Hodge Integrals and Tau-Symmetric Integrable Hierarchies of Hamiltonian Evolutionary PDEs

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Outline of the talk

- Introduction
- Hodge potential of a semisimple Frobenius manifold
- The Hodge Hierarchy
- Some examples
- A universality property of the Hodge hierarchy

1. Introduction

We consider the class of hierarchies of integrable evolutionary PDEs that are related to 2d topological field theory. The unknown functions of these evolutionary PDEs are given by the special two-point correlation functions

$$w^\alpha = \epsilon^2 \eta^{\alpha\gamma} \frac{\partial^2 \mathcal{F}(t)}{\partial t^{1,0} \partial t^{\gamma,0}}, \quad \alpha = 1, \ldots, n.$$

Here $\mathcal{F}(t)$ is the free energy which has the genus expansion

$$\mathcal{F}(t) = \epsilon^{-2} \mathcal{F}_0(t) + \mathcal{F}_1(t) + \epsilon^2 \mathcal{F}_2(t) + \ldots.$$

$t = (t^{\alpha,p}), \alpha = 1, \ldots, n; p \geq 0$ are the coupling constants of the fields $\tau_p(\phi_\alpha)$.

For topological sigma models

The genus g free energies $\mathcal{F}_g(t)$ are defined by the generating functions of genus g Gromov-Witten invariants

$$\langle\prod_{j=1}^{m}\tau_{p_j}(\phi_{\alpha_j})\rangle_{g,\beta} = \int_{[\overline{M}_{g,m}(X,\beta)]^{\text{virt}}}\prod_{j=1}^{m} ev_j^*(\phi_{\alpha_j})\wedge c_1^{p_j}(\mathcal{L}_j)$$

by the formula

$$\mathcal{F}_g(t) = \sum_{\beta\in H_2(X,\mathbb{Z})}\langle e^{\sum t^{\alpha,p}\tau_p(\phi_\alpha)}\rangle_{g,\beta}\, q^{\beta}.$$

Here X is a smooth projective variety with $H^{\text{odd}}(X,\mathbb{C})=0$. $\overline{M}_{g,m}(X,\beta)$ is the moduli space of stable maps of degree $\beta\in H_2(X,\mathbb{Z})$ with target X of curves of genus g with m marked points. $\mathcal{L}_j$ are the tautological line bundles over $\overline{M}_{g,m}(X,\beta)$, and $ev_i:\ \overline{M}_{g,m}(X,\beta)\to X$ are the evaluation maps.

The appearance of integrable hierarchies:

Consider the genus zero two point correlation functions

$$v^\alpha(t) = \eta^{\alpha\gamma} \frac{\partial^2 \mathcal{F}_0(t)}{\partial t^{1,0} \partial t^{\gamma,0}}, \quad \alpha = 1, \dots, n.$$

Here the matrix $(\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}$ is the inverse to the Poincaré pairing matrix. Redenote $t^{1,0} = x$, then we have the equations

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\gamma} \frac{\partial}{\partial x} \left(\frac{\partial^2 \mathcal{F}_0(t)}{\partial t^{\gamma,0} \partial t^{\beta,q}} \right), \quad \alpha, \beta = 1, \dots, q \geq 0.$$

The Principal Hierarchy

As observed by Dijkgraaf & Witten (1990) the two-point functions that appear in the r.h.s. of the above equation can be represented as functions of $v^1(t), \ldots, v^n(t)$ (the constitutive relation)

$$\frac{\partial^2 \mathcal{F}_0(t)}{\partial t^{\alpha,0} \partial t^{\beta,q}} = \Omega_{\alpha,0;\beta,q}(v(t)) = \frac{\partial \theta_{\beta,q+1}(v)}{\partial v^\alpha}.$$

A hierarchy of infinite dimensional Hamiltonian systems

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = P^{\alpha\gamma} \frac{\delta H_{\beta,q}}{\delta v^\gamma}, \quad t^{1,0} = x$$

with Hamiltonian operator P and pairwise commutative Hamiltonians given by

$$P^{\alpha\beta} = \eta^{\alpha\beta} \frac{\partial}{\partial x}, \quad H_{\beta,q} = \int \theta_{\beta,q+1}(v(x)) dx.$$

Integrable hierarchy that characterizes the full genera free energy $\mathcal{F}(t)$

Theorem

For a 2d topological field theory associated to a semisimple Frobenius manifold, the full genera free energy $\mathcal{F}(t)$ is given by the logarithm of a particular tau function of a certain integrable Hamiltonian deformation of the Principal Hierarchy

$$\frac{\partial w^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\gamma}\frac{\partial}{\partial x}\left(\frac{\partial\theta_{\beta,q+1}(w)}{\partial v^\gamma}\right) + \sum_{g\geq 1}\epsilon^{2g}K^\alpha_{\beta,q;g}(w, w_x, \ldots, w^{(2g+1)}), \quad \alpha,\beta = 1,\ldots,n, q\geq 0.$$

Here $K^\alpha_{\beta,q;g}$ are homogeneous polynomials of $w_x^\gamma, \ldots, \partial_x^{2g+1}w^\gamma$ of degree $2g+1$.

(Constructed by Dubrovin, Z.; polynomiality proved by Buryak, Posthuma, Shadrin)

The case of one-dimensional Frobenius manifold: 2d topological gravity

$$\mathcal{F}_g = \sum \frac{1}{k!} t_{p_1} \cdots t_{p_k} \int_{\overline{M}_{g,k}} \psi_1^{p_1} \wedge \cdots \wedge \psi_k^{p_k}, \quad \psi_j^{p_j} = c_1^{p_j}(\mathcal{L}_j).$$

Witten-Kontsevich Theorem: The integrable hierarchy is the KdV hierarchy.

The first three flows:

$$\begin{aligned} w_{t_0} &= w_x, \quad t_0 := x \\ w_{t_1} &= w w_x + \frac{\epsilon^2}{12} w_{xxx} \quad (\text{KdV equation}) \\ w_{t_2} &= \frac{1}{2} w^2 w_x + \frac{\epsilon^2}{12} \left(2 w_x w_{xx} + w w_{xxx}\right) + \frac{\epsilon^4}{240} w^{(5)} \end{aligned}$$

The bihamiltonian structure

$$\frac{\partial w}{\partial t_p} = P_1 \frac{\delta H_p}{\delta w} = (p + \frac{1}{2})^{-1} P_2 \frac{\delta H_{p-1}}{\delta w}, \quad p \geq 0$$

with the bihamiltonian structure

$$P_1 = \frac{\partial}{\partial x},$$
$$P_2 = w(x)\frac{\partial}{\partial x} + \frac{1}{2}\, w_x(x) + \frac{\epsilon^2}{8}\,\frac{\partial^3}{\partial x^3}$$

Densities of the Hamiltonians

$$h_{-1} = w, \quad h_0 = \frac{w^2}{2} + \epsilon^2 \frac{w_{xx}}{12}, \ldots$$

Hodge integrals

The genus g Hodge integrals of a smooth projective variety X

$$
\begin{aligned}
&\langle \prod_{i=1}^{l} \mathrm{ch}_{k_i}(\mathbb{E}) \prod_{j=1}^{m} \tau_{p_j}(\phi_{\alpha_j}) \rangle_{g,\beta} \\
&\qquad = \int_{[\overline{M}_{g,m}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{l} \mathrm{ch}_{k_i}(\mathbb{E}) \wedge \prod_{j=1}^{m} ev_j^*(\phi_{\alpha_j}) \wedge c_1^{p_j}(\mathcal{L}_j).
\end{aligned}
$$

Here $\mathbb{E}$ is the rank g Hodge bundle over $\overline{M}_{g,m}(X,\beta)$, $\mathrm{ch}_k(\mathbb{E})$ are the homogenous components of the Chern character of $\mathbb{E}$.

The Hodge potential

The genus g Hodge potential of X

$$\mathcal{H}_g(t;s) = \sum_{\beta \in H_2(X,\mathbb{Z})} \langle e^{\sum s_{2k-1} ch_{2k-1}(\mathbb{E})} e^{\sum t^{\alpha,p}\tau_p(\phi_\alpha)} \rangle_{g,\beta}\, q^\beta.$$

The Hodge potential of X is defined by

$$\mathcal{H}(t;s;\epsilon) = \sum_{g=0}^{\infty} \epsilon^{2g-2} \mathcal{H}_g(t;s).$$

The Hodge potential satisfy the initial condition

$$\mathcal{H}_g(t;0) = \mathcal{F}_g(t), \quad \mathcal{H}(t;0;\epsilon) = \mathcal{F}(t;\epsilon).$$

Reconstruction of the Hodge integrals from the Gromov-Witten invariants

The partition function for the Hodge integrals

$$Z_{\mathbb{E}}(t;s;\epsilon) = e^{\epsilon^{-2}\mathcal{H}_0+\mathcal{H}_1+\epsilon^2\mathcal{H}_2+\dots}$$

satisfies (Faber, Pandharipande 2000)

$$\frac{\partial Z_{\mathbb{E}}}{\partial s_k} = \left(\sum_{p\geq 0}\tilde{t}^{\alpha,p}\frac{\partial}{\partial t^{\alpha,p+2k-1}} - \frac{\epsilon^2}{2}\sum_{p=0}^{2k-2}(-1)^p\eta^{\alpha\beta}\frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,2k-2-p}}\right) Z_{\mathbb{E}},$$

$$Z_{\mathbb{E}}(t;0;\epsilon) = Z(t;\epsilon).$$

Here we redenote $-\frac{B_{2k}}{(2k)!}s_{2k-1}$ by s_k, B_{2k} are the Bernoulli numbers. $Z(t;\epsilon) = e^{\mathcal{F}(t;\epsilon)}$ is the partition function for Gromov–Witten invariants of X, and $\tilde{t}^{\alpha,p} = t^{\alpha,p} - \delta_1^\alpha\delta_1^p$.

Problem: To study integrable hierarchies that are satisfied by the two point functions

$$w^\alpha = \epsilon^2 \eta^{\alpha\gamma} \frac{\partial^2 \mathcal{H}(t; s; \epsilon)}{\partial t^{1,0} \partial t^{\gamma,0}}, \quad \alpha = 1, \dots, n.$$

We will study this problem starting from any semisimple Frobenius manifold.

2. Hodge potential of a semisimple Frobenius manifold

Frobenius manifold structure (Dubrovin)

Encodes the properties of the primary free energy
$F = F(v^1, \dots, v^n) = \mathcal{F}_0(t)|_{t^{\alpha,0}=v^\alpha,\ t^{\alpha,p\geq 1}=0}$:

$$\frac{\partial^3 F}{\partial v^1 \partial v^\alpha \partial v^\beta} = \eta_{\alpha\beta} = \text{constant}, \quad (\eta_{\alpha\beta}): \ \text{nondegenerate},$$

$$c^\gamma_{\alpha\beta} = \eta^{\gamma\xi} \frac{\partial^3 F}{\partial v^\xi \partial v^\alpha \partial v^\beta}: \quad \begin{array}{c}\text{structure constants of}\\ \text{an associative algebra}\end{array}$$

$$\partial_E F = (3-d)F + \text{quadratic terms in } v^\alpha,$$

$$\text{here the Euler vector } E = \sum_{\alpha=1}^n (d_\alpha v^\alpha + r_\alpha) \frac{\partial}{\partial v^\alpha}.$$

They are the WDVV equations of associativity.

The deformed flat connection of a FM $(M^n;\ \cdot\ ;\langle\ ,\ \rangle;\ e;\ E)$:

$$\tilde{\nabla}_a b = \nabla_a b + z\, a \cdot b$$

Extend it to $M \times \mathbb{C}^*$ by

$$\tilde{\nabla}_{\frac{d}{dz}} b = \partial_z b + E \cdot b - \frac{1}{z}\mu\, b$$

with $\mu = \frac{2-d}{2} - \nabla E$.

The deformed flat coordinates $\tilde{v}_1(v;z), \ldots, \tilde{v}_n(v;z)$ satisfying

$$\tilde{\nabla} d\tilde{v}_\alpha(v;z) = 0, \quad \alpha = 1, \ldots, n.$$

The functions $\theta_{\beta,q}(v)$

The deformed flat coordinates have the form

$$(\tilde{v}_1(v;z),\ldots,\tilde{v}_n(v;z)) = (\theta_1(v;z),\ldots,\theta_n(v;z))z^{\mu}z^{R}$$

Here $\theta_1(v;z),\ldots,\theta_n(v;z)$ are analytic at $z=0$ with Taylor expansion

$$\theta_\alpha(v;z) = \sum_{p\geq 0}\theta_{\alpha,p}(v)z^p$$

satisfying the normalization conditions

$$\begin{gathered}\theta_\alpha(v;0) = \eta_{\alpha\beta}v^\beta, \quad \alpha = 1,\ldots,n\\ \langle\nabla\theta_\alpha(v;-z),\nabla\theta_\beta(v;z)\rangle = \eta_{\alpha\beta}.\end{gathered}$$

$\mu, R:$ monodromy data at $z=0$.

The Principal Hierarchy

A hierarchy of evolutionary PDEs

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\gamma}\frac{\partial}{\partial x}\left(\frac{\partial\theta_{\beta,q+1}(v)}{\partial v^\gamma}\right), \quad \alpha,\beta = 1,\dots,n,\ q\geq 0.$$

It possesses a bihamitonian structure of hydrodynamic type

$$\{v^\alpha(x), v^\beta(y)\}_1 = \eta^{\alpha\beta}\delta'(x-y),$$
$$\{v^\alpha(x), v^\beta(y)\}_2 = g^{\alpha\beta}\delta'(x-y) + \Gamma^{\alpha\beta}_\gamma(v)v^\gamma_x\delta(x-y).$$

with Hamiltonians

$$H_{\beta,q} = \int\theta_{\beta,q+1}(v(x))dx, \quad \beta = 1,\dots,n, q\geq 0.$$

The Principal Hierarchy is a bihamiltonian integrable hierarchy

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = \{v^\alpha(x), H_{\beta,q}\}_1$$

with bihamiltonian recursion relation

$$\begin{aligned}\{v^\alpha(x), H_{\beta,q-1}\}_2 =& (q+\frac{1}{2}+\mu_\beta)\{v^\alpha(x), H_{\beta,q}\}_1 \\ &+ \sum_{k=1}^{m} (R_k)^\gamma_\beta \{v^\alpha(x), H_{\gamma,q-k}\}_1.\end{aligned}$$

The genus zero part of the associated 2d TFT is characterized by the Principal Hierarchy in the sense that the genus zero free energy $\mathcal{F}_0(t)$ can be represented by a particular (called topological) solution of the Principal Hierarchy

Topological solution of the Principal Hierarchy

Particular solution $v^\alpha = v^\alpha(t)$ of the integrable hierarchy that satisfies the string equation is obtained by solving the equations (generalized hodograph transformation)

$$\sum_{q\geq 0} \tilde{t}^{\beta,q}\,\nabla\theta_{\beta,q}(v) = 0, \quad \tilde{t}^{\beta,q} = t^{\beta,q} - \delta^\beta_1\delta^q_1.$$

The genus zero free energy

$$\mathcal{F}_0(t) = \frac{1}{2}\sum_{\alpha,p;\beta,q} \tilde{t}^{\alpha,p}\tilde{t}^{\beta,q}\,\Omega_{\alpha,p;\beta,q}(v)|_{v=v(t)},$$

$$\sum \Omega_{\alpha,p;\beta,q}(v)\, z^p w^q = \frac{< \nabla\theta_\alpha(v;z), \nabla\theta_\beta(v;w) > -\eta_{\alpha\beta}}{z+w}.$$

Construct the higher genera free energies $\mathcal{F}_g(t)$

We (Dubrovin, Z. 2001) use the properties of the Virasoro symmetries of the Principal Hierarchy to determine $\mathcal{F}_g$.

The first symmetry is the Galilean symmetry:

$$v^\alpha \mapsto v^\alpha + \epsilon \left(\sum_{\beta,q} t^{\beta,q} \frac{\partial v^\alpha}{\partial t^{\beta,q-1}} + \delta^\alpha_1 \right) + \mathcal{O}(\epsilon^2),$$

$$\tau \mapsto \tau + \epsilon L_{-1}\tau + \mathcal{O}(\epsilon^2)$$

with the tau function τ and the operator L_{-1} defined by

$$\tau = e^{\mathcal{F}_0(t)}, \quad L_{-1} = \sum_{q\geq 1} t^{\beta,q} \frac{\partial}{\partial t^{\beta,q-1}} + \frac{1}{2}\,\eta_{\alpha\beta}\, t^{\alpha,0} t^{\beta,0}.$$

Virasoro symmetries acting on the tau function

$$\tau \mapsto \tau + \epsilon \left(a_m^{\alpha,p;\beta,q} \frac{1}{\tau} \frac{\partial \tau}{\partial t^{\alpha,p}} \frac{\partial \tau}{\partial t^{\beta,q}} \right.$$
$$\left. + b_{m;\alpha,p}^{\beta,q} t^{\alpha,p} \frac{\partial \tau}{\partial t^{\beta,q}} + c_{m;\alpha,p\beta,q} t^{\alpha,p} t^{\beta,q} \tau \right) + \mathcal{O}(\epsilon^2)$$

with the Virasoro operators (for $m \geq -1$)

$$L_m = \epsilon^2 a_m^{\alpha,p;\beta,q} \frac{\partial^2}{\partial t^{\alpha,p} \partial t^{\beta,q}} + b_{m;\alpha,p}^{\beta,q} t^{\alpha,p} \frac{\partial}{\partial t^{\beta,q}}$$
$$+ \epsilon^{-2} c_{m;\alpha,p,\beta,q} t^{\alpha,p} t^{\beta,q} + \kappa_0 \, \delta_{m,0}, \quad m \geq -1.$$

Linearization of the Virasoro symmetries

Note that the higher genera free energy $\mathcal{F}_g(t)$ can be represented as functions of the two-point functions

$$\mathcal{F}_g(t) = F_g(v, v_x, v_{xx}, \dots)|_{v=v(t)}$$

We require that the infinitesimal Virasoro symmetries act linearly on the full genera tau function

$$\tau = e^{\epsilon^{-2}\mathcal{F}_0 + \sum_{g\geq 1} \epsilon^{2g-2} F_g(v, v_x, v_{xx}, \dots)|_{v=v(t)}}.$$

in the following way

$$\tau \mapsto \tau + \epsilon\, L_m \tau + \mathcal{O}(\epsilon^2), \quad m \geq -1.$$

Here the parameter ϵ is called the string coupling constant.

The loop equation

The condition of linearization of the Virasoro symmetries is equivalent to a system of equations, called the loop equation, for the functions $F_g, g \geq 1$. For example, when $n = 1$ we have

$$
\begin{aligned}
&\sum_r \frac{\partial \Delta\mathcal{F}}{\partial v^{(r)}} \partial_x^r \frac{1}{v-\lambda} + \sum_{r\geq 1} \frac{\partial \Delta\mathcal{F}}{\partial v^{(r)}} \sum_{k=1}^{r} \binom{r}{k} \partial_x^{k-1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{r-k+1} \frac{1}{\sqrt{v-\lambda}} \\
&= \frac{1}{16\,\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2} \\
&\quad + \frac{\epsilon^2}{2} \sum \left[\frac{\partial^2 \Delta\mathcal{F}}{\partial v^{(k)} \partial v^{(l)}} + \frac{\partial \Delta\mathcal{F}}{\partial v^{(k)}} \frac{\partial \Delta\mathcal{F}}{\partial v^{(l)}} \right] \partial_x^{k+1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{l+1} \frac{1}{\sqrt{v-\lambda}} \\
&\quad - \frac{\epsilon^2}{16} \sum \frac{\partial \Delta\mathcal{F}}{\partial v^{(k)}} \partial_x^{k+2} \frac{1}{(v-\lambda)^2}.
\end{aligned}
$$

Here $\Delta\mathcal{F} = \sum_{g\geq 1} \epsilon^{2g-2} F_g$.

The ϵ^0's coefficients of the loop equation give the equation for $\mathcal{F}_1$:

$$\frac{1}{v-\lambda}\frac{\partial F_1}{\partial v} - \frac{3}{2}\frac{v'}{(v-\lambda)^2}\frac{\partial F_1}{\partial v'} = \frac{1}{16\,\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2}.$$

This implies that

$$\kappa_0 = \frac{1}{16}, \quad F_1 = \frac{1}{24}\log v'.$$

Similarly, the coefficients of ϵ^2 give the squations for F_2, from which we obtain

$$F_2 = \frac{v^{(4)}}{1152\,v'^2} - \frac{7\,v''v'''}{1920\,v'^3} + \frac{v''^3}{360\,v'^4}.$$

For a general semisimple Frobenius manifold, the ϵ^0 terms of the loop equation yield the equation

$$\sum_{i=1}^n \frac{\partial F_1}{\partial u_i}\frac{1}{u_i-\lambda} - \sum_{i=1}^n \frac{\partial F_1}{\partial u_i'}\frac{u_i'}{(u_i-\lambda)^2} + \sum \frac{\partial F_1}{\partial v_x^\gamma}\partial_1 p_\alpha G^{\alpha\beta}\partial_x\partial^\gamma p_\beta$$
$$= -\frac{1}{16}\sum_{i=1}^n \frac{1}{(\lambda-u_i)^2} - \frac{1}{2}\sum_{i<j}\frac{V_{ij}^2}{(\lambda-u_i)(\lambda-u_j)} + \frac{1}{4\,\lambda^2}\,\mathrm{tr}\left(\frac{1}{4}-\hat{\mu}^2\right) - \frac{\kappa_0}{\lambda^2}.$$

Here $u_1, \ldots, u_n$ are the canonical coordinates of the semisimple Frobenius manifold M, which have the property

$$\frac{\partial}{\partial u_i}\cdot\frac{\partial}{\partial u_j} = \delta_{i,j}\frac{\partial}{\partial u_i}.$$

The genus one free energy

$$F_1(v, v_x) = \frac{1}{24} \log\det\left(c_{\alpha\beta\gamma}(v) v_x^\gamma\right) + G(v)$$
$$= \frac{1}{24} \log\det\left(\frac{\partial^3 \mathcal{F}_0(t)}{\partial t^{1,0} \partial t^{\alpha,0} \partial t^{\beta,0}}\right) + G(v).$$

(Witten-Dijkgraaf and Getzler)

The function $G(v)$ has the form (Dubrovin, Z.)

$$G(v) = \log \frac{\tau_I(v)}{J^{1/24}(v)}\,.$$

Defintion of the Hodge potentials for semisimple Frobenius manifolds

We require that the partition function

$$Z_{\mathbb{E}}(t;s) = e^{\epsilon^{-2}\mathcal{H}_0+\mathcal{H}_1+\epsilon^2\mathcal{H}_2+\dots}$$

satisfies

$$\frac{\partial Z_{\mathbb{E}}}{\partial s_k} = \left(\sum_{p\ge 0}\tilde{t}^{\alpha,p}\frac{\partial}{\partial t^{\alpha,p+2k-1}} - \frac{\epsilon^2}{2}\sum_{p=0}^{2k-2}(-1)^p\eta^{\alpha\beta}\frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,2k-2-p}}\right) Z_{\mathbb{E}},$$

$$Z_{\mathbb{E}}(t;0;\epsilon) = Z(t;\epsilon).$$

Here $Z(t;\epsilon)$ is the partition function for the Frobenius manifold

$$Z(t;\epsilon) = e^{\epsilon^{-2}\mathcal{F}_0(t;\epsilon)+\mathcal{F}_1(t;\epsilon)+\epsilon^2\mathcal{F}_2(t;\epsilon)+\dots}.$$

Defining equations for $\mathcal{H}_g(t; s)$

We introduce the operators

$$D_k = \sum \tilde{t}^{\alpha,p} \frac{\partial}{\partial t^{\alpha,2k-1+p}} - \sum_{p=0}^{2k-2} (-1)^p \eta^{\alpha\beta} \frac{\partial \mathcal{F}_0(t)}{\partial t^{\alpha,p}} \frac{\partial}{\partial t^{\beta,2k-2-p}}$$

Then from the above equations we know that

$$\begin{aligned}
\frac{\partial \mathcal{H}_g}{\partial s_k} &= D_k \mathcal{H}_g - \frac{1}{2} \sum_{p=0}^{2k-2} (-1)^p \eta^{\alpha\beta} \frac{\partial^2 \mathcal{H}_{g-1}}{\partial t^{\alpha,p} \partial t^{\beta,2k-2-p}} \\
&\quad - \frac{1}{2} \sum_{p=0}^{2k-2} \sum_{m=1}^{g-1} (-1)^p \eta^{\alpha\beta} \frac{\partial \mathcal{H}_m}{\partial t^{\alpha,p}} \frac{\partial \mathcal{H}_{g-m}}{\partial t^{\beta,2k-2-p}}, \\
\mathcal{H}_g(t; 0) &= \mathcal{F}_g(t),
\end{aligned}$$

Theorem

For an arbitrary calibrated semisimple Frobenius manifold and an arbitrary solution to the associated Principal Hierarchy there exists a unique Hodge potential determined by the system of equations mentioned above. It can be represented in the form

$$\begin{aligned}
&\mathcal{H}_0 = \mathcal{F}_0,\\
&\mathcal{H}_1 = \mathcal{F}_1 - \frac{1}{2} s_1 \eta^{\alpha\beta} \partial_{v^\alpha} \partial_{v^\beta} F(v),\\
&\mathcal{H}_g = \mathcal{H}_g\left(v; v_x, v_{xx}, \dots, v^{(3g-3)}; s_1, \dots, s_g\right) \quad \text{for} \quad g \geq 2,
\end{aligned}$$

where F is the potential of the Frobenius manifold, $\mathcal{H}_g$ $(g \geq 2)$ is a polynomial in s_1, ..., s_g, v^α_{xx}, ..., $v^{\alpha,(3g-3)}$ and a rational function in v^α_x, and

$$\overline{\deg}\Delta\mathcal{H}_g \leq 3g-3$$

with $\Delta\mathcal{H}_g = \mathcal{H}_g - \mathcal{F}_g$.

For the example of one-dimensioanl Frobenius manifold

$$\begin{aligned}
\mathcal{H}_0(t;s) &= \mathcal{F}_0(t),\\
\mathcal{H}_1(t;s) &= \mathcal{F}_1(t) - \frac{1}{2}s_1 v = \frac{1}{24}\log v_x - \frac{1}{2}s_1 v,\\
\mathcal{H}_2(t;s) &= \mathcal{F}_2(t) + s_1\left(\frac{11 v_{xx}^2}{480 v_x^2} - \frac{v_{xxx}}{40 v_x}\right) + \frac{7}{40}s_1^2 v_{xx} - \left(\frac{s_1^3}{10} + \frac{s_2}{48}\right) v_x^2.
\end{aligned}$$

Here

$$F_1 = \frac{1}{24}\log v_x.$$

$$F_2 = \frac{v^{(4)}}{1152\, {v_x}^2} - \frac{7\, v_{xx} v_{xxx}}{1920\, {v_x}^3} + \frac{{v_{xx}}^3}{360\, {v_x}^4}.$$

Corollaries of the theorem

We consider the intersection numbers of the form

$$\int_{\overline{M}_{g,m}} \lambda_{i_1} \dots \lambda_{i_k} \psi_1^{p_1} \dots \psi_m^{p_m}, \quad \lambda_i = c_i(\mathbb{E})$$

from the above theorem it follows that they vanish unless

$$i_1 + \cdots + i_k \leq 3g - 3.$$

We introduce the generating function

$$H_g(\lambda_{i_1} \dots \lambda_{i_k}; t) = \sum_{m \geq 0} \frac{1}{m!} \sum_{p_1, \dots, p_m \geq 0} t_{p_1} \dots t_{p_m} \int_{\overline{M}_{g,m}} \lambda_{i_1} \dots \lambda_{i_k} \psi_1^{p_1} \dots \psi_m^{p_m}.$$

Corollary

Let $v(t)$ denote the topological solution of the dispersionless KdV hierarchy

$$v(t) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_1+\ldots+p_k=k-1} \frac{t_{p_1}}{p_1!} \cdots \frac{t_{p_k}}{p_k!},$$

and $v_k = \partial_x^k v(t)$. Then following formulae hold true:

$$H_g(\lambda_g\,\lambda_{g-1}\,\lambda_{g-2};t) = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g}v_1^{2g-2}(t),$$

$$H_1(\lambda_1;t) = \frac{1}{24}v,$$

$$H_2(\lambda_1;t) = \frac{1}{480}\frac{v_3}{v_1} - \frac{11}{5760}\frac{v_2^2}{v_1^2},$$

$$H_2(\lambda_2;t) = \frac{7}{5760}v_2, \quad H_2(\lambda_1\lambda_2;t) = \frac{1}{5760}v_1^2,$$

$$
\begin{aligned}
H_3(\lambda_1;t) &= \frac{131 v_2^5}{45360 v_1^6} - \frac{9343 v_2^3 v_3}{1451520 v_1^5} + \frac{869 v_2 v_3^2}{322560 v_1^4} + \frac{185 v_2^2 v_4}{96768 v_1^4} \\
&\quad - \frac{689 v_3 v_4}{967680 v_1^3} - \frac{383 v_2 v_5}{967680 v_1^3} + \frac{7 v_6}{138240 v_1^2}, \\
H_3(\lambda_2;t) &= -\frac{19 v_2^4}{53760 v_1^4} + \frac{151 v_2^2 v_3}{207360 v_1^3} - \frac{61 v_3^2}{322560 v_1^2} - \frac{373 v_2 v_4}{1451520 v_1^2} + \frac{41 v_5}{580608 v_1}, \\
H_3(\lambda_3;t) &= \frac{31 v_4}{967680}, \\
H_3(\lambda_1\lambda_2;t) &= \frac{v_2^3}{36288 v_1^2} - \frac{19 v_2 v_3}{483840 v_1} + \frac{23 v_4}{193536}, \\
H_3(\lambda_1\lambda_3;t) &= \frac{31 v_2^2}{1451520} + \frac{41 v_1 v_3}{1451520}, \dots \\
H_3(\lambda_2\lambda_3;t) &= \frac{v_1^2 v_2}{120960}, \quad H_3(\lambda_1\lambda_2\lambda_3;t) = \frac{v_1^4}{1451520}, \dots .
\end{aligned}
$$

3. The Hodge Hierarchy

Let us construct a new hierarchy of integrable Hamiltonian PDEs associated with the calibrated semisimple n-dimensional Frobenius manifold under consideration. The equations of the hierarchy will depend on the parameters s_1, s_2, ...Logarithms of tau-functions of the new hierarchy are Hodge potentials, $\log \tau = \mathcal{H}$. That is, the solutions w_1, ..., w_n are given by the second derivatives of the Hodge potential

$$w_\alpha(t; s; \epsilon) = \epsilon^2 \frac{\partial^2 \mathcal{H}(t; s; \epsilon)}{\partial x \partial t^{\alpha,0}} = v_\alpha + \sum_{g \geq 1} \epsilon^{2g} \frac{\partial^2 \mathcal{H}_g(t; s)}{\partial x \partial t^{\alpha,0}}.$$

Due to properties of the Hodge potentials we can represent w_α in the form

$$w_\alpha = v_\alpha + \sum_{g \geq 1} \epsilon^{2g} V_\alpha^{[g]} \left(v; v_x, \ldots, v^{(3g)}; s_1, \ldots, s_g \right).$$

We apply the substitution (the quasi-Miura transformation)

$$v_\alpha \mapsto w_\alpha = v_\alpha + \sum_{g\geq 1} \epsilon^{2g} V_\alpha^{[g]}\left(v; v_x, \dots, v^{(3g)}; s_1, \dots, s_g\right), \quad \alpha = 1, \dots, n$$

to the equations of the Principal Hierarchy

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\gamma} \frac{\partial}{\partial x}\left(\frac{\partial \theta_{\beta,q+1}(v)}{\partial v^\gamma}\right), \quad \alpha, \beta = 1, \dots, n,\ q \geq 0.$$

Theorem

The Hodge Hierarchy associated with an arbitrary semisimple calibrated Frobenius manifold is a tau-symmetric integrable hierarchy of Hamiltonian evolutionary PDEs.

For the example of one-dimensional Frobenius manifold

$$
\begin{aligned}
\frac{\partial w}{\partial t_0} &= \tilde{P}\frac{\delta \tilde{H}_0}{\delta w(x)} = w_x,\\
\frac{\partial w}{\partial t_1} &= \tilde{P}\frac{\delta \tilde{H}_1}{\delta w(x)} = w w_x + \epsilon^2\left(\frac{w_{xxx}}{12} - w_x w_{xx} s_1\right)\\
&\quad + \epsilon^4\left[-\frac{w_5}{60}s_1 + \left(w_2 w_3 + \frac{1}{5}w_1 w_4\right)s_1^2 + \left(-\frac{8}{5}w_1 w_2^2 - \frac{4}{5}w_1^2 w_3\right)s_1^3\right.\\
&\quad \left. + \left(-\frac{1}{3}w_1 w_2^2 - \frac{1}{6}w_1^2 w_3\right)s_2\right] + \mathcal{O}(\epsilon^6),\\
\frac{\partial w}{\partial t_q} &= \tilde{P}\frac{\delta \tilde{H}_q}{\delta w(x)}, \quad q \geq 2.
\end{aligned}
$$

The deformed Hamiltonian operator is given by

$$
\tilde{P} = \partial_x - \epsilon^2 s_1\,\partial_x^3 + \frac{3}{5}\,\epsilon^4\,s_1^2\,\partial_x^5 + \mathcal{O}(\epsilon^6)
$$

Tau-symmetric integrable Hamiltonian deformation of the Principal Hierarchy

Let M be a Frobenius manifold. A hierarchy of Hamiltonian evolutionary PDEs

$$\frac{\partial w^\alpha}{\partial t^{\beta,q}} = \{w^\alpha(x), H_{\beta,q}\} = P^{\alpha\gamma}\frac{\delta H_{\beta,q}}{\delta w^\gamma(x)}, \qquad q \geq 0$$

is called a tau-symmetric integrable Hamiltonian deformation of the Principal Hierarchy of M if the flow $\frac{\partial}{\partial t^{1,0}}$ is given by the translation along the spatial variable x and the following conditions are satisfied:

1) Integrability: for $\beta = 1, \ldots, n$, $q \geq 0$ the functionals $H_{\beta,q}$ are conserved quantities for each flow of the hierarchy.

2) $H_{\beta,-1} = \int h_{\beta,-1}(w(x))\,dx$ are Casimirs of the Hamiltonian operator.

3) Polynomiality: the Hamiltonian operator and the densities of the Hamiltonians take the forms

$$P^{\alpha\beta} = \eta^{\alpha\beta}\partial_x + \sum_{k\geq 1}\epsilon^k \sum_{l=0}^{k+1} P^{\alpha\beta}_{k,l}(w; w_x, \ldots, w^{(k+1-l)})\partial_x^l$$

$$h_{\beta,q} = \theta_{\beta,q+1}(w) + \sum_{k\geq 1}\epsilon^k h_{\beta,q;k}(w; w_x, \ldots, w^{(k)}), \quad q \geq 0,$$

4) Tau-symmetry:

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}}, \qquad p, q \geq 0,$$

where $h_{\alpha,-1} = w_\alpha = \eta_{\alpha\gamma} w^\gamma$.

Tau function of the tau-symmetric integrable Hamiltonian hierarchy

The Tau-symmetry condition ensures the existence of a tau function $\tau(t;\epsilon)$ for any solution $w_\alpha(t;\epsilon)$ of the tau-symmetric integrable Hamiltonian hierarchy such that

$$\tilde{\Omega}_{\alpha,p;\beta,q} = \epsilon^2 \frac{\partial^2 \log \tau(t;\epsilon)}{\partial t^{\alpha,p} \partial t^{\beta,q}}, \quad \frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \partial_x \tilde{\Omega}_{\alpha,p;\beta,q} = \partial_x \tilde{\Omega}_{\beta,q;\alpha,p}.$$

In particular

$$\begin{aligned} & w_\alpha(t;\epsilon) = \epsilon^2 \frac{\partial^2 \log \tau(t;\epsilon)}{\partial x \partial t^{\alpha,0}}, \\ & h_{\alpha,p}(w(t;\epsilon); w_x(t;\epsilon), \ldots; \epsilon) = \epsilon^2 \frac{\partial^2 \log \tau(t;\epsilon)}{\partial x \, \partial t^{\alpha,p+1}}, \quad p \geq -1. \end{aligned}$$

4. Some examples

We consider the special case of Hodge integrals of a point, which correspond to the one-dimensional Frobenius manifold.

Example 1. Let us assume that the parameters s_k take the following form:

$$s_k = -\frac{B_{2k}}{2k(2k-1)} s^{2k-1}, \quad \text{for} \quad k \geq 1,$$

where s is an arbitrary parameter. Then we have the Chern polynomial

$$e^{\sum s_{2k-1}\mathrm{ch}_{2k-1}(\mathbb{E})} = \prod_{i=1}^{g}(1 + s\,x_i) = 1 + s\,\lambda_1 + \cdots + s^g\lambda_g =: \Lambda_g(s)$$

Here we use standard notations for the Chern classes of Hodge bundle

$$\lambda_i = c_i(\mathbb{E}), \quad i = 1, \ldots, g.$$

The Hodge potential of a point specifies to

$$\mathcal{H} \mapsto \sum_g \epsilon^{2g-2} \sum_{n\geq 0} \sum_{k_1,\ldots,k_n} \frac{t_{k_1}\ldots t_{k_n}}{n!} \int_{\overline{M}_{g,n}} \Lambda_g(s)\psi_1^{k_1}\ldots\psi_n^{k_n}.$$

Here $\psi_j^{k_j} = c_1^{k_j}(\mathcal{L}_j)$. Buryak proved that the function

$$u = w + \sum_{g\geq 1} \frac{(-1)^g}{2^{2g}(2g+1)!}\epsilon^{2g} s^g w_{2g}$$

with $w = \epsilon^2 \frac{\partial^2 \mathcal{H}}{\partial x \partial x}$ satisfies the Intermediate Long Wave (ILW) equation

$$u_{t_1} = u\, u_x + \sum_{g\geq 1} \epsilon^{2g} s^{g-1} \frac{|B_{2g}|}{(2g)!} u_{2g+1}.$$

The Hamiltonian operator of the hierarchy has the explicit expression

$$\tilde{P} = \partial_x + \sum_{g\geq 1} \frac{(2g-1)|B_{2g}|}{(2g)!} s^g \epsilon^{2g} \partial_x^{2g+1}.$$

The generating function of the linear Hodge integrals was also shown by Kazarian (2009) to be the logarithm of the tau-function of a family of solutions to the KP hierarchy.

Example 2. Now let us consider a particular choice of the parameters s_k such that the resulting Hodge hierarchy of a point possesses a bihamiltonian structure. We require that the parameters are given by

$$s_k = (4^k - 1)\frac{B_{2k}}{2k(2k-1)}s^{2k-1}, \quad k \geq 1.$$

Here, as in the above example, s is an arbitrary parameter. Then the Hodge potential is reduced to

$$\mathcal{H} \mapsto \sum_g \epsilon^{2g-2} \sum_{n\geq 0} \sum_{k_1,\dots,k_n} \frac{t_{k_1} \dots t_{k_n}}{n!} \int_{\overline{M}_{g,n}} \Lambda_g(s)\, \Lambda_g(-2\,s)\, \Lambda_g(-2\,s)\, \psi_1^{k_1} \dots \psi_n^{k_n}.$$

Consider the following combination

$$\frac{\partial w}{\partial t} := 2\sum_{k=0}^{\infty}(2s)^k\frac{\partial w}{\partial t_k}$$

of the flows of the Hodge hierarchy. It has the expression

$$\frac{\partial w}{\partial t} = 2e^{2s\,w}w_x + \frac{\epsilon^2}{3}e^{2s\,w}\left(-s^3w_x^3 + s^2w_xw_{xx} + s\,w_{xxx}\right) + \mathcal{O}(\epsilon^4).$$

Making a rescaling

$$w \to \frac{w}{2s},\quad \partial_x \to \frac{1}{\sqrt{s}}\partial_x,\quad \partial_t \to \frac{1}{\sqrt{s}}\partial_t,$$

we obtain the equation

$$w_t = 2e^w w_x + \frac{\epsilon^2}{3}e^w\left(-\frac{1}{4}w_x^3 + \frac{1}{2}w_xw_{xx} + w_{xxx}\right) + \mathcal{O}(\epsilon^4).$$

Now performing a Miura-type transformation

$$u = w + \sum_{k=1}^{\infty} \epsilon^{2k} \frac{3^{2k+2} - 1}{(2k+2)!4^{k+1}} w_{2k},$$

we can check up to the ϵ^{12}-approximations that the above equation is transformed to the *discrete KdV equation*

$$u_t = \frac{1}{\epsilon}\left(e^{u(x+\epsilon)} - e^{u(x-\epsilon)}\right) = 2e^u u_x + \frac{\epsilon^2}{3} e^u \left(u_x^3 + 3u_x u_{xx} + u_{xxx}\right) + \mathcal{O}(\epsilon^4),$$

At the same order of approximation, we find that, apart from the KdV case, this is the only specification of the Hodge hierarchy of a point which possesses a bihamiltonian structure

$$\{u(x), u(y)\}_1 = \frac{\delta(x-y+\epsilon)-\delta(x-y-\epsilon)}{\epsilon},$$

$$\{u(x), u(y)\}_2 = \left[e^{u(x)} + e^{u(y)}\right] \frac{\delta(x-y+\epsilon)-\delta(x-y-\epsilon)}{\epsilon} + \frac{1}{\epsilon}\left[e^{u(x+\epsilon)}\delta(x-y+2\epsilon) - e^{u(y+\epsilon)}\delta(x-y-2\epsilon)\right].$$

Alternative generating functions of the cubic Hodge integrals were constructed by Jian Zhou (2010), he showed that they are tau-functions of a family of solutions to the KP hierarchy.

Example 3. Let us consider a special choice of the parameters $s_k, k \geq 1$ such that the Hamiltonian operator $\tilde{P}$ of the Hodge hierarchy of a point has the form

$$\tilde{P} = \partial_x + \sum_{g \geq 1} \tilde{P}_k(s_1, \ldots, s_g)\partial_x^k,$$

where the coefficients $\tilde{P}_k$ do not depend on w and its x-derivatives. We conjecture that this requirement is equivalent to the following choice of the parameters s_k:

$$s_k = -\frac{B_{2k}}{2k(2k-1)}\left(p^{2k-1} + q^{2k-1} - \left(\frac{pq}{p+q}\right)^{2k-1}\right), \quad k \geq 1.$$

Here p and q are arbitrary complex numbers such that $p + q \neq 0$.

We also conjecture a closed formula for the Hamiltonian operator

$$\tilde{P} = \frac{\frac{p}{2\sqrt{p+q}}\epsilon\partial_x}{\sin\left(\frac{p}{2\sqrt{p+q}}\epsilon\partial_x\right)} \circ \frac{\frac{q}{2\sqrt{p+q}}\epsilon\partial_x}{\sin\left(\frac{q}{2\sqrt{p+q}}\epsilon\partial_x\right)} \circ \frac{\frac{\sqrt{p+q}}{2}\epsilon\partial_x}{\sin\left(\frac{\sqrt{p+q}}{2}\epsilon\partial_x\right)} \circ \partial_x.$$

This two-parameter family of the Hodge hierarchy corresponds to the cubic Hodge potential

$$\mathcal{H} = \sum_g \epsilon^{2g-2} \sum_{n\geq 0} \sum_{k_1,\dots,k_n} \frac{t_{k_1}\dots t_{k_n}}{n!} \int_{\overline{M}_{g,n}} \Lambda_g(p)\,\Lambda_g(q)\,\Lambda_g\left(-\frac{pq}{p+q}\right)\psi_1^{k_1}\dots\psi_n^{k_n}.$$

If we set $a_1 = p$, $a_2 = q$, $a_3 = -\frac{pq}{p+q}$, then they satisfy

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 0,$$

which is exactly the local Calabi–Yau condition that appears in the localization calculation of Gromov–Witten invariants.

5. A universality property of the Hodge hierarchy

Consider deformations of the dispersionless KdV hierarchy

$$\frac{\partial v}{\partial t_q} = \frac{\partial}{\partial x}\frac{\delta H_q}{\delta v(x)} = \frac{v^q}{q!} v_x, \quad q \geq 0.$$

(also called the *Riemann hierarchy*) with

$$H_q = \int \frac{v^{q+2}}{(q+2)!} dx.$$

Conjecture

Any tau-symmetric integrable Hamiltonian deformation of the Riemann hierarchy is equivalent, under a normal Miura-type transformation, to the canonical *tau-symmetric integrable deformation of the form*

$$\frac{\partial w}{\partial t^q} = \frac{\partial}{\partial x}\left(\frac{\delta H_q}{\delta w(x)}\right), \quad q \geq 0$$

which is uniquely determined by the following standard form of the density h_1 of the Hamiltonian H_1:

$$\begin{aligned} h_1 = \frac{w^3}{6} - \frac{\epsilon^2}{24} a_0 w_1^2 + \epsilon^4 a_1 w_2^2 + \epsilon^6 (a_2 w_2^3 + b_1 w_3^2) \\ + \epsilon^8 \left(a_3 w_2^4 + b_2 w_2 w_3^2 + b_3 w_4^2\right) \\ + \epsilon^{10} \left(a_4 w_2^5 + b_4 w_2^2 w_3^2 + b_5 w_2 w_4^2 + b_6 w_5^2\right) + \dots. \end{aligned}$$

In the above conjecture, $w_k = \partial_x^k w$, $a_0, a_i, b_i, i \geq 1$ are certain constants and, starting from ϵ^4, the terms appearing in this standard form are selected by the following two rules:

i) The factor with the highest order derivative in each monomial is nonlinear.

ii) Each of these terms does not contain any w_x factor.

In this standard form, the coefficient of $\epsilon^2\, w_1^2$ is denoted by $-\frac{a_0}{24}$; the coefficient of $\epsilon^{2k} w_2^k$ is denoted by a_{k-1}; other coefficients are denoted by b_1, b_2, ... Moreover, in the case $a_0 = 0$ all coefficients $a_j, b_j, j \geq 1$ must vanish. In the case $a_0 \neq 0$, the coefficients b_j with $j \geq 1$ are uniquely determined by $a_0, a_1, a_2 \dots$.

Conjecture

Any nontrivial tau-symmetric integrable Hamiltonian deformation of the Riemann hierarchy is equivalent, under a normal Miura-type transformation, to the Hodge hierarchy of a point with a certain particular choice of the parameters $s_k, k \geq 1$.

Thanks