrix} 0 & f_0^2 \\ f_0^{-2} & 0 \end{pmatrix} \right].$$

Vary x_0 and write (x, f, f') instead of $(x_0, f_0, \widetilde{f}_0)$. Then:

Flaschka-Newell 1980: One obtains an *isomonodromic family*
$\iff$ f is a solution of $P_{III}(0, 0, 4, -4)$.

TERP structures

A TERP structure consists of the data $(H \to \mathbb{P}^1, \nabla, H'_{\mathbb{R}}, \tau, P)$.

$H \to \mathbb{P}^1$ is a hol vector bundle of some rank $n \in \mathbb{N}$.

∇ is a (flat) hol connection on $H' := H|_{\mathbb{C}^*}$ and has on H poles of order ≤ 2 at 0 and at ∞.

$H'_{\mathbb{R}} \subset H'$ is a flat real subbundle with $H'_z = H'_{\mathbb{R},z} \oplus iH'_{\mathbb{R},z} \ \forall \ z \in \mathbb{C}^*$.

$\tau : H_z \to H_{1/\bar{z}}$ is $\forall \ z \in \mathbb{P}^1$ a $\mathbb{C}$-antilinear involution, flat on $\mathbb{C}^*$, holomorphic, the restriction to S^1 is the complex conjugation from $H'_{\mathbb{R}}$ (thus $H'_{\mathbb{R}}$ and τ determine one another.)

$P : H_z \times H_{-z} \to \mathbb{C} \ \forall \ z \in \mathbb{P}^1$ is a $\mathbb{C}$-bilinear nondegenerate hol flat (on $\mathbb{C}^*$) pairing and compatible with the real structure:

$$P(\tau a, \tau b) = \overline{P(a, b)}, \quad P : H'_{\mathbb{R},z} \times H'_{\mathbb{R},-z} \to \mathbb{R}.$$

TERP structures II

A TERP structure is *pure* if $H \to \mathbb{P}^1$ is trivial bundle.

Then $P(., \tau.)$ induces a nondegenerate hermitian pairing $h := P(., \tau.)$ on $\Gamma(\mathbb{P}^1, H)(\cong \mathbb{C}^n)$.

A TERP structure is *pure* and *polarized* if it is pure and if h is positive definite.

Flaschka-Newell's bundles can be enriched to TERP structures:

$$\tau(\underline{v}(z)) = \underline{v}(1/\overline{z}) \cdot \begin{pmatrix} 0 & f_0 \\ f_0^{-1} & \end{pmatrix}, \quad P(\underline{v}(z), \underline{v}(-z)) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

It is pure. It is polarized $\iff$ $f_0 > 0$, *anti-polarized* if $f_0 < 0$.

Special 1-par families of TERP structures

Basic fact: Any TERP structure $TERP_{x_0}$ induces a special 1-par isomonodromic family $(TERP_x)_{x>0}$ on $\mathbb{R}_{>0}$.

In the case of the Flaschka-Newell TERP structures, the real solutions f on $\mathbb{R}_{>0}$ of $P_{III}(0,0,4,-4)$ (with or without singularities) correspond to the 1-par families of TERP structures which extend the Flaschka-Newell TERP structures.

For x_0 in the 4 walls of M_{ini}^{real}, the TERP structure is non-pure, with

$$\mathcal{O}(H) \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Thus a solution f is smooth [and positive] on $U \subset \mathbb{R}_{>0} \iff$ the TERP structures $(TERP_x)_{x\in U}$ are all pure [and polarized].

Limit theorems for TERP structures

Theorem: Start with 1 TERP structure $TERP_{x_0}$ and its 1-par family $(TERP_x)_{x>0}$.

(i) (T. Mochizuki 2008, conjecture of Hertling 2003)
All members $TERP_x$ for $x \gg 1$ are pure and polarized
$\iff$ $TERP_{x_0}$ is a *mixed* TERP structure.

(ii) (Hertling-Sevenheck 2006)
All members $TERP_x$ for $0 < x \ll 1$ are pure and polarized
$\iff$ some candidate is a polarized mixed Hodge structure.

For the Flaschka-Newell TERP structures,
mixed TERP means $(b_1, b_2) = (1, 0)$,
and the candidate is a PMHS $\iff$ $|s| \leq 2,\ b_1 > 0$.

Thus (i) reproves Thm 2(a), and (ii) reproves Thm 1(a).

M_{mon}^{real}: the parameter s

$$I^{left} := \{z \in \mathbb{C}^* \mid \Im(z) < 0\}, \qquad I^{right} := \{z \in \mathbb{C}^* \mid \Im(z) > 0\},$$
$$I_0^+ = I_\infty^- := \mathbb{C}^* - (-i)\cdot\mathbb{R}_{>0}, \qquad I_0^- = I_\infty^+ := \mathbb{C}^* - i\cdot\mathbb{R}_{>0}.$$

$$\begin{aligned}
\underline{v}|_{I_0^\pm} &= \underline{e}_0^\pm \cdot \begin{pmatrix} e^{-x/z} & 0 \\ 0 & e^{x/z} \end{pmatrix} \cdot \mathcal{A}_0^\pm(z) \quad \text{with } \mathcal{A}_0^\pm(0) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \\
\underline{v}|_{I_\infty^\pm} &= \underline{e}_\infty^\pm \cdot \begin{pmatrix} e^{-xz} & 0 \\ 0 & e^{xz} \end{pmatrix} \cdot \mathcal{A}_\infty^\pm(z) \quad \text{with } \mathcal{A}_\infty^\pm(0) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \\
\underline{e}_0^-|_{I^{left}} &= \underline{e}_0^+|_{I^{left}} \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \qquad \underline{e}_0^-|_{I^{right}} = \underline{e}_0^+|_{I^{right}} \cdot \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \\
\underline{e}_\infty^-|_{I^{left}} &= \underline{e}_\infty^+|_{I^{left}} \cdot \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}, \qquad \underline{e}_\infty^-|_{I^{right}} = \underline{e}_\infty^+|_{I^{right}} \cdot \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}.
\end{aligned}$$

M_{mon}^{real}: the parameters (b_1, b_2)

$$\begin{aligned} \underline{e}_\infty^- &= \underline{e}_0^+ \cdot B, \qquad B = \text{connection matrix}, \\ BS &= S(B^t)^{-1},\ B = \overline{B}^{-1} \Rightarrow B = \begin{pmatrix} b_1 + \frac{1}{2} isb_2 & -ib_2 \\ ib_2 & b_1 - \frac{1}{2} isb_2 \end{pmatrix} \end{aligned}$$

$$M_{mon}^{real} = \{(s, b_1, b_2) \in \mathbb{R}^3 \mid b_1^2 + (\frac{s^2}{4} - 1)b_2^2 \quad (= \det B) = 1\}$$

$$\tau(\underline{e}_0^\pm(z)) = \underline{e}_\infty^\mp(\frac{1}{\overline{z}}), \qquad \tau(\underline{e}_\infty^\pm(z)) = \underline{e}_0^\mp(\frac{1}{\overline{z}})$$

Mixed TERP structure $\Rightarrow$ real and Stokes structure are compatible
$\iff$ the Stokes structures at 0 and ∞ are compatible
$\iff$ $(b_1, b_2) = (\pm 1, 0)$.

Some positivity in a *mixed* TERP structure $\Rightarrow (b_1, b_2) = (1, 0)$.

Main results of the paper with Martin Guest

- All possible sequences of zeros and poles of real solutions on $\mathbb{R}_{>0}$ of $P_{III}(0,0,4,-4)$: 14 types.
- Vector bundles with rich additional structure for all meromorphic multivalued solutions on $\mathbb{C}^*$ of $P_{III}(0,0,4,-4)$: P_{3D6}-TEJPA bundles.
- Normal forms for the families of vector bundles which correspond to zeros and poles. They are families of bundles of type $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.
- Complete classification of semisimple rank 2 TERP structures and their isomonodromic families.