# Algebraic entropy

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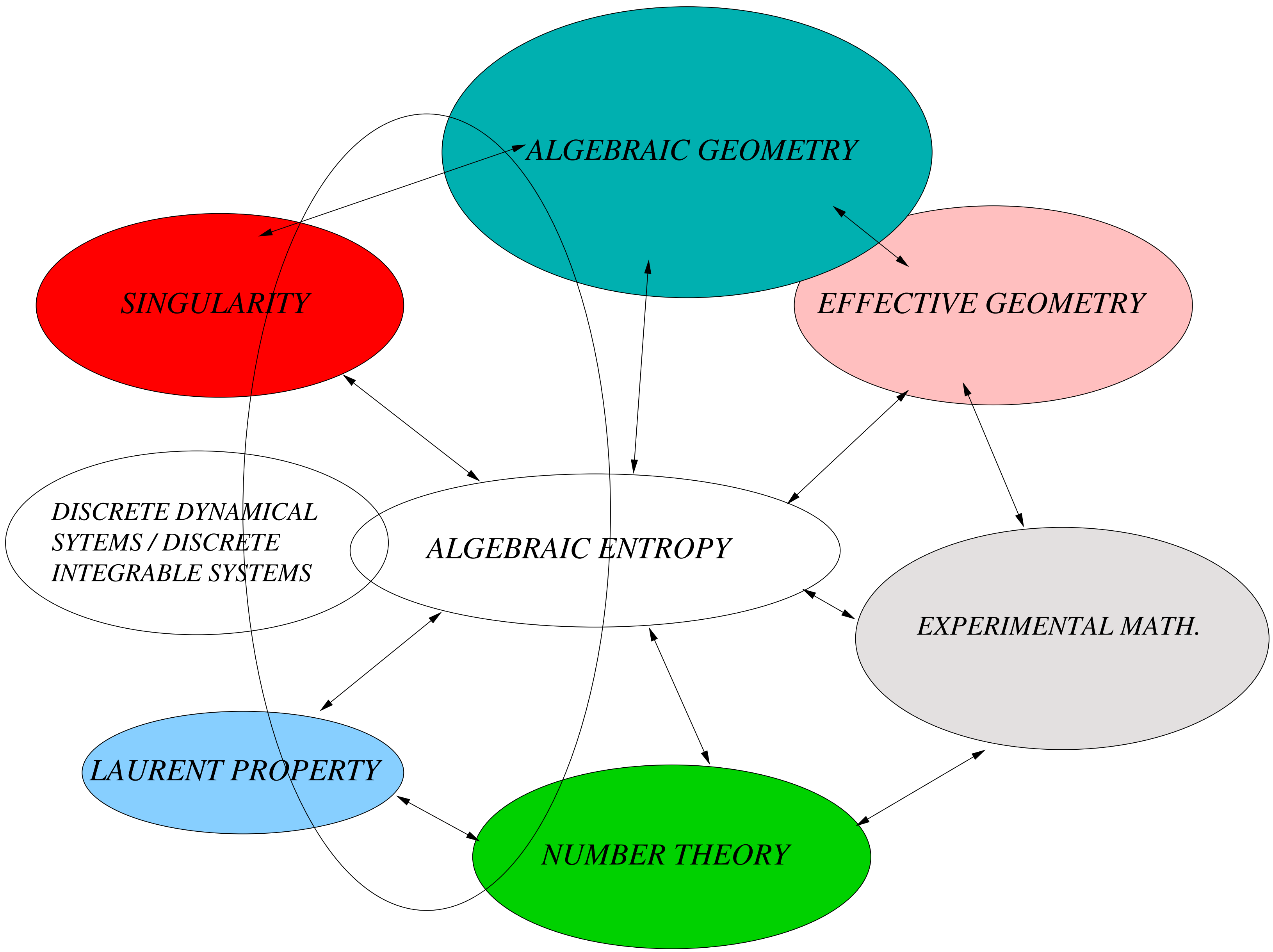
Contributions from many people, see in particular the series of SIDE conferences (Symmetries and Integrability of Difference Equations) http://side-conferences.net/

# Mathematical environment



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# An elementary illustration of the difference integrable / not integrable

Consider the following plane (bi)-rational maps $\varphi_5$ and $\varphi_7$:

$$\varphi_5 : (u,v) \longrightarrow \left( u' = \frac{-1-u-v+u^2+v^2+uv}{u^2-v^2+u-uv}, v' = \frac{-1-u-v+u^2+v^2+uv}{v^2-u^2+v-uv} \right)$$

$$\varphi_7 : (u,v) \longrightarrow \left( u' = \frac{1+2\,u+2\,v-u^2-v^2-3\,uv}{2\,v^2-u-u^2}, v' = \frac{1+2\,u+2\,v-u^2-v^2-3\,uv}{2\,u^2-v-v^2} \right)$$

Both maps are birational maps of infinite order.

The simplest thing is to draw a few generic orbits of respectively $\varphi_5$ and $\varphi_7$.

# Elementary illustration

Although the two maps are constructed in a very similar way (inversions of cyclic matrices) and come from comparable algebraic structures of bialgebras, the behaviour of the iterates is completely different.

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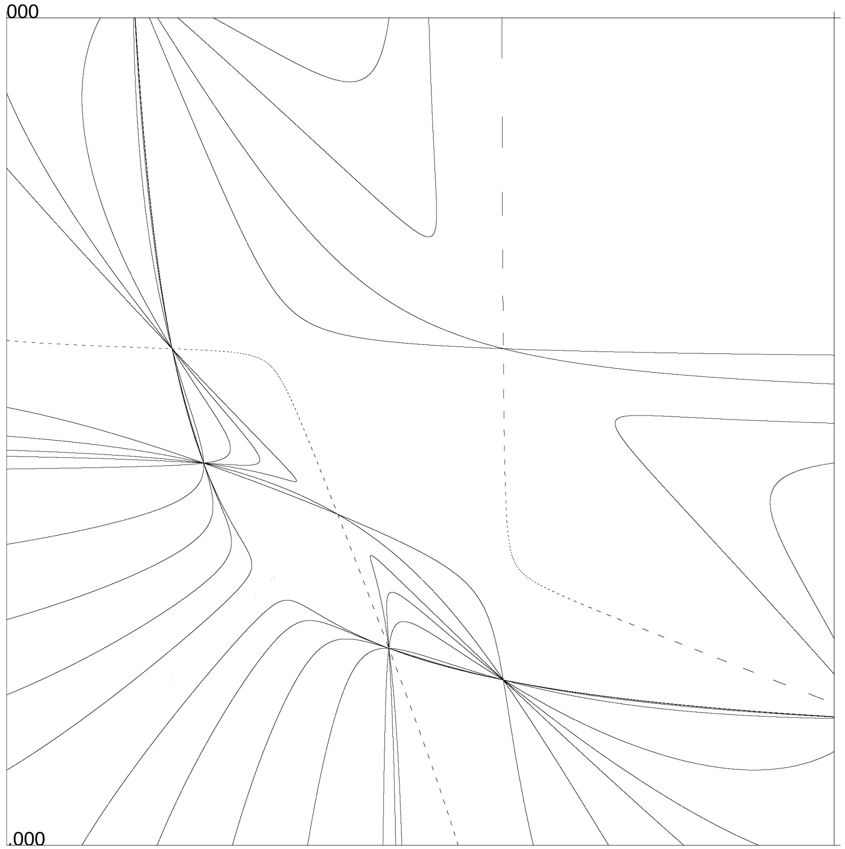


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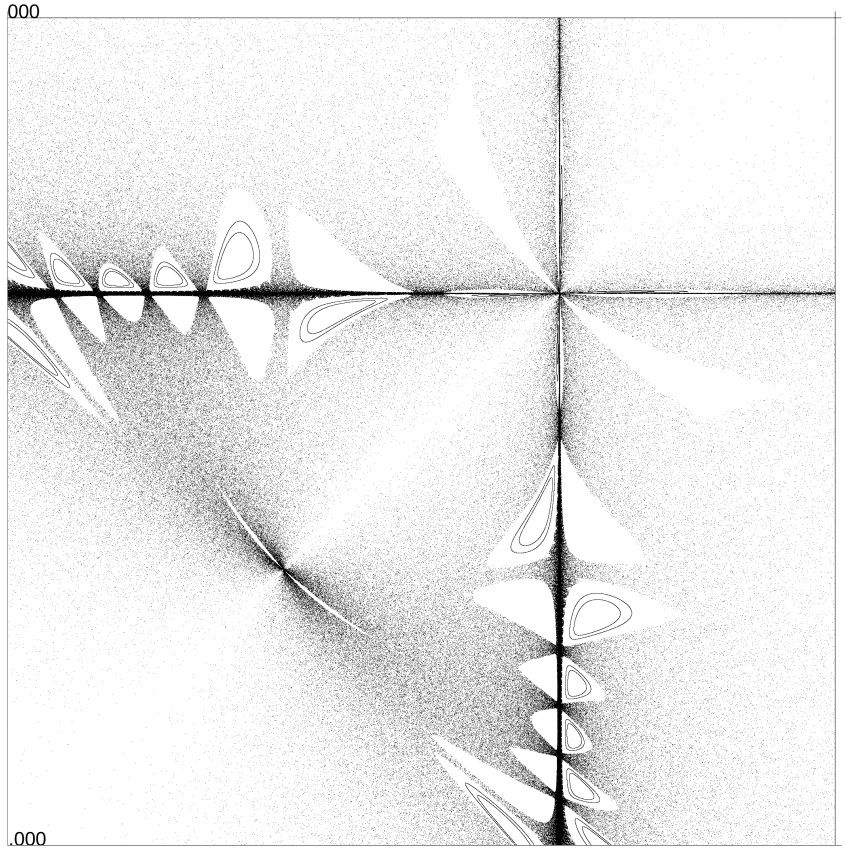

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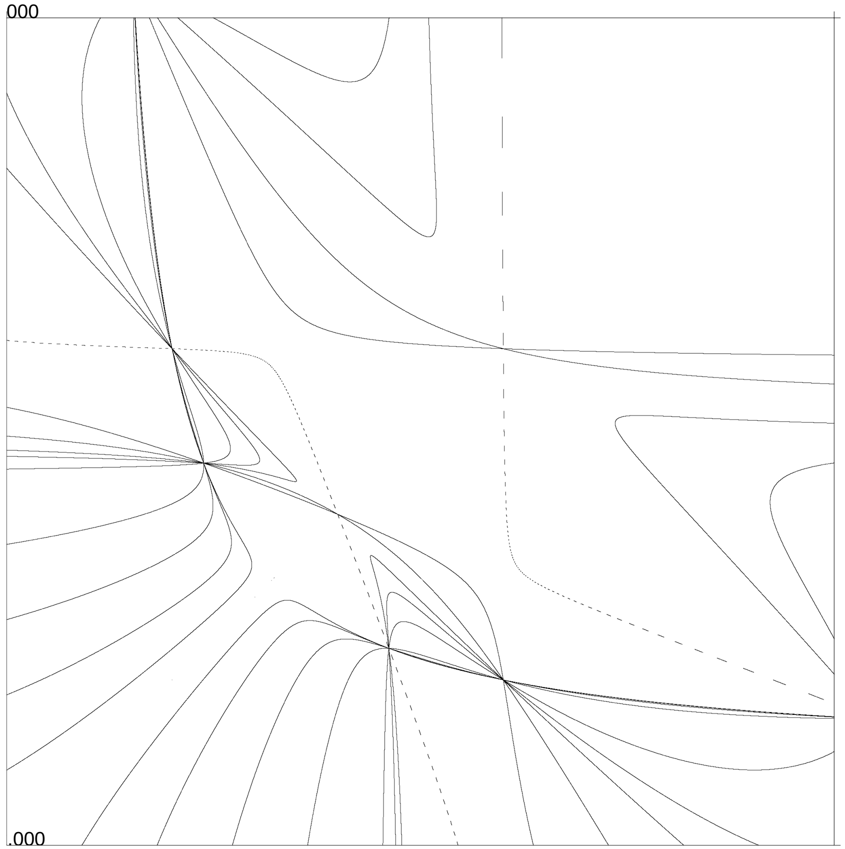


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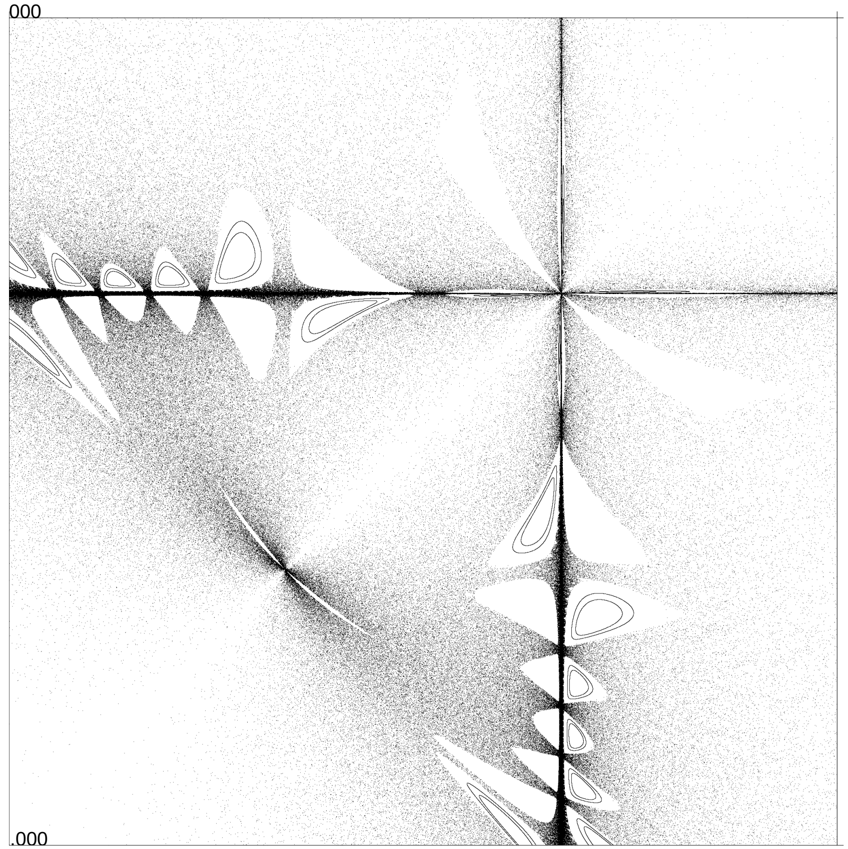


"z7"

We need an instrument to characterise the different behaviours.

# Basics: Birational transformations, singularities

We use complex projective spaces as spaces of initial conditions $\mathcal{I}$, with homogeneous coordinates.

Suppose for simplicity that $\mathcal{I}$ is of dimension $N$, with $N+1$ homogeneous coordinates and call $\varphi$ is the forward map, and $\psi$ the backward map. Then

$$\varphi : [x_0, x_1, \ldots, x_N] \to [x'_0, x'_1, \ldots, x'_N]$$
$$\psi : [y_0, y_1, \ldots, y_N] \to [y'_0, y'_1, \ldots, y'_N]$$

The evolution step will always be given by a birational map, so that both $\varphi$ and $\psi$ are polynomial maps, of degree $d_\varphi$ and $d_\psi$ respectively (usually $d_\varphi = d_\psi > 1$).

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Since $[0, 0, \ldots, 0]$ is forbidden, there are singular points of the transformations: the ones for which $x'_j = 0, j = 0 \ldots n$, (resp. $y'_j = 0, j = 0 \ldots n$). We know that the sets of singular points are algebraic varieties of dimension $\leq N - 2$.

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The idea is to measure the complexity of the evolution by the rate of growth of the sequence of degrees $\{d_n\}$ of the successive iterates (A. Veselov, G. Falqui – CMV).

Define the algebraic entropy (M. Bellon – CMV), characterising its asymptotic behaviour:

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Since $d_{n+m} \leq d_n \cdot d_m$, this limit always exists, and it is canonically associated to the map (invariant by changes of coordinates).

The above inequality is straightforward: composing $\varphi^n$ and $\psi^m$ leads to polynomial expressions of degree $d_n \cdot d_m$ at first. If there is any common factor, they ought to be removed, there is a drop of the overall degree, and we get a strict inequality. One basic question is to evaluate precisely what is the possible drop.

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This is where contact is made with the singularity structure: singularities are causing this drop, as we will see.

NB: The exponential of $\epsilon$ is called dynamical degree. $\hookleftarrow$

Two important objects: the multipliers $\kappa_\varphi$ and $\kappa_\psi$.

Since '$\psi$ is the inverse of $\varphi$' means that the composition $\varphi \cdot \psi$ appears as a multiplication of all coordinates by a common factor, we have the two basic relations

$$\boxed{\psi \cdot \varphi \simeq \kappa_\varphi \cdot id, \qquad \varphi \cdot \psi \simeq \kappa_\psi \cdot id}$$

The two polynomials $\kappa_\varphi$ and $\kappa_\psi$, both of degree $d_\varphi \, d_\psi - 1$, may be decomposable.

$$\boxed{\kappa_\varphi = \prod_{j=1}^{p} K_j^+, \qquad \kappa_\psi = \prod_{j=1}^{q} K_j^-}$$

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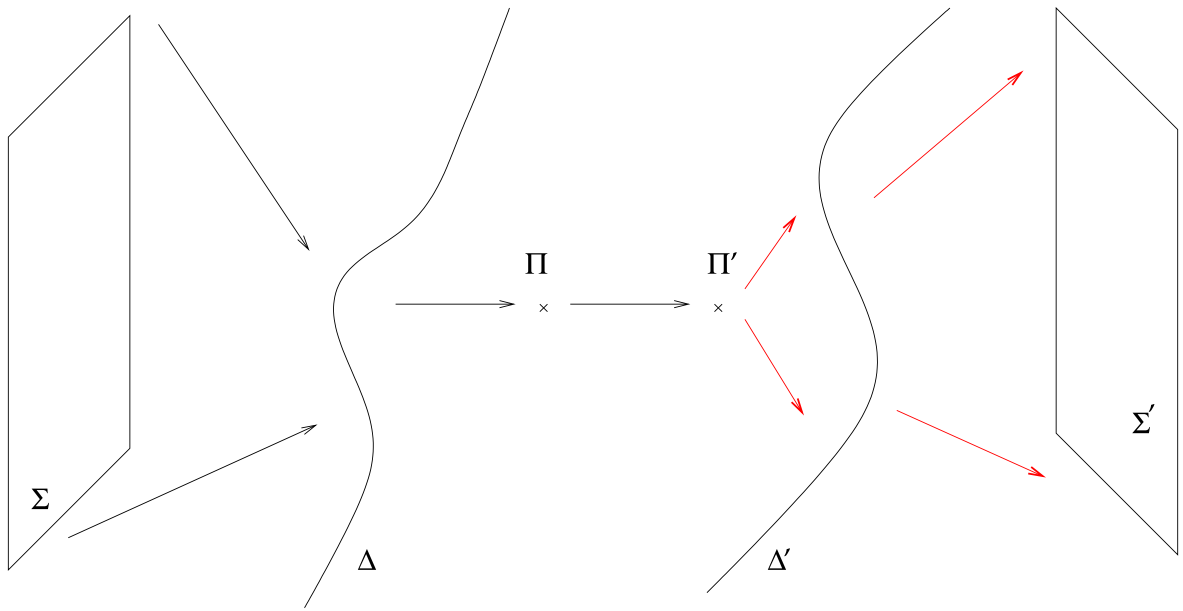
There is a simple relation between the varieties $\kappa_\varphi$ and $\kappa_\psi$ and the singular locus: the varieties of equation $K_j^+ = 0$ are blown down by $\varphi$ and their image is entirely made of singular points of $\psi$. This reflects the fact that one cannot take one step forward and then a step backward when starting from a point on $\kappa_\varphi = 0$. The same applies to $\psi$ mutatis mutandis.

$$\boxed{\psi \cdot \varphi \simeq \kappa_\varphi \cdot id, \qquad \varphi \cdot \psi \simeq \kappa_\psi \cdot id}$$

Let $\Sigma$ be the surface of equation $\kappa_\varphi = 0$ (or some factor of $\kappa_\varphi$ if it is not indecomposable). As soon as the evolution is non-linear, the degree of $\kappa$ is positive, and there is a non empty variety $\Sigma$. In the case of the following pattern




it is easy to see that the equation of $\Sigma$ factors from the fourth iterate of the map: there will be a drop of degree.

$\hookleftarrow$

# More about the multipliers $\kappa$

Suppose $\Sigma$ is an indecomposable variety of codimension $1$ of equation $E_\Sigma = 0$. The pullback by $\varphi$ of the equation of $\Sigma$ gives the equation $E_{\Sigma'}$ of the image $\Sigma'$ of $\Sigma$ by $\psi$. The important point is that this pullback may contain additional factors, which are part of the total transform, and are not part of the proper transform.

$$\boxed{\varphi^*(E_\Sigma) = E_{\Sigma'}\,(K_1^+)^{n_1}\,(K_2^+)^{n_2}\ldots(K_p^+)^{n_p}}$$

The exponents $n_j$ remaining to be determined: they depend on which singular varieties are embedded in the variety $\Sigma$.

We will use this later.

$\hookleftarrow$

# Three ways to calculate the entropy exactly

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1– The first method is heuristic. Evaluate as many terms as possible of the sequence of degrees and analyse the piece of the sequence one obtains. One could of course try to calculate the iterates explicitly, but this is unrealistic. Relying on the simple geometric interpretation of the degree of the iterates, we can make the calculations simpler by evaluating the images of a line (go from multivariate to univariate calculations). This is a particular case of a notion of complexity of maps introduced by Arnold for diffeomorphisms.

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2– The second method is very useful in two dimension (order two recurrences) and has been at the source of the "classification" of discrete Painlevé equations, inspired by the older work of Okamoto, and leading to affine Lie algebra structures. It consists in augmenting the space of initial conditions by a sufficient number of blow-ups, and using the paraphernalia available from the theory of intersection of curves to obtain the entropy as an eigenvalue of the map induced on the Picard group of the variety constructed by this process.

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3– The third method is based on the observation of the factorisation properties of the successive iterates. This leads to define a "derived map" acting on new variables which are the indecomposable blocks appearing in the iterates. The method has the advantage of being applicable to all dimensions, including infinite number of dimensions (e.g. delay-Painlevé, lattice maps). $\hookleftarrow$

## Method 1: Calculate as many as possible of the degrees

Calculating the iterates directly, but this a rapidly growing process, and your favourite formal calculation software will not suffice.

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[1] (p = 541, 1999, 3571, 815837627809793)

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The problem is to extract an information on the asymptotic behaviour of the sequence of degrees from a finite number of terms. As we will see, this is possible.

Going back to the two maps pictured earlier:

- z5: iterates of $\varphi_5$ (the integrable one)
  1, 2, 4, 7, 12, 18, 25, 34, 44, 55, 68, 82, 97, 114, ...
- z7: iterates of $\varphi_7$ (the chaotic one)
  1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, 376, 609, 986, 1596, 2583 ...

The most efficient way is to write down the generating function of the sequence

$$g(s) = \sum_{k=0}^{\infty} d_k \, s^k$$

and try to fit it with a Padé approximant.

For these examples:

$$g_5 = \frac{1 + s^2 + 2\,s^4}{(s^2 + s + 1)\,(1 - s)^3}, \qquad g_7 = \frac{1}{(1 - s)\,(1 - s - s^2)},$$

The remarkable result is that this method works in many cases and the generating function we find is a rational fraction with integer coefficients.

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One should nevertheless keep in mind that the sequence of degrees is not canonical: it is not invariant by (birational) changes of coordinates. The same applies to the generating function $g(s)$.

The modulus of the smallest modulus of the poles of $g(s)$ is canonical. Moreover, all the poles lie on the unit circle (vanishing entropy), the order of $s = 1$ is also canonical: it gives the polynomial rate of growth of the degrees. The main experimental observation is that the coefficients $\delta_m$ are integer and $\delta_q = 1$. As a consequence entropy is the log of an algebraic integer.

For most maps, the generating function is rational. It is unfortunately not always true.

Suppose we take the monomial map (J. Propp):

$$[x, y, z, t] \longrightarrow [yt, zt, x^2, xt]$$

we get the list:

1, 2, 3, 4, 6, 9, 12, 17, 25, 33, 45, 65, 85, 112, 159, 215, 262, 365, 524, 627, 833, 1198, 1404, 1760, 2537, 3415, 3937, 5507, 8481, 11455, 16881, 25281, 33681, 47426, 69571, 91716, 124470, 179369, 234268, 307249, 435129, 593006,...

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It is not possible to fit the sequence with a rational fraction. It is however possible to show that the entropy the log of an eigenvalue of the matrix of exponents defining the map (and thus of an algebraic integer). It is the log of the largest modulus of the roots of $s^3 + s + s^2 - 1$. One peculiarity is that the entropy of the forward map is different from the one of the backward map.

We still have the conjecture, with no counterexample yet:

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# A case study: an illustration of method 1

Consider the recurrence

$$x_{n+1} \cdot x_{n-1} = x_n + \frac{1}{x_n} + a$$

This may be written , with homogeneous coordinates $[x, y, z]$ as a map $\varphi : \; P_2 \to P_2$:

$$x \longrightarrow x^2 z + z^3 + a \, x z^2, \quad y \longrightarrow y x^2, \qquad z \longrightarrow xyz$$

$\varphi$ is the product of two involutions. It is reversible and preserves the measure

$$\omega = \frac{du \wedge dv}{uv}, \quad u = \frac{x}{z}, \; v = \frac{y}{z}$$

Calculating the first degrees of the iterates, one finds

$$1, \; 3, \; 5, \; 12, \; 25, \; 53, \; 112, \; 233, \; 487, \; 1013, \; 2111, \; 4393, \; 9144, \; 19029, \; 39601, \; 82412, \; 171501, \; 356899, \; 742713, \; 1545603, 3216429, \; 6693452, \; 13929201 \; \ldots$$

The sequence clearly has an exponential growth

To show the experimental side of the analysis with Padé approximants: starting from the sequence, one tries to fit the generating function with rational fractions $M/N$, $M$ and $N$ being polynomials of degree $m$ and $n$ respectively, with $m + n =$ length of the sequence - 1.

# The hit or miss appears in a striking way, when the Padé approximant happens to be exact!

$[5, 17], \left(35192908 s^5 + 211096 s^4 - 72235952 s^3 - 13265329 s^2 + 29714412 s + 10217575\right) \Big/ \left(68710233 s^{17} + 36527378 s^{16} + 24466071 s^{15} - 6804768 s^{14} + 32587091 s^{13} + 3011012 s^{12} - 25383840 s^{11} - 66013720 s^{10} - 28283499 s^9 - 6821824 s^8 + 3547362 s^7 - 18160864 s^6 + 42248064 s^5 + 80344278 s^4 - 5540492 s^3 - 61538265 s^2 - 938313 s + 10217575\right)$

$[6, 16], -\left(68710233 s^6 - 36527378 s^5 - 7262483 s^4 + 30212836 s^3 - 39053326 s^2 + 18297978 s + 21712771\right) \Big/ \left(4324360 s^{16} - 12795924 s^{15} - 11794980 s^{14} - 14218484 s^{13} - 1581992 s^{12} + 8303256 s^{11} + 8502856 s^{10} + 14553696 s^9 + 12949384 s^8 + 26546031 s^7 - 53831783 s^6 - 27007968 s^5 + 27966219 s^4 - 25149787 s^3 + 7096176 s^2 + 46840335 s - 21712771\right)$

$$[7, 15], \frac{1081090 s^7 + 3198981 s^6 - 2085576 s^5 + 76709 s^4 + 973352 s^3 - 1870122 s^2 + 1376016 s + 1202627}{912588 s^{15} + 864780 s^{14} + 838068 s^{13} - 40776 s^{12} - 614952 s^{11} - 761352 s^{10} - 1031072 s^9 - 1135638 s^8 - 607457 s^7 + 3363511 s^6 + 1085146 s^5 - 1060693 s^4 + 1264139 s^3 - 1187662 s^2 - 2231865 s + 1202627}$$

$$[8, 14], -\frac{76049 s^8 - 72065 s^7 - 5814 s^6 + 36283 s^5 - 60423 s^4 + 49472 s^3 + 30975 s^2 - 21911 s - 8167}{3600 s^{14} + 22104 s^{13} + 8568 s^{12} + 7824 s^{11} - 10472 s^{10} + 12587 s^9 - 77120 s^8 - 9462 s^7 + 48858 s^6 - 28112 s^5 + 34746 s^4 + 57594 s^3 - 64040 s^2 - 2590 s + 8167}$$

$$[9, 13], \frac{50 s^9 - 307 s^8 + 245 s^7 + 52 s^6 - 169 s^5 + 239 s^4 - 146 s^3 - 125 s^2 + 73 s + 31}{72 s^{13} + 24 s^{12} + 32 s^{11} - 46 s^{10} + 91 s^9 - 260 s^8 - 66 s^7 + 194 s^6 - 116 s^5 + 78 s^4 + 242 s^3 - 220 s^2 - 20 s + 31}$$

HIT

$$[10, 12], -\frac{3 s^{10} - s^9 - s^8 + 2 s^7 + s^6 - s^5 + 2 s^4 + s^3 - 2 s^2 + s + 1}{(s-1)\left(s^4 - s^3 - 2 s^2 - s + 1\right)\left(s^6 + s^3 + 1\right)}$$

CHECK

$$[11, 11], -\frac{3 s^{10} - s^9 - s^8 + 2 s^7 + s^6 - s^5 + 2 s^4 + s^3 - 2 s^2 + s + 1}{(s-1)\left(s^4 - s^3 - 2 s^2 - s + 1\right)\left(s^6 + s^3 + 1\right)}$$

$$[12, 10], \frac{72 s^{12} + 144 s^{11} + 360 s^{10} - 138 s^9 - 113 s^8 + 407 s^7 + 156 s^6 - 67 s^5 + 381 s^4 + 10 s^3 - 255 s^2 + 227 s + 157}{286 s^{10} + 65 s^9 - 332 s^8 + 42 s^7 + 246 s^6 - 284 s^5 + 114 s^4 + 270 s^3 - 308 s^2 - 244 s + 157}$$

$$[13, 9], -\frac{528 s^{13} - 120 s^{12} + 288 s^{11} - 600 s^{10} - 672 s^9 + 2733 s^8 - 1309 s^7 - 942 s^6 + 1791 s^5 - 1955 s^4 + 64 s^3 + 1635 s^2 - 459 s - 467}{1399 s^9 - 1536 s^8 - 1118 s^7 + 1906 s^6 - 1280 s^5 - 118 s^4 + 2474 s^3 - 1144 s^2 - 942 s + 467}$$

$$[14, 8], \frac{72748 s^{14} + 79872 s^{13} + 66296 s^{12} + 66968 s^{11} + 40488 s^{10} + 160848 s^9 + 242496 s^8 - 94490 s^7 + 179907 s^6 + 12496 s^5 - 91357 s^4 + 243696 s^3 + 63426 s^2 + 4840 s + 20545}{210734 s^8 - 30171 s^7 - 5855 s^6 + 58562 s^5 - 242491 s^4 - 112127 s^3 + 131086 s^2 - 56795 s + 20545}$$

$$[15, 7], \frac{842936 s^{15} + 120684 s^{14} + 806128 s^{13} - 166904 s^{12} + 230952 s^{11} + 368616 s^{10} - 142512 s^9 + 992576 s^8 + 845216 s^7 - 3489637 s^6 + 1929178 s^5 + 422167 s^4 - 1849340 s^3 + 2208110 s^2 - 616618 s - 1042379}{2217271 s^7 - 2583363 s^6 - 1223532 s^5 + 1589367 s^4 - 1558731 s^3 - 111552 s^2 + 2510519 s - 1042379}$$

$[16, 6], -\frac{1}{10000744 s^6 + 729089 s^5 - 6883561 s^4 + 1380223 s^3 - 2675862 s^2 - 5578567 s + 3370651}\left(2217271 s^{16} + 2583363 s^{15} + 431467 s^{14} - 482620 s^{13} - 1675988 s^{12} - 262988 s^{11} - 504540 s^{10} - 2223884 s^9 - 1819964 s^8 - 4292013 s^7 - 9285193 s^6 + 5950495 s^5 - 1201269 s^4 - 5907614 s^3 + 2558308 s^2 - 4533386 s - 3370651\right)$

$[17, 5], \frac{1}{10227671 s^5 - 12329151 s^4 - 11933935 s^3 + 7950334 s^2 + 4719743 s - 2384283}\left(10000744 s^{17} - 729089 s^{16} + 4669131 s^{15} - 2130933 s^{14} + 11535076 s^{13} + 4776972 s^{12} + 3891220 s^{11} + 285700 s^{10} + 1310036 s^9 + 10996428 s^8 - 718853 s^7 + 7695071 s^6 + 601127 s^5 - 11349445 s^4 + 6904386 s^3 + 10188148 s^2 - 2433106 s - 2384283\right)$

$[18, 4], -\frac{1}{21124132 s^4 + 8915037 s^3 - 18019631 s^2 - 9566724 s + 6642520}\left(10227671 s^{18} + 12329151 s^{17} + 6140918 s^{16} + 6626979 s^{15} + 9462147 s^{14} + 19461336 s^{13} + 8471928 s^{12} + 5104432 s^{11} + 1185584 s^{10} + 10632711 s^9 + 11737863 s^8 + 2440862 s^7 + 963803 s^6 - 4597469 s^5 - 9033400 s^4 + 13267236 s^3 + 13507203 s^2 - 10360836 s - 6642520\right)$

We find (or rather guess) that the generating function for the sequence of degrees is

$$g(s) = \frac{1 + s - 2\,s^2 + s^3 + 2\,s^4 - s^5 + s^6 + 2\,s^7 - s^8 - s^9 + 3\,s^{10}}{(s^4 - s^3 - 2\,s^2 - s + 1)\,(s^6 + s^3 + 1)\,(1 - s)}$$

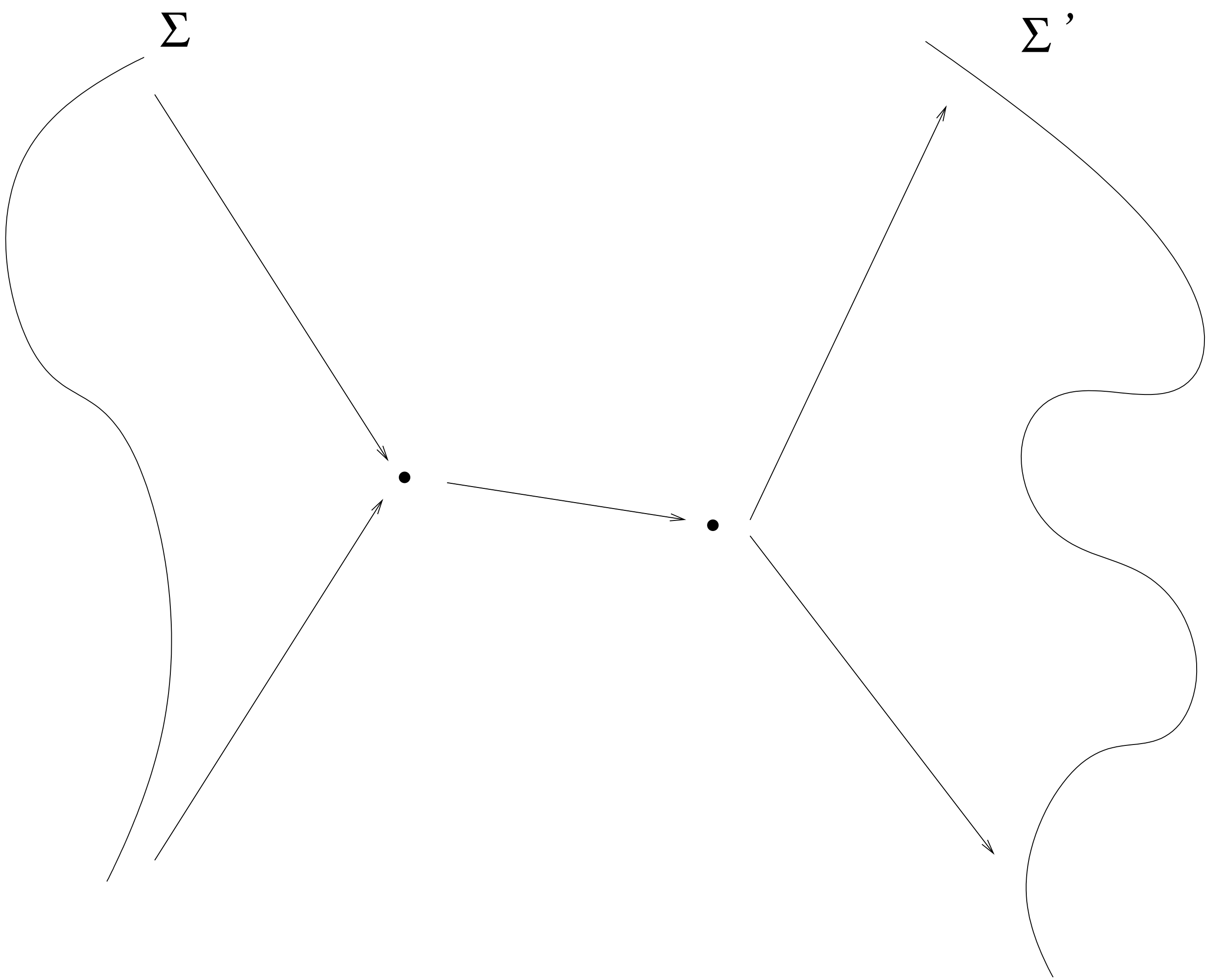
meaning that the entropy would be $\simeq log(2.0810)$, definitely non vanishing.

We need a proof that this is the correct value.

# Method 2 – Augment the space of initial conditions (blow-ups)

In the two dimensional case, if the map verifies ‘singularity confinement’, it is possible to construct, by the action of a finite number of blow-ups of points, a rational variety where the map is a diffeomorphism. Moreover we can use the theory of intersection of curves (and the non degenerate scalar product on the Picard group (intersection number)).

# Method 2 – Augment the space of initial conditions (blow-ups)




Show a blow-up

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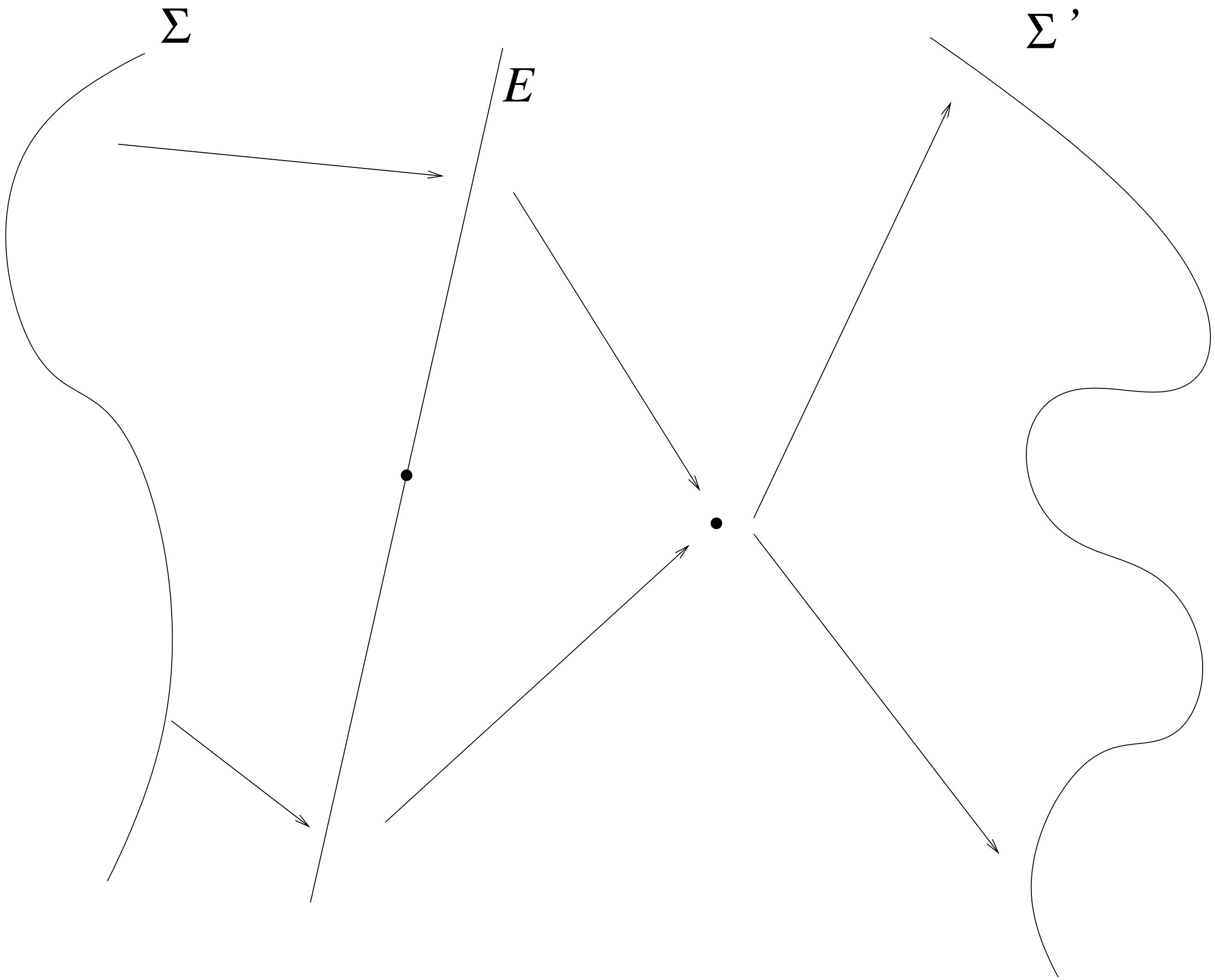


Show a blow-up

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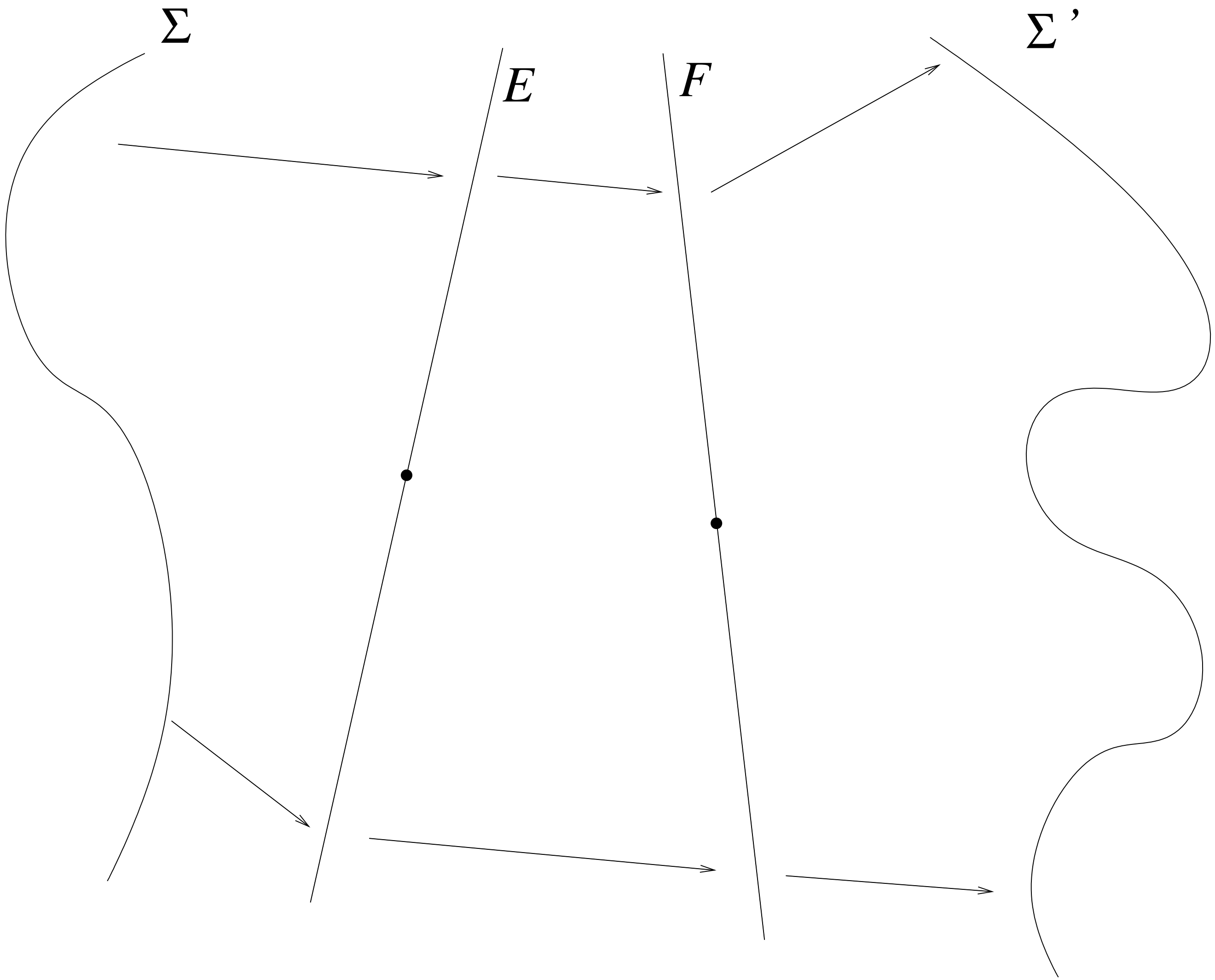


Show a blow-up

# Method 2 – Augment the space of initial conditions (blow-ups)

The linear map induced on the Picard group gives the algebraic entropy as the log of the largest eigenvalue of its characteristic polynomial.

This, in passing demonstrate for maps in two dimensions the fundamental conjecture that the entropy is the log of an algebraic integer!

Moreover the fact that the induced map is an isometry for the metric given by the intersection numbers imposes additional constraints on the roots of the characteristic polynomial.

This approach has lead to a characterisation of integrable maps in two dimensions, via the link to affine Lie algebras, in the autonomous (QRT) as well as the non-autonomous case (Sakai).

# Back to the case study

In order to prove that the value of the entropy found for the case study is correct, one may use the singularity structure of the map $\varphi$ since it verifies "singularity confinement".

In that case one needs to blow up 18 points, some coinciding, getting a rational variety $V$ where the map is completely desingularised.

The characteristic polynomial of the map induced by $\varphi$ on the Picard group of $V$ is:

$$(1+s)\left(1-s+s^2\right)\left(s^4-s^3-2\,s^2-s+1\right)\left(s^6+s^3+1\right)(1-s)^2\left(1+s+s^2\right)^2.$$

This confirms the rational fit for the generating function of the sequence of degrees, found earlier:

$$g(s)=\frac{1+s-2\,s^2+s^3+2\,s^4-s^5+s^6+2\,s^7-s^8-s^9+3\,s^{10}}{(s^4-s^3-2\,s^2-s+1)\,(s^6+s^3+1)\,(1-s)}$$

Recall that the latter was found from the calculated 22 first terms of the sequence of degrees! Having calculated 23 terms was already giving a check, but not a proof.

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Recall that the latter was found from the calculated 22 first terms of the sequence of degrees! Having calculated 23 terms was already giving a check, but not a proof.

The growth $\lambda$ of the degree is given by the largest root (actually also the inverse of the smallest root) of $s^4-s^3-2\,s^2-s+1$ that is to say $\lambda \simeq 2.0810$. This indicates[2] non-integrability.

[2]Actually it proves the non-existence of an algebraic invariant

# Method 3 - Stabilisation of the iterates, derived map, derived recurrence

We have observed a stabilisation of the form of the iterates of birational maps. The idea is to look at the factorisation properties of these iterates, defining indecomposable “blocks”, which can be taken as new variables.

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Using these variables yields recurrences which have remarkable polynomial factorisation properties. The latter are an extension of the Laurent property and they provide us with interesting results on the algebraic entropy. We may call them derived maps (or recurrences).

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Consider the recurrence, a prototype of confining chaotic map (J. Hietarinta –CMV)

$$u_{n+1} + u_{n-1} = u_n + \alpha/u_n^2$$

Here

$$\begin{aligned}
\varphi : [x, y, z] &\longrightarrow [x^3 + \alpha z^3 - x^2 y, x^3, x^2 z] \\
\psi : [x, y, z] &\longrightarrow [y^3, -xy^2 + y^3 + \alpha z^3, y^2 z] \\
\kappa_\varphi = x^3, & \qquad\qquad \kappa_\psi = y^3
\end{aligned}$$

This map has been shown to have positive entropy originally by method 1, then by the construction of a rational surface over $P_2$ where the singularities are resolved (method 2, cf. Takenawa).

The successive iterates have the following form

$$
\begin{aligned}
p_0 &= [x, y, z] \\
p_1 &= [A_1, x^3, x^2 z] \\
p_2 &= [A_2, A_1^3, x^2 z A_1^2] \\
p_3 &= [x^3 A_3, A_2^3, x^2 z A_1^2 A_2^2] \\
p_4 &= [A_1^3 A_4, x A_3^3, z A_1^2 A_2^2 A_3^2] \\
p_5 &= [A_2^3 A_5, A_1 A_4^3, z A_2^2 A_3^2 A_4^2] \\
\ldots
\end{aligned}
$$

The form of the iterates stabilises from $k = 4$ to:

$$p_k = [A_{k-3}^3\, A_k, A_{k-4}\, A_{k-1}^3, z\, A_{k-3}^2\, A_{k-2}^2\, A_{k-1}^2]$$

and the recurrence relation between the blocks $A$ becomes

$$\boxed{A_k^3\, A_{k-3}^3 + \alpha\, z^3\, A_{k-1}^6\, A_{k-2}^6 - A_{k-1}^3\, A_{k-4}\, A_k^2 = A_{k-3}^2\, A_{k-2}^3\, \mathbf{A_{k+1}}}$$

extending over a string of length $6$: its order (5) is not the same as the initial one (2). This is the same equation as the one A. Hone obtained from the singularity structure of the map.

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extending over a string of length $6$: its order (5) is not the same as the initial one (2). This is the same equation as the one A. Hone obtained from the singularity structure of the map.

We are talking of a change of description, not the (birational) change of coordinates we are used to in this game.

Proving the validity of the derived recurrence is done by recursion.

The key ingredient is that the pull-back of $A_k$ by $\varphi$ is necessarily of the form

$$\varphi^*(A_k) = x^{\nu(k)}\, A_{k+1}$$

It is then easy to determine the sequence $\nu(k)$, and the stability of the form of the iterates will basically come from the fact that the factors $x^{\nu(k)}$ all get removed after the fourth iterate.

Another way of stating this is to say each $A_{k+1}$ is the proper transform of $A_k$.

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The derived recurrence verifies two properties:

– It has the Laurent property: the iterates are Laurent polynomials in the initial variables.

– If the initial conditions are some explicit blocks obtained from the original recurrences (i.e. specific polynomials in $x, y, z$), then the further iterates are polynomials in the same variables.

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$\hookleftarrow$

## Going to infinite dimension: an integrable differential-difference equation

Consider the following equation:

$$a\,u(t) - b\,\partial_t u(t) = u(t)\,\left(u(t+1) - u(t-1)\right)$$

where $\partial_t u$ means time derivative. One may rather consider the differential difference equation, or recurrence of order two defined on functional space:

$$a\,u_n(t) - b\,\partial_t u_n(t) = u_n(t)\,\left(u_{n+1}(t) - u_{n-1}(t)\right)$$

This equation for $a = 0$, aka Volterra chain, appears in the works of Manakov, Kaup-Paine (Born-Green-Yvon). It was obtained by reduction of the Kac-van Moerbeke/discrete KdV in Quispel-Capel-Sahadevan. This type of equations appears in Joshi, Joshi-Spicer, and Grammaticos-Ramani-Moreira, see also Halburd-Korhonen for a "Nevanlinna approach".

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The (vanishing entropy) maps $\varphi$ and $\psi$ associated to the equation are:

$$\begin{aligned}
\varphi : [x, y, z] &\longrightarrow [a\, xz - b\,(x'z - xz') + xy,\ x^2,\ xz] \\
\psi : [x, y, z] &\longrightarrow [y^2,\ -a\, yz + b\,(y'z - yz') + xy,\ yz]
\end{aligned}$$

were prime ($'$) means derivative.

For this map $\qquad \kappa_\varphi([x, y, z]) = x^3 \qquad$ and $\qquad \kappa_\psi([x, y, z]) = y^3,$

so that the only factors which could appear when pulling back by $\varphi$ (resp. $\psi$) any differential polynomial in $\{x, y, z\}$ are powers of $x$ (resp. $y$).

We are working with an infinite dimensional space of initial conditions. The three homogeneous coordinates $[x(t), y(t), z(t)]$ of the points in this space should be understood as infinite sequences, say

$$x(t) = \begin{pmatrix} \cdot \\ X_3 \\ X_2 \\ X_1 \\ X_0 \end{pmatrix}, \quad y(t) = \begin{pmatrix} \cdot \\ Y_3 \\ Y_2 \\ Y_1 \\ Y_0 \end{pmatrix}, \quad z(t) = \begin{pmatrix} \cdot \\ Z_3 \\ Z_2 \\ Z_1 \\ X_0 \end{pmatrix},$$

with

$$X_n = \partial_t^n(x(t)), \quad Y_n = \partial_t^n(y(t)), \quad Z_n = \partial_t^n(z(t)),$$

so that for example

$$\begin{aligned}
\varphi : [x, y, z] \;&\to\; [a\, X_0 Z_0 + (X_0 Z_1 - X_1\, Z_0)\, c + X_0 Y_0, {X_0}^2, Z_0 X_0] \\
\varphi^2 : [x, y, z] \;&\to\; [a^2 {Z_0}^2 {X_0}^2 + a\, c\, Z_0 X_0\, (Z_1\, X_0 - Z_0 X_1) + c\, {X_0}^2\, (Z_1\, X_0 - Z_0 X_1 + Y_0 Z_1 - Z_0 Y_1) \\
&\quad + a\, Z_0 {X_0}^2\, (Y_0 + X_0) + c^2 \left({Z_1}^2 {X_0}^2 - Z_0 {X_0}^2 Z_2 + {Z_0}^2 X\, X_2 - {Z_0}^2 {X_1}^2\right) + Y_0 {X_0}^3, \\
&\quad a^2 Z_0^2 {X_0}^2 + 2\, a\, c\, Z_0 X_0\, (Z_1\, X_0 - Z_0 X_1) + 2\, a\, Y_0 Z_0 {X_0}^2 + c^2\, (-Z_1\, X_0 + Z_0 X_1)^2 \\
&\quad + 2\, c\, X_0 Y_0\, (Z_1\, X_0 - Z_0 X_1) + {X_0}^2 {Y_0}^2, \\
&\quad a\, Z^2 {X_0}^2 + c\, (Z_1\, X_0 - Z_0 X_1)\, Z_0 X_0 + Y_0 Z_0 {X_0}^2] \\
&\cdots
\end{aligned}$$

The surface $\{X = 0\}$ is blown down to a variety $U$ of smaller dimension (codimension $\geq 3$)

$$\varphi : \quad \{X_0 = 0\} \to U = \{Y_0 = 0,\ Y_1 = 0,\ Z_0 = 0\,,\ \ldots\}$$

but there are singular varieties of $\psi$ of codimension $2$:

$$V = \{Y_0 = 0,\ Z_0 = 0\} \text{ and } W = \{Y_0 = 0,\ Y_1 = 0\}.$$

For instance the family of surfaces

$$\Sigma = \alpha\, y(t)^2 + \beta(\partial_t\, y(t))^2 + \gamma\, z(t)^2 \simeq \alpha\, Y_0^2 + \beta Y_1^2 + \gamma\, Z_0^2 = 0$$

contains the image $U$ of $\{X_0 = 0\}$ by $\varphi$, but not $V$ or $W$, and

$$\begin{aligned}
&\varphi^*(\Sigma) = x(t)^2\, \Sigma' \\
&\Sigma' = \alpha\, x(t)^2 + 4\,\beta\,(\partial_t\, x(t))^2 + \gamma\, z(t)^2
\end{aligned}$$

One can show that the surface $\{X_0 = 0\}$ is sent after four steps into the surface $\{Y_0 = 0\}$.

We will not try to analyse the singularity structure further, but rather look at the factorisation properties of the iterates, starting from a generic point $p_0$.

$$
\begin{aligned}
p_0 &= [A_0, B_0, C_0] \\
p_1 &= [\ A_1,\ A_0^2,\ A_0C_0\ ] \\
p_2 &= [\ A_2,\ A_1^2,\ A_0A_1C_0\ ] \\
p_3 &= [\ A_0^2A_3,\ A_2^2,\ A_0A_1A_2C_0\ ] \\
p_4 &= [\ A_1^2A_4,\ A_0A_3^2,\ A_1A_2A_3C_0\ ] \\
&\dots \\
p_k &= [\ A_{k-3}^2\ A_k,\ A_{k-4}\ A_{k-1}^2,\ A_{k-3}\ A_{k-2}A_{k-1}\ C_0\ ]
\end{aligned}
$$

$$\boxed{p_k = [\, A_{k-3}^2\, A_k,\; A_{k-4}\, A_{k-1}^2,\; A_{k-3}\, A_{k-2} A_{k-1}\, C_0 \,]}$$

The coordinates of the iterates split into indecomposable blocks $A_k$.

One may calculate $A_k$ in terms of the previous ones: the blocks verify a “derived” recurrence.

$$\boxed{a\, C_0\, A_k\, A_{k+1}\, A_{k+2}\, A_{k+3} + A_{k-1}\, A_{k+3}\, A_{k+2}^2 + c\, A_k^2 A_{k+3}^2\, \partial_t \left( \frac{C_0\, A_{k+1}\, A_{k+2}}{A_k\, A_{k+3}} \right) = A_k\, A_{k+1}^2\, \mathbf{A_{k+4}}}$$

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Remark 1: This relation extends over a string of length $6$. This means that we completely changed the description of the map. This is not a mere change of coordinates.

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Remark 1: This relation extends over a string of length $6$. This means that we completely changed the description of the map. This is not a mere change of coordinates.

Remark 2: Although expressible as a ratio of differential polynomials in the $5$ previous $A_k$'s, $A_{k+4}$ is a differential polynomial in the initial conditions. There is however no formula giving $A_{k+4}$ as a differential polynomial in the $5$ previous $A_k$'s.

How to prove the previous relation?

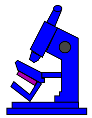
## Sketch of the proof: pull-back and watch for the effect of singularities

We know from the value of $\kappa_\varphi$ that

$$\varphi^*(A_k) = A_0^{\nu(k)}\, A_{k+1}.$$

The first values of the exponent $\nu$ are obtained by an explicit calculation, as well as the degrees $\delta(k)$ of $A_k$.

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\nu$ | 0 | 0 | 2 | 3 | 6 | 8 |
| $\delta$ | 1 | 2 | 4 | 6 | 9 | 12 |



One key point of the proof is that

$$\varphi^*\left(\frac{C_0\, A_{k+1}\, A_{k+2}}{A_k\, A_{k+3}}\right) = \frac{C_0\, A_{k+2}\, A_{k+3}}{A_{k+1}\, A_{k+4}},$$

so that the pullback of the derivative term appearing in the above recurrence does not contain $A_0$ factors. Another important fact is that the surface $X_0 = 0$ never comes back to itself after the fourth iterate.

The recurrence relation is verified for k=1. Operating by pull-back yields the next level of the recurrence, since all $A_0$ factors disappear, as well as the next values of $\nu$ and $\delta$.

We have the following relations on $\delta(k)$ and $\nu(k)$.

$$
\begin{aligned}
1+\delta(k)+\delta(k+1)+\delta(k+2)+\delta(k+3) \;&=\; \delta(k-1)+2\,\delta(k+2)+\delta(k+3) \\
&=\; \delta(k)+2\,\delta(k+1)+\delta(k+4).
\end{aligned}
$$

The two sequences $\delta$ and $\nu$ have different initial conditions and are calculable:

$$
\begin{aligned}
\nu(k) \;&=\; 3/8\left((-1)^k-1\right)+k/2+k^2/4 \\
\delta(k) \;&=\; 1/8\left((-1)^k+7\right)+k+k^2/4
\end{aligned}
$$

One then easily gets the the sequence of degrees of $p_n$

$$
\begin{aligned}
\delta(p_n) \;&=\; \delta(n)+2\,\delta(n-3)=2\,\delta(n-1)+\delta(n-4) \\
&=\; 1+\delta(n-3)+\delta(n-2)+\delta(n-1) \\
&=\; \frac{1}{8}\left(6\,n^2+9-(-1)^n\right),
\end{aligned}
$$

ensuring the vanishing of the entropy.

We have the following relations on $\delta(k)$ and $\nu(k)$.

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The two sequences $\delta$ and $\nu$ have different initial conditions and are calculable:

$$\begin{aligned} \nu(k) &= 3/8\left((-1)^k-1\right)+k/2+k^2/4 \\ \delta(k) &= 1/8\left((-1)^k+7\right)+k+k^2/4 \end{aligned}$$

One then easily gets the the sequence of degrees of $p_n$

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What more have we got?

Going back to the non-homogeneous form of the stable form of the iterates

$$p_k \;=\; [\; A_{k-3}^2\, A_k,\; A_{k-4}\, A_{k-1}^2,\; A_{k-3}\, A_{k-2} A_{k-1}\, C_0 \;]$$

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Known solutions of the equation (for $a = 0$) are indeed of this form, with $\tau$ an entire function !

$$\tau(t) \;=\; t + \alpha, \quad C_0 = -\frac{2}{c} \text{ a rational limit of the soliton solution}$$

$$\tau(t) \;=\; \cosh(\frac{\kappa\, t + \delta}{2}), \quad C_0 = -\frac{2\,\sinh(\kappa)}{c\,\kappa}.$$

We guess that the above form persists when $a \neq 0$.

This leads to the idea that the $A_k$ are the $\tau$ functions of the model, taking us back to original idea of Painlevé, with a link between Hirota form and the notion of proper transforms.

$\hookleftarrow$ Concl. $\hookrightarrow$

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We can play with what makes the degree drop: the singularity structure.

We can also play with the dimension of the space (order of the relation for recurrences) (see later).

$\hookleftarrow$

# Exercise in two dimension (d'après McMullen)

Consider the basic following plane (bi)-rational map:

$$\varphi := [x, y, z] \to [(az + y)\, x, (bx + y)\, z, zx]$$

For generic values of $a$ and $b$, the sequence of degrees is

$$1, 2, 3, 4, 5, 6, 8, 10, 13, 17, 22, 29, 38, 50, 66, 87, 115, 152, \ldots$$

which is fitted by the generating function

$$\frac{s^3 + s^4 + s^5}{(s - 1)\,(s^3 + s^2 - 1)}$$

so that the entropy if the log of the largest root of $1 + s - s^3$ that is to say $s_P \simeq 1.3247$[3].

But we can force singularity confinement by properly choosing the values of $a$ and $b$. This goes as follows:

[3] This is known as the smallest Pisot number, that is to say the smallest algebraic integer bigger than 1 and whose conjugates are smaller than one in absolute value

## Confinement at will

The projective inverse of the map is

$$\psi : [x, y, z] \longrightarrow [-(x - az)\, z, (x - az)(-y + zb), (-y + zb)\, z]$$

so that $\psi\varphi$ is the multiplication by $xyz$.The map $\varphi$ sends the three lines $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$ to points:

$$\begin{aligned}\{x = 0\} &\to [0, 1, 0]\\ \{y = 0\} &\to [a, b, 1]\\ \{z = 0\} &\to [1, 0, 0]\end{aligned}$$

The points $[0, 1, 0]$ and $[1, 0, 0]$ are singular under $\varphi$. If $a$ and $b$ are generic, the point $[a, b, 1]$ has an infinite sequence of images. Imposing singularity confinement is to force, by a proper choice of $a$ and $b$, this point to coincide with the third singular point of $\varphi$, i.e. $[0, 0, 1]$.

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We may choose the iterate of $\varphi$ for which this happens, and get different maps, all having a different entropy: if $\varphi^{\nu}(\{y = 0\}) = [0, 0, 1]$, the entropy is the log of the largest root of the polynomial

$$P_\nu(s) = \frac{t^\nu(t^3 - t - 1) + (t^3 + t^2 - 1)}{t - 1}$$

If $1 \leq \nu \leq 6$, the map is periodic (of period 6, 5, 8, 12, 18, 30) and the entropy vanishes.

If $\nu = 7$, (i.e. the map may be desingularised with 9 blow-ups) It is of infinite order, but the entropy still vanishes. This is where the integrability of infinite order maps appears.

If $\nu = 8$ (10 blow-ups to regularise the map), the entropy jumps to a positive value $\epsilon_{Lehmer}$,

$$\epsilon_{Lehmer} = Log(s_{Lehmer})$$

where $s_{\text{Lehmer}}$ verifies[4]

$$s^{10} + s^9 - s^7 - s^6 - s^5 - s^4 - s^3 + s + 1 = 0$$

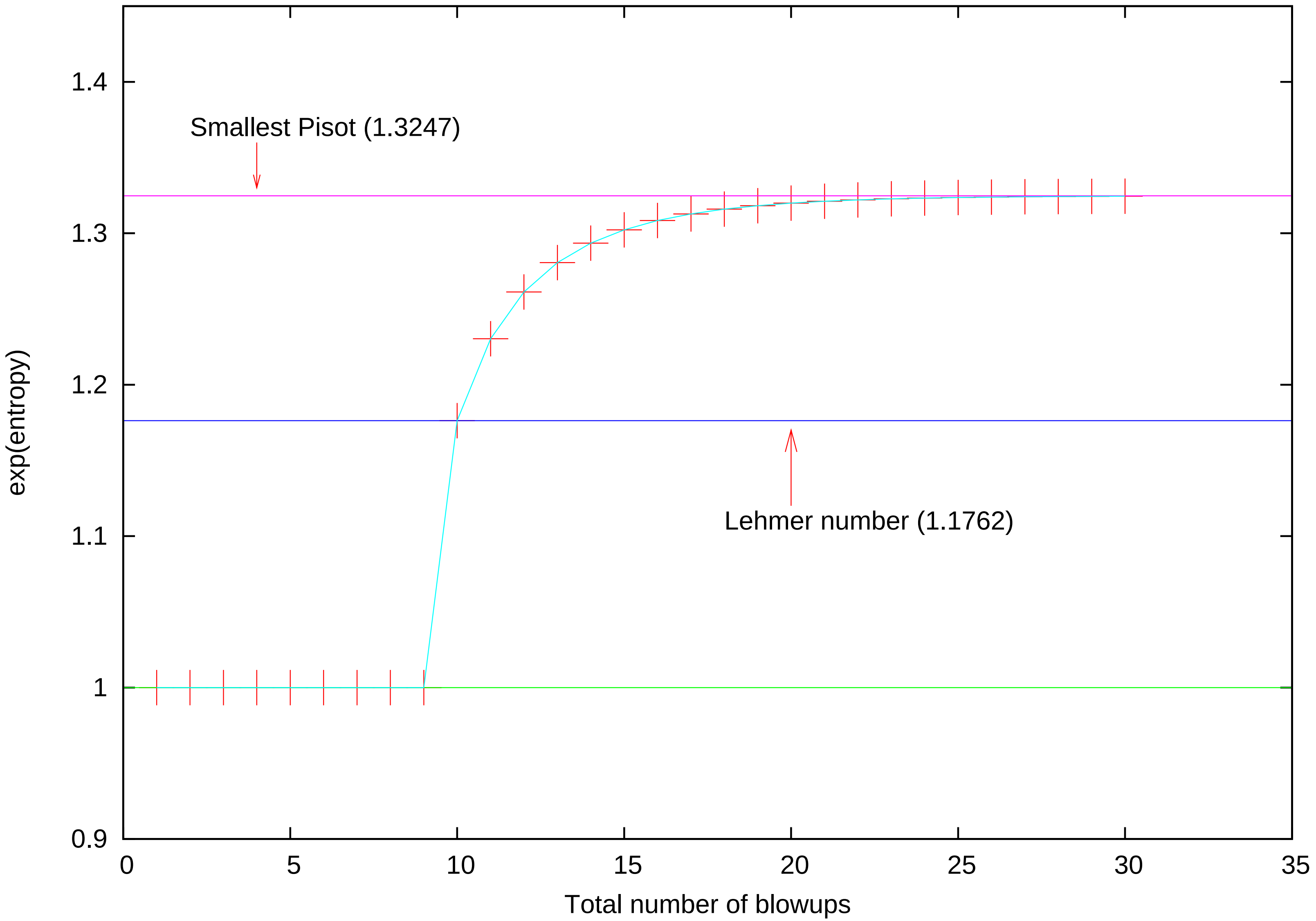
Then for $\nu > 8$, $\epsilon$ increases towards $\epsilon_\infty = \epsilon_P$.

The sequence of values of $s = \log \epsilon$ we get is:

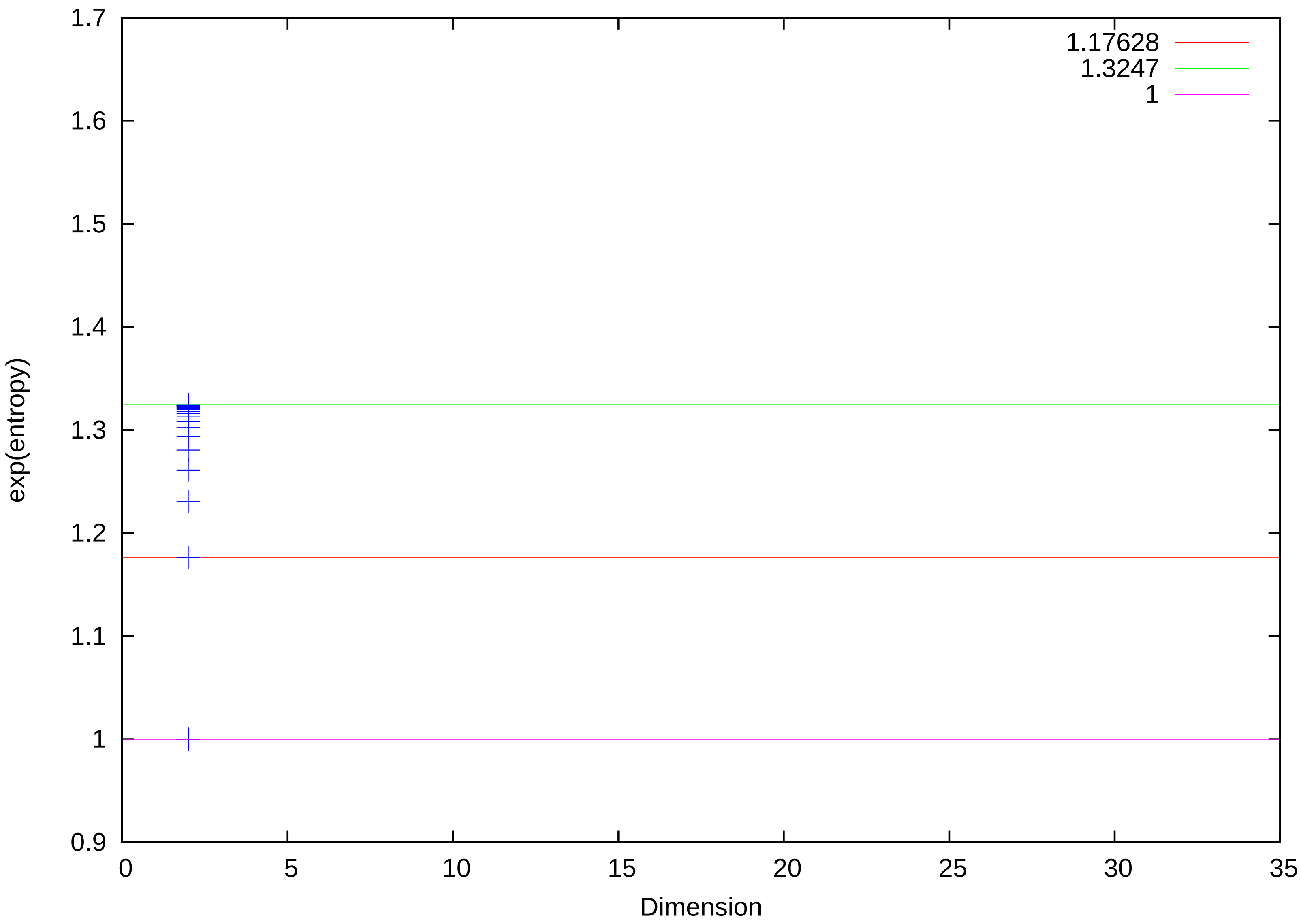
$$\begin{gathered} 1, 1, 1, 1, 1, 1 \\ 1, \\ 1.176280818 \\ 1.230391434 \\ 1.261230961 \\ 1.280638156 \\ \dots \\ 1.324717957 \end{gathered}$$

↩

[4] A Salem number is an algebraic integer whose conjugates have a norm which is smaller than or equal to one (one at least having norm 1). Lehmer's number is supposed to be the smallest Salem number. Pisot if all conjugates are smaller than 1.

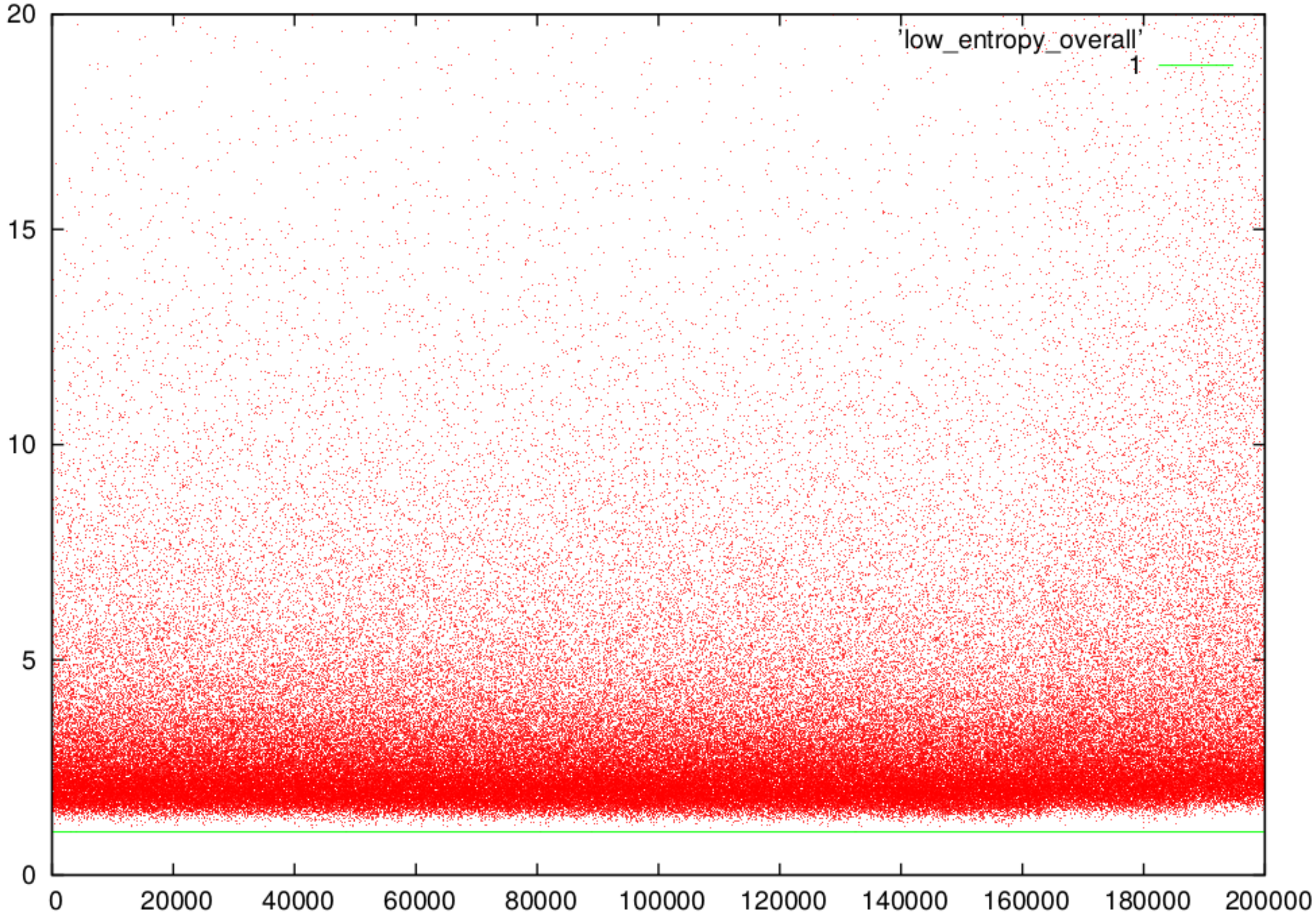

Smallest Pisot (1.3247)
Lehmer number (1.1762)
exp(entropy)
Total number of blowups
0.9
1
1.1
1.2
1.3
1.4
0
5
10
15
20
25
30
35

↩



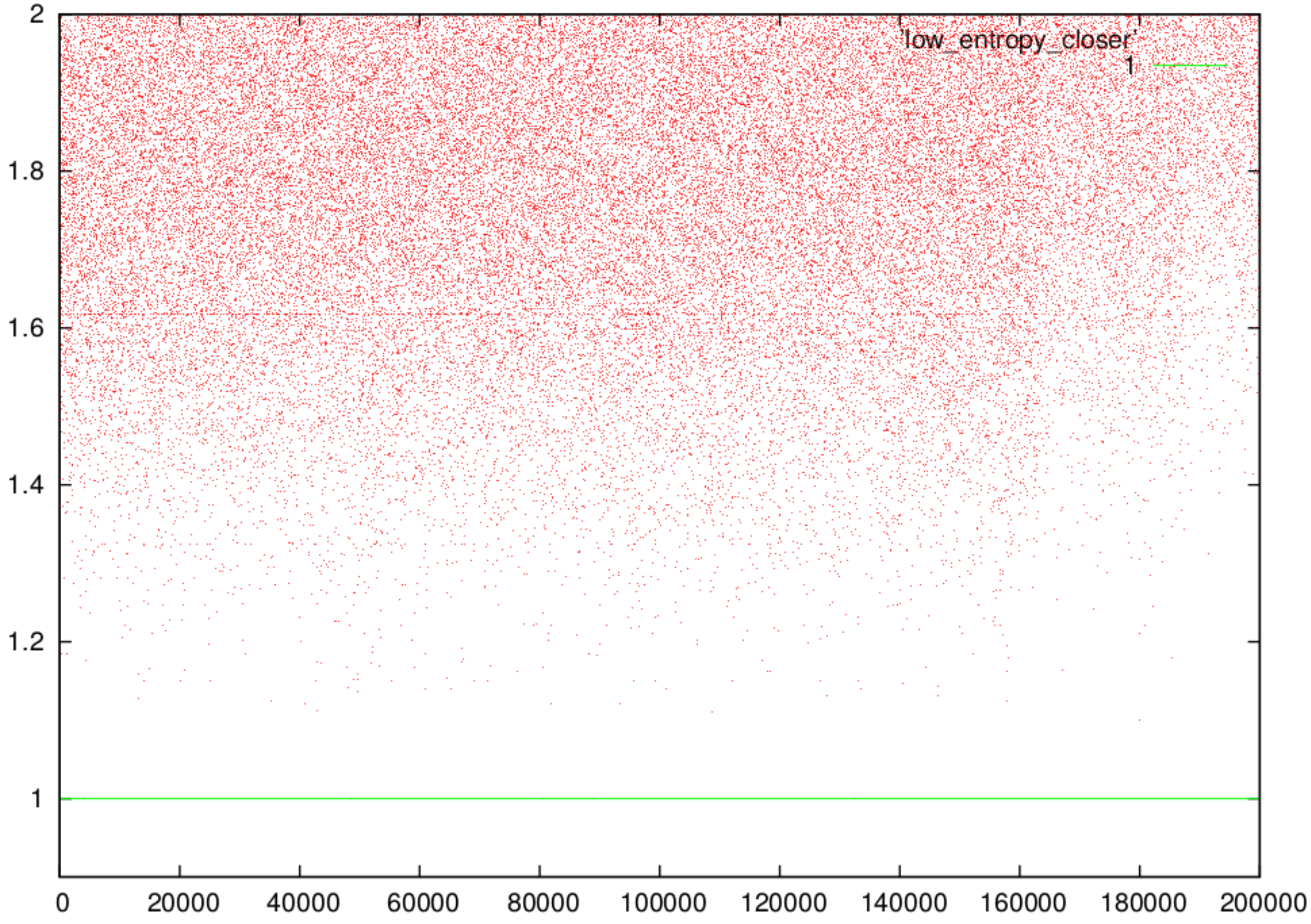

We would like to complete this picture. $\hookleftarrow$ $\hookrightarrow$

# A few values



'low_entropy_overall'
1
20
15
10
5
0
0
20000
40000
60000
80000
100000
120000
140000
160000
180000
200000

# A minimum?



'low_entropy_closer'
1
2
1.8
1.6
1.4
1.2
1
0
20000
40000
60000
80000
100000
120000
140000
160000
180000
200000

# A minimum ?

Let

$$\mu(n) = \mathsf{min}(\exp(\epsilon))$$

be the minimum of $\exp(\epsilon)$ over birational maps of $P_n$.

- This minimum exists since $\epsilon \geq 0$
- $\mu(n) \geq 1$
- $\mu(n+1) \leq \mu(n)$
- $\mu(2) \leq s_{\mathsf{Lehmer}}$

In dimension 2, there is almost a proof that the minimum is actually $s_{\mathsf{Lehmer}}$.

What about higher dimensions?

$\hookleftarrow$

# Large dimensions

Consider the monomial maps of $P_n$

$$\varphi : [x_0, x_1, \dots, x_n] \longrightarrow [x_1 x_0,\ x_2, x_n,\ x_3 x_n, \dots,\ x_0 x_n,\ x_n^2]$$

Notice that the map is almost

$$\sigma : [x_0, x_1, \dots, x_n] \longrightarrow [x_1 x_n,\ x_2 x_n,\ x_3 x_n, \dots,\ x_0 x_n,\ x_n^2]$$

which would just be the permutation

$$x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots x_{n-1} \rightarrow x_0$$

We have mixed this permutation with a quadratic map acting on a few coordinates. The entropy may be calculated from the characteristic polynomial of the matrix associated to $\varphi$. It is the maximal root of the polynomial

$$Q_n = s^n - s^{n-1} - 1$$

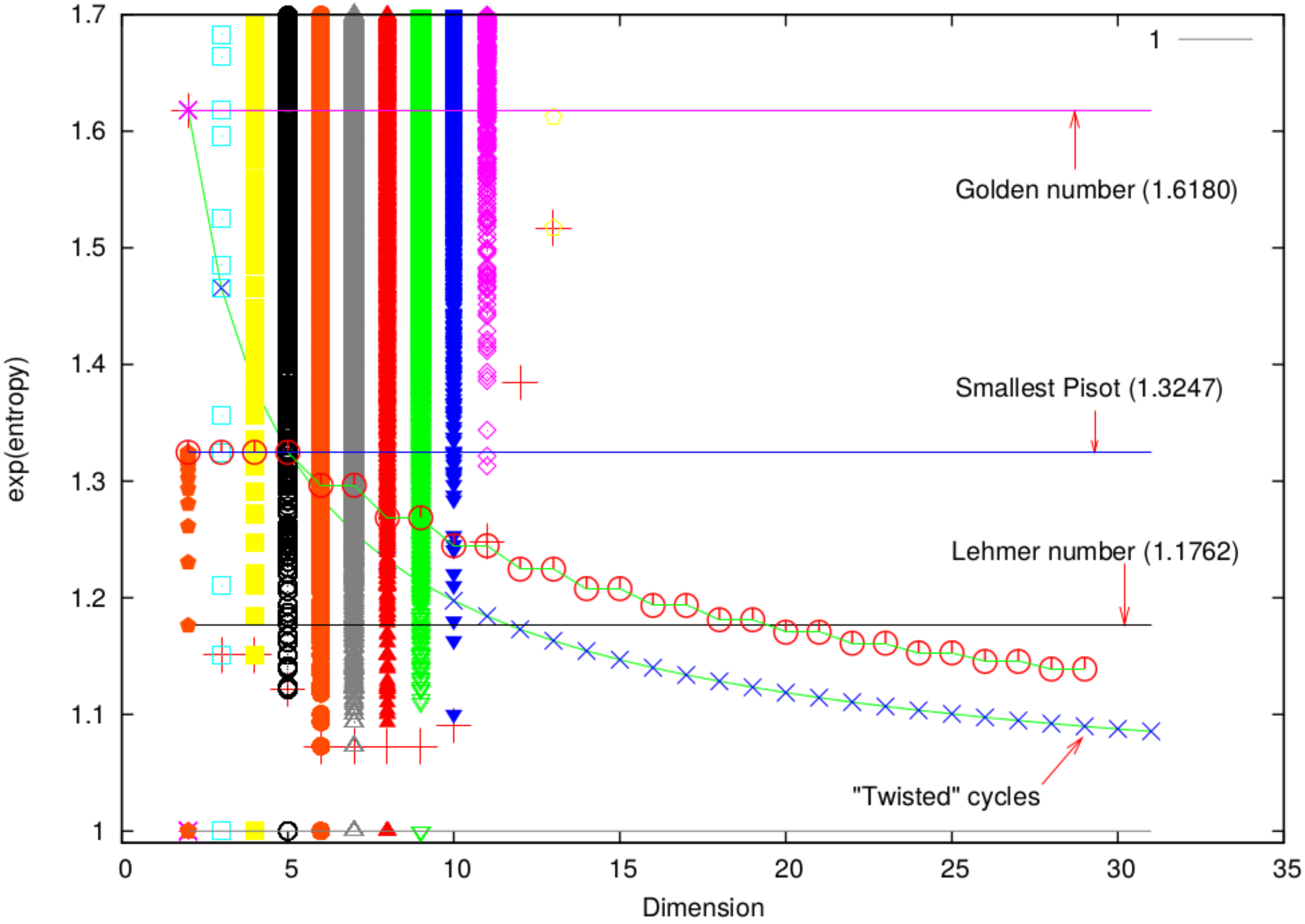
We find a sequence of entropies

$$\epsilon_n \simeq \frac{\log(n)}{n}$$

This shows that

$$\lim_{n \to \infty} \mu(n) = 0$$

# More values



1.7
1.6
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30
35
exp(entropy)
Dimension
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Golden number (1.6180)
Smallest Pisot (1.3247)
Lehmer number (1.1762)
"Twisted" cycles

## Conclusion and perspectives

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The entropy is to maps what the genus is to curves.

Its definition extends way beyond the case of maps in finite dimensional spaces.

For birational evolutions, it is the best integrability detector to date.

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0 – Explore the various polynomial identities provided by the derived recurrences[5], their reductions to integers (growth of height) and finite fields, their ultradiscrete limits.

[5] Somos sequences are essentially fixed points of the derivation process

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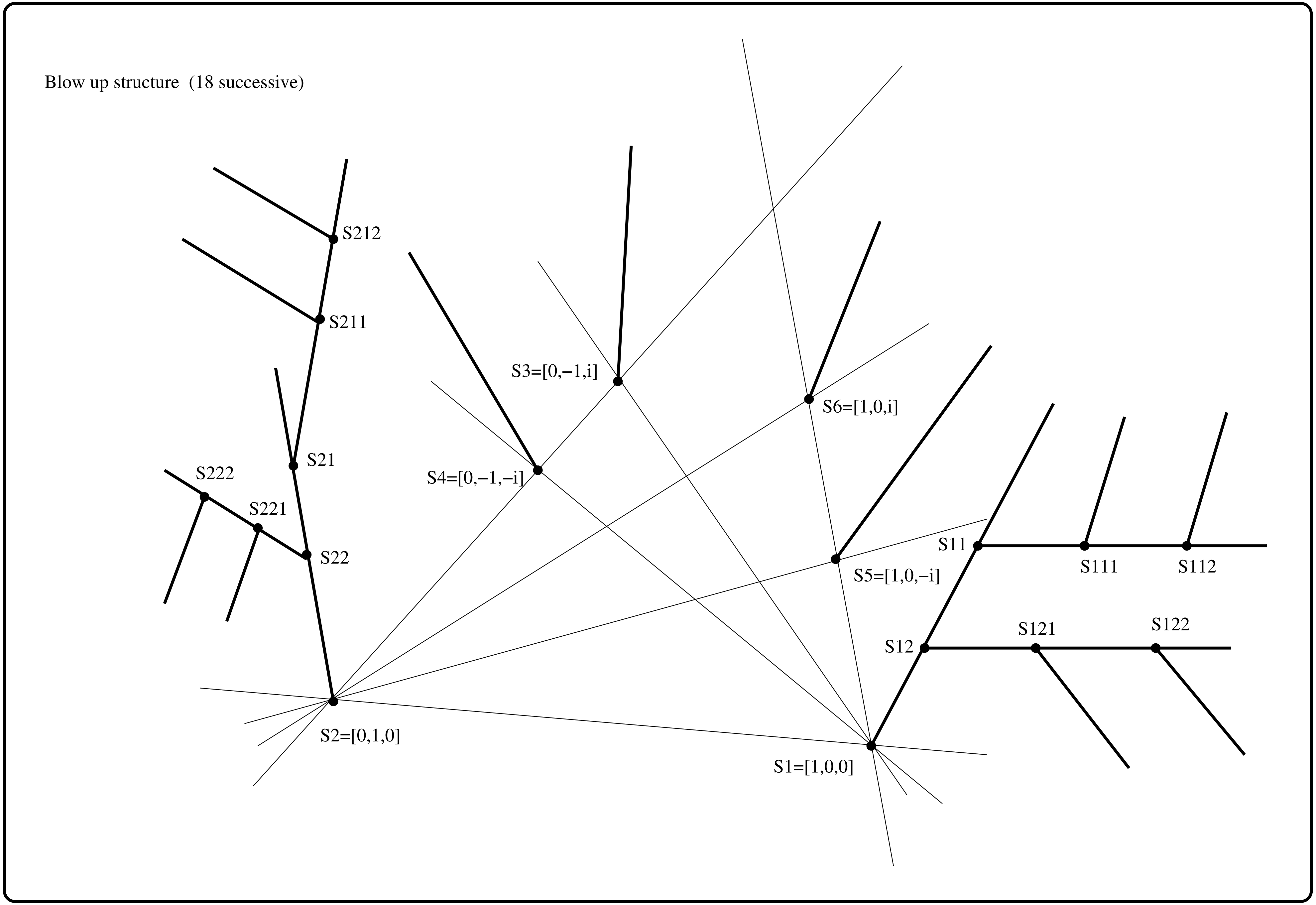
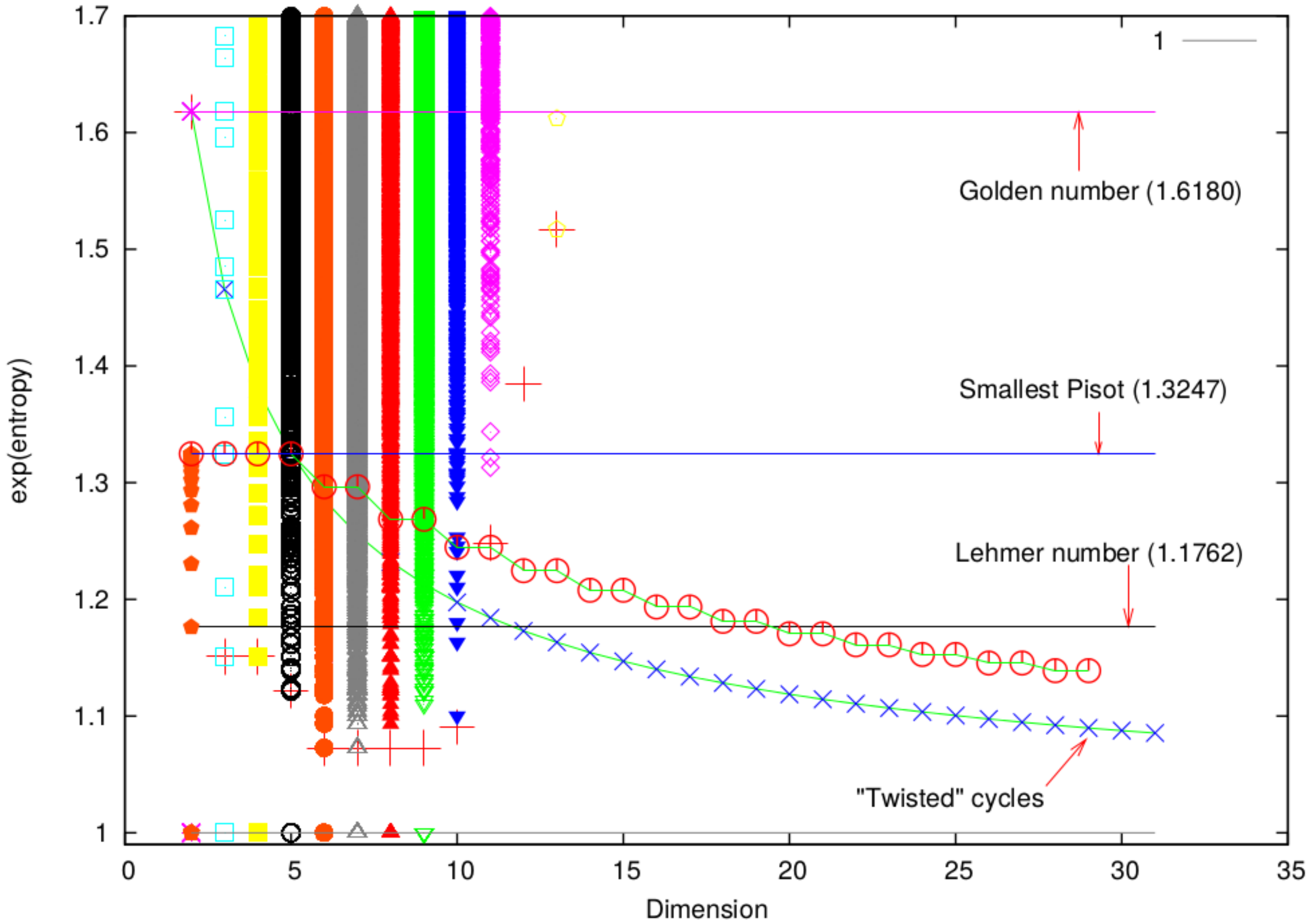
4 – Reconsider all other characteristic features of integrablity ( Lax pairs, symmetries, conserved quantities, ... ) in terms of the new variables we introduced.

$\longleftarrow$

# THANK YOU


Blow up structure (18 successive)
S212
S211
S21
S222
S221
S22
S2=[0,1,0]
S3=[0,−1,i]
S4=[0,−1,−i]
S6=[1,0,i]
S5=[1,0,−i]
S1=[1,0,0]
S11
S111
S112
S12
S121
S122



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