Double Affine Hecke Algebras and Character Varieties of Knots

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Plan

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References

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2. Y.B., P. Samuelson, *Affine cubic surfaces and character varieties of knots*, preprint (to appear).

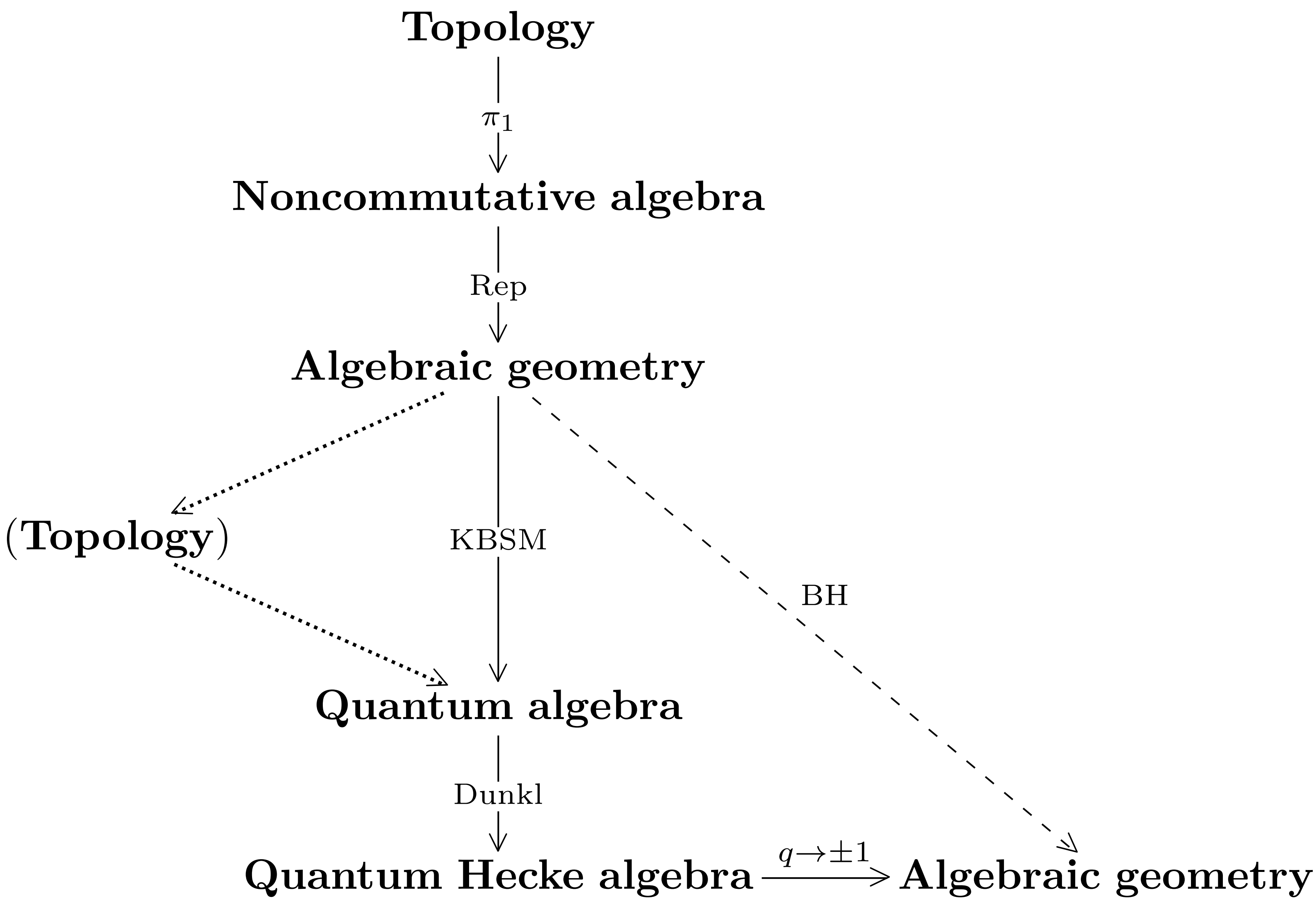
Vista

Let $K \subset S^3$ be a knot, G be a (complex reductive) algebraic group.

- We associate to K a module (representation) $\mathcal{M}_{q,t}(K)$ over the double affine Hecke algebra $\mathcal{H}_{q,t}(G)$ of type G.

- $\mathcal{M}_{q,t}(K)$ is a topological invariant that allows one to define a $multivariable$ generalization ('t-deformation') of the colored Jones polynomials $J_K(n;q)$ (Witten-Reshetikhin-Turaev invaraints).

- When $q \to \pm 1$, the module $\mathcal{M}_{q,t}(K)$ still makes sense and defines an interesting algebro-geometric invariant of K. In the case $G = \mathrm{SL}_2(\mathbb{C})$, $\mathcal{M}_{-1,t}(K)$ determines a family of algebraic curves in classical cubic surfaces that arise in the theory of integrable systems (Painlevé VI).


Topology
π_1
Noncommutative algebra
Rep
Algebraic geometry
(Topology)
KBSM
BH
Quantum algebra
Dunkl
Quantum Hecke algebra
$q \to \pm 1$
Algebraic geometry

2. Knot groups and their character varieties

A *knot* K in S^3 is (the ambient isotopy class of) a smooth embedding $S^1 \hookrightarrow S^3$. We'll deal with *oriented* knots (i.e., fix an orientation on S^1).

Write $S^3\backslash K$ for the *complement* of (a small tubular nghd of) K in S^3. This is a compact 3-manifold with a torus boundary $\partial(S^3\backslash K) \cong T^2$.

The most natural algebraic invariant of K is the *knot group*

$$\pi(K) := \pi_1(S^3\backslash K, *)$$

This is a powerful and effective invariant, but *not* complete: there exist non-equivalent knots with isomorphic knot groups (Fox, 1952).

One can refine $\pi(K)$ by considering it together with the *peripheral map*

$$\alpha : \ \pi_1[\partial(S^3 \backslash K)] \to \pi(K)$$

induced by the inclusion $\partial(S^3 \backslash K) \hookrightarrow S^3 \backslash K$. This map is injective (unless K is trivial), so $\pi_1[\partial(S^3 \backslash K)]$ can be identified with $\mathrm{Im}(\alpha)$.

For an oriented K, we can choose simple loops in $S^3 \backslash K$: the *meridian* m and the *longitude* l, unique up to (basepoint-free) isotopy, representing two generators of $\pi_1[\partial(S^3 \backslash K)]$. The triple $(\pi(K), m, l)$ is called the *peripheral system* of K.

Theorem (Waldhausen). *Two knots* $K, K' \subset S^3$ *are ambiently isotopic iff there is an isomorphism* $\phi : \pi(K) \to \pi(K')$ *such that* $\phi(m) = m'$ *and* $\phi(l) = l'$.

Examples

1. Unknot: $\pi(K) \cong \langle a \rangle$, $m = a$, $l = 1$

2. Trefoil knot:

$$\pi(K) \cong \langle a, b \,|\, aba = bab \rangle \ , \quad m = a \ , \quad l = baaba^{-4}$$

3. Torus knots:

$$\pi(K) \cong \langle a, b \,|\, a^p = b^q \rangle \ , \quad m = a^n b^{-k} \ , \quad l = b^q m^{-pq}$$

where n and k are integers satisfying $-pk + qn = 1$. (Note that m and l are independent of the choice of (n, k).)

4. 'Figure 8' knot:

$$\pi(K) = \langle a, b \mid aba^{-1}ba = bab^{-1}ab \rangle \ , \quad m = a, \quad l = ba^{-1}b^{-1}a^2b^{-1}a^{-1}b$$

Character varieties

The peripheral map α is quite complicated: it is natural to 'simplify' it by replacing fundamental groups with their linear representations.

Fix a complex reductive algebraic group G. For any (discrete) group π, let

$$\mathrm{Rep}(\pi, G) := \text{space of all representations } \pi \to G$$

This is naturally an algebraic variety (more precisely, an affine scheme) called the *representation variety* of π in G.

The *character variety* of π is the algebro-geometric quotient

$$\mathrm{Char}(\pi, G) := \mathrm{Rep}(\pi, G)//\mathrm{Ad}(G)$$

with coordinate ring $\mathcal{O}\mathrm{Char}(\pi, G) = \mathbb{C}[\mathrm{Rep}(\pi, G)]^G$.

For a knot group, the peripheral map α induces a morphism

$$\alpha^*: \ \mathrm{Char}(\pi(K), G) \to \mathrm{Char}(\pi_1[\partial(S^3 \backslash K)], G)$$

Identifying $\pi_1[\partial(S^3 \backslash K)] \cong \mathbb{Z}^2$ via (m, l), we can compute the target of α^*.

Let $\mathbb{T} \subset G$ be a maximal torus in G, and W the corresponding Weyl group.

Then

$$\mathbb{T} \times \mathbb{T} = \mathrm{Rep}(\mathbb{Z}^2, \mathbb{T}) \hookrightarrow \mathrm{Rep}(\mathbb{Z}^2, G) \twoheadrightarrow \mathrm{Char}(\mathbb{Z}^2, G)$$

induces

$$(\mathbb{T} \times \mathbb{T})/W \to \mathrm{Char}(\mathbb{Z}^2, G)$$

In general, this map is injective, and for 'many' G, it is known to be an isomorphism of schemes (e.g., for $G = \mathrm{SL}_2(\mathbb{C})$ or any simply connected G of classical type, see [Sikora, 2014]). In this case,

$$\mathcal{O}\mathrm{Char}(\mathbb{Z}^2, G) \cong \mathbb{C}[\mathbb{T} \times \mathbb{T}]^W$$

Many interesting algebro-geometric invariants of knots arise from α^*: e.g. the Alexander polynomial $\Delta_K(t)$, and the so-called A-polynomial $A_K(m,l)$.

Let $G = \mathrm{SL}_2(\mathbb{C})$. Then $\mathrm{Char}(\mathbb{Z}^2, G) \cong (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$, and

$$\begin{array}{ccc} & & \mathbb{C}^* \times \mathbb{C}^* \\ & & \Big\downarrow p \\ \mathrm{Char}(\pi(K), \mathrm{SL}_2) & \xrightarrow{\alpha^*} & (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2 \end{array}$$

Let $X_K \subseteq \overline{\mathrm{im}(\alpha^*)}$ denote the union of 1-dimensional components in the (Zariski) closure of the image of α^*. It is known that $X_K \neq \emptyset$ (Thurston), and X_K (or rather its inverse image under p) is called the A-*curve* of K.

The A-*polynomial* $A_K(m,l)$ is a polynomial in $\mathbb{C}[m^{\pm 1}, l^{\pm 1}]$ defining $p^{-1}(X_K) \subset \mathbb{C}^* \times \mathbb{C}^*$ (Copper, Culler, Gillet, Long, Shalen, 1994).

3. Topological quantization of character varieties

Does α^* determine 'quantum' invariants of knots, e.g. Jones polynomials? The answer is 'yes', but one needs to deform or 'quantize' the map α^*.

The KBSM construction

A *framed link* in an oriented 3-manifold M is $\coprod_i (S^1 \times [0,1])_i \hookrightarrow M$.

Let $\mathfrak{L}(M)$ be the $\mathbb{C}$-vector space spanned by the (ambient isotopy classes of) framed (unoriented) links in M (including $\varnothing$). For $q \in \mathbb{C}^*$, let $\mathfrak{L}'_q(M)$ be the smallest subspace of $\mathcal{L}(M)$ containing all 'skein expressions':

$$\times \; - \; q \asymp \; - \; q^{-1} \;)(\qquad , \qquad L \sqcup \bigcirc + (q^2 + q^{-2})\, L$$

(The links "$\times$", "$\asymp$", and "$)($" are identical outside of a small 3-ball B embedded in M and inside B they appear as in the above skein expressions.)

Definition (Przytycki). The *Kauffman bracket skein module* of M is

$$\mathcal{M}_q(M) := \mathfrak{L}(M)/\mathfrak{L}'_q(M)$$

In general, $\mathcal{M}_q(M)$ is just a vector space (with a distinguished element $\varnothing \in \mathfrak{L}(M)$). However, if M has extra structure, then $\mathcal{M}_q(M)$ has also extra structure.

Properties:

1. If F is a surface, then $\mathcal{A}_q(F) := \mathcal{M}_q(F \times [0,1])$ is an associative algebra with multiplication given by 'stacking links.' We call $\mathcal{A}_q(F)$ the *skein algebra* of F.

2. If M is a manifold with boundary, then $\mathcal{M}_q(M)$ is a module over $\mathcal{A}_q(\partial M)$. The action is given by 'pushing links from the boundary into the manifold.'

3. An oriented embedding $M \hookrightarrow N$ of 3-manifolds induces a linear map $\mathcal{M}_q(M) \to \mathcal{M}_q(N)$. Hence $\mathcal{M}_q(-)$ is a functor on the category of oriented 3-manifolds, with morphisms being oriented embeddings.

4. If $q = \pm 1$, then $\mathcal{M}_q(M)$ is a *commutative algebra* (for any M). The multiplication is given by 'disjoint union of links.' (This makes sense because when $q = \pm 1$, the skein relations allow strands to 'pass through' each other.)

Remark. $\mathcal{M}_q(S^3 \backslash K)$ is different from other knot invariants in a fundamental way. Many knot invariants are defined combinatorially, in the sense that they assign certain data to each crossing in a diagram of K and then combine these data to produce an invariant that does not depend on the choice of diagram. In contrast, the module $\mathcal{M}_q(S^3 \backslash K)$ depends on the global topology of $S^3 \backslash K$.

Relation to character varieties

An (unbased) loop $\gamma : S^1 \to M$ determines a conjugacy class in $\pi_1(M)$. Since the trace of a matrix is invariant on conjugacy classes, we can define a *trace function* $\mathrm{Tr}(\gamma) \in \mathcal{O}\mathrm{Char}(\pi_1(M), G)$ for any matrix group G.

Theorem (Bullock, Przytycki-Sikora). *For* $G = \mathrm{SL}_2(\mathbb{C})$*, the assignment* $\gamma \mapsto -\mathrm{Tr}(\gamma)$ *extends to an algebra isomorphism*

$$\mathcal{M}_{q=-1}(M) \stackrel{\sim}{\to} \mathcal{O}\mathrm{Char}(\pi_1(M), G)$$

Remark. The key observation here is that for $q = -1$, the skein relation becomes the Hamilton-Cayley identity for matrices in $\mathrm{SL}_2(\mathbb{C})$:

$$\mathrm{Tr}(A)\,\mathrm{Tr}(B) = \mathrm{Tr}(AB) + \mathrm{Tr}(AB^{-1})$$

The skein algebra of the torus

For $q \in \mathbb{C}^*$, define the quantum Weyl algebra

$$A_q := \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}\rangle / (XY - q^2 YX)$$

Note that $\mathbb{Z}_2$ acts by automorphisms on A_q by $(X, Y) \mapsto (X^{-1}, Y^{-1})$.

Theorem (Frohman-Gelca). *There is a natural isomorphism of algebras*

$$\mathcal{A}_q(T^2) \xrightarrow{\sim} A_q^{\mathbb{Z}_2}$$

The above isomorphism can be written quite explicitly. Under this isomorphism, the simple curves on T^2 representing the meridian m and the longitude l correspond to the elements:

$$m \mapsto X + X^{-1}\ , \qquad l \mapsto Y + Y^{-1}$$

Topological pairing and Jones polynomials

The above results suggest that $\mathcal{M}_q(S^3 \setminus K)$ should be viewed as a 'quantization' of the SL_2-character variety of the knot group $\pi(K)$. ni If $q \neq \pm 1$, $\mathcal{M}_q(S^3 \setminus K)$ is *not* an algebra but a (left) module over $\mathcal{A}_q(T^2)$. This should be thought of as a 'quantization' of the peripheral map α^*.

It turns out that $\mathcal{M}_q(S^3 \setminus K)$ determines the $\mathfrak{sl}_2$-colored Jones polynomials $J_K(n, q) \in \mathbb{C}[q, q^{-1}]$, originally defined by Witten, Reshetikhin-Turaev.

The key fact is that $\mathcal{M}_q(S^3 \setminus K)$ comes with a natural pairing:

$$\langle -, - \rangle_K : \; \mathcal{M}_q(D^2 \times S^1) \otimes_{\mathcal{A}_q(T^2)} \mathcal{M}_q(S^3 \setminus K) \to \mathbb{C}$$

induced by gluing a solid torus $D^2 \times S^1$ to the complement $S^3 \setminus K$ along the common boundary $T^2 = S^1 \times S^1$:

$$(D^2 \times S^1) \amalg_{T^2} (S^3 \setminus K) \xrightarrow{\sim} S^3$$

Recall the isomorphism $\mathcal{A}_q(T^2) \cong A_q^{\mathbb{Z}_2}$, mapping

$$l \mapsto L := Y + Y^{-1}$$

Theorem (Kirby-Melvin). *For any knot $K \subset S^3$,*

$$J_K(n;q) = (-1)^{n-1} \langle \varnothing,\ S_{n-1}(L) \cdot \varnothing \rangle_K \ ,$$

where S_n are the Chebyshev polynomials of the second kind.

Remark. Note that L is an (undeformed) *Macdonald operator*. Moreover, in the simplest case when K is the unknot, the topological pairing *coincides* with (undeformed) symmetric Dunkl-Cherednik pairing: in particular,

$$\langle \varnothing,\ S_{n-1}(L) \cdot \varnothing \rangle_K = \langle \varnothing \cdot S_{n-1}(x),\ \varnothing \rangle_K$$

This is not a coincidence!

4. Double Affine Hecke Algebras

The peripheral morphism of G-character varieties can written in dual terms as a map of commutative algebras

$$\alpha_* : \ \mathcal{O}\mathrm{Char}(\mathbb{Z}^2, G) \to \mathcal{O}\mathrm{Char}(\pi(K), G)$$

Recall, for a simply connected (classical) G, we have a natural isomorphism

$$\mathcal{O}\mathrm{Char}(\mathbb{Z}^2, G) \cong \mathbb{C}[(\mathbb{T} \times \mathbb{T})/W] = \mathbb{C}[\mathbb{T} \times \mathbb{T}]^W$$

Thus, for any knot $K \subset S^3$,

$$\alpha_* : \ \mathbb{C}[\mathbb{T} \times \mathbb{T}]^W \to \mathcal{O}\mathrm{Char}(\pi(K), G)$$

Now, the invariant ring $\mathbb{C}[\mathbb{T} \times \mathbb{T}]^W$ has very interesting (noncommutative) deformations, which have been studied extensively in recent years.

These deformations are related to the so-called *double affine Hecke algebras* usually abbreviated as DAHA (Cherednik, 1995).

Consider the canonical (non-unital) algebra homomorphism

$$\mathbb{C}[\mathbb{T}\times\mathbb{T}]^W \hookrightarrow \mathbb{C}[\mathbb{T}\times\mathbb{T}]\rtimes W\ , \quad a \mapsto \boldsymbol{e}\cdot a\cdot \boldsymbol{e}$$

where $\boldsymbol{e} := 1/|W| \sum_{w\in W} w$ is the symmetrizing idempotent of W.

This homomorphism is injective and its image equals $\boldsymbol{e}(\mathbb{C}[\mathbb{T}\times\mathbb{T}]\rtimes W])\boldsymbol{e}$, which is called the *spherical subalgebra* $\mathcal{A}(W)$ of $\mathbb{C}[\mathbb{T}\times\mathbb{T}]\rtimes W$. Thus

$$\mathbb{C}[\mathbb{T}\times\mathbb{T}]^W \cong \boldsymbol{e}(\mathbb{C}[\mathbb{T}\times\mathbb{T}]\rtimes W])\boldsymbol{e} =: \mathcal{A}(W)$$

The DAHA of type W is a two-parameter family $\mathcal{H}_{q,t}(W)$ of deformations of $\mathbb{C}[\mathbb{T}\times\mathbb{T}]\rtimes W$, depending on $q\in\mathbb{C}^*$ and $t\in(\mathbb{C}^*)^r$, where r is the number of conjugacy classes of reflections in W.

The symmetrizer $\boldsymbol{e} \in W$ 'deforms' to a distinguished idempotent $\boldsymbol{e}_{q,t}$ in $\mathcal{H}_{q,t}(W)$, called the Bernstein-Zelevinsky idempotent, and the subalgebra $\mathcal{A}(W)$ of $\mathbb{C}[\mathbb{T} \times \mathbb{T}] \rtimes W$ 'deforms' to the subalgebra of $\mathcal{H}_{q,t}$:

$$\mathcal{A}_{q,t}(W) := \boldsymbol{e}_{q,t}\, \mathcal{H}_{q,t}(W)\, \boldsymbol{e}_{q,t}$$

called the *spherical* DAHA of type W. In particular, when $q = t = 1$, there is a natural algebra isomorphism $\mathcal{A}_{1,1}(W) \cong \mathbb{C}[\mathbb{T} \times \mathbb{T}]^W$.

Remark. The above construction gives a flat family of deformations of $\mathcal{A}(W)$, that is actually *universal* (i.e., 'maximal possible' from the deformation theory point of view). It is remarkable that these deformations can be realized algebraically in terms of generators and relations.

The double affine Hecke algebra of rank one

The rank one DAHA $\mathcal{H}_{q,t}(\mathbb{Z}_2)$ (of type $C^\vee C_1$) has the following presentation (Sahi, 1999; Noumi-Stokman, 2004):

$$\mathcal{H}_{q,t}(\mathbb{Z}_2) = \mathbb{C}\langle T_1,\, T_2,\, T_3,\, T_4 \rangle$$

with T_1, T_2, T_3, T_4 satisfying the relations

$$\begin{aligned}
(T_1 - t_1)\,(T_1 + t_1^{-1}) &= 0 \\
(T_2 - t_2)\,(T_2 + t_2^{-1}) &= 0 \\
(T_3 - t_3)\,(T_3 + t_3^{-1}) &= 0 \\
(T_3 - t_4)\,(T_4 + t_4^{-1}) &= 0 \\
T_4\,T_3\,T_2\,T_1 &= q
\end{aligned}$$

Remarks

1. $\mathcal{H}_{q,t}(\mathbb{Z}_2)$ was originally introduced to study the Askey-Wilson orthogonal polynomials, and the Hecke parameters (t_1, t_2, t_3, t_4) are algebraically related to the Askey-Wilson coefficients (a, b, c, d).

2. $\mathcal{H}_{q,t}(\mathbb{Z}_2)$ can be viewed topologically as a (flat) deformation of the *orbifold* fundamental group algebra $\mathbb{C}\pi_1^{\mathrm{orb}}(\Sigma, *)$ of the orbifold Riemann surface $\Sigma = \mathbb{C}/\Gamma$, where $\Gamma := (\mathbb{Z} \oplus i\mathbb{Z}) \rtimes \mathbb{Z}_2$ acts by translations-reflections.

3. For $t_1 = t_2 = t_4 = 1$ and $t_3 = t$, $\mathcal{H}_{q,t}(\mathbb{Z}_2)$ specializes to Cherednik's DAHA $\mathbf{H}_{q,t}$ of type A_1.

Spherical DAHA

Choose a B.-Z. idempotent in $\mathcal{H}_{q,t}(\mathbb{Z}_2)$, say $\boldsymbol{e} := (T_3 + t_3)/(t_3 + t_3^{-1})$, and consider the corresponding spherical DAHA

$$\mathcal{A}_{q,t}(\mathbb{Z}_2) := \boldsymbol{e}\,\mathcal{H}_{q,t}(\mathbb{Z}_2)\,\boldsymbol{e}$$

Theorem (Oblomkov). Let A_q be the quantum Weyl algebra.

1. $\mathcal{H}_{q,t}(\mathbb{Z}_2)$ is a universal deformation of $A_q \rtimes \mathbb{Z}_2$

2. $\mathcal{A}_{q,t}(\mathbb{Z}_2)$ is a universal deformation of $A_q^{\mathbb{Z}_2}$

Lemma. If q is not a root of unity (or $q = \pm 1$ and t generic), the projection functor $M \mapsto \boldsymbol{e}\,M$ is an equavalence of categories

$$\mathcal{H}_{q,t}(\mathbb{Z}_2)\text{-Mod} \ \overset{\sim}{\to}\ \mathcal{A}_{q,t}(\mathbb{Z}_2)\text{-Mod}$$

Theorem (Koornwinder). The algebra $\mathcal{A}_{q,t}(\mathbb{Z}_2)$ is generated by

$$x := (T_4T_3 + (T_4T_3)^{-1})\,\boldsymbol{e}$$
$$y := (T_3T_2 + (T_3T_2)^{-1})\,\boldsymbol{e}$$
$$z := (T_3T_1 + (T_3T_1)^{-1})\,\boldsymbol{e}$$

subject to the relations

$$\begin{aligned}
[x, y]_q &= (q^2 - q^{-2})z - (q - q^{-1})\gamma \\
[y, z]_q &= (q^2 - q^{-2})x - (q - q^{-1})\alpha \\
[z, x]_q &= (q^2 - q^{-2})y - (q - q^{-1})\beta \\
\Omega &= (\bar{t}_1)^2 + (\bar{t}_2)^2 + (\overline{qt_3})^2 + (\bar{t}_4)^2 - \bar{t}_1\bar{t}_2(\overline{qt_3})\bar{t}_4 + (q + q^{-1})^2
\end{aligned}$$

where $\bar{t}_i := t_i - t_i^{-1}$ $(i = 1, 2, 3, 4)$ and

$$\alpha := \bar{t}_1\bar{t}_2 + (\overline{qt_3})\bar{t}_4, \quad \beta := \bar{t}_2\bar{t}_4 + (\overline{qt_3})\bar{t}_1, \quad \gamma := \bar{t}_1\bar{t}_4 + (\overline{qt_3})\bar{t}_2$$

Remarks.

1. Note that the element

$$\Omega := -qyzx + q^2x^2 + q^2y^2 + q^{-2}z^2 - q\alpha x - q\beta y - q^{-1}\gamma z$$

is central in $\mathcal{A}_{q,t}(\mathbb{Z}_2)$ for all q, t.

2. For $q = \pm 1$, the algebra $\mathcal{A}_{\pm 1,t}(\mathbb{Z}_2)$ is commutative, and it is isomorphic to the coordinate ring of an affine cubic in $\mathbb{C}^3$:

$$xyz + x^2 + y^2 + z^2 + Ax + Dy + Cz + D = 0$$

which, for generic t's, is actually smooth.

Dunkl embedding

The most useful and important property of $\mathcal{H}_{q,t}$ is the exsitence of an *injective* algebra homomorphism

$$\Theta_{q,t}: \ \mathcal{H}_{q,t} \hookrightarrow D_q := \mathbb{C}(X)[Y^{\pm 1}] \rtimes \mathbb{Z}_2$$

whose image is the subalgebra of D_q generated by X, X^{-1} and the following operators (Sahi, 1999; Noumi-Stokman, 2004):

$$T_{\mathrm{DC}} := t_1 s Y + \frac{q\bar{t}_1 X + \bar{t}_2}{qX - q^{-1}X^{-1}}\,(1-sY)\,, \quad T_{\mathrm{DL}} := t_3 s + \frac{\bar{t}_3 X^{-1} + \bar{t}_4}{X^{-1} - X}\,(1-s)\,,$$

called the Dunkl-Cherednik and Demazure-Lusztig operators, respectively. Explicitly (in our notation), $\Theta_{q,t}$ is given by

$$T_1 \mapsto q\, T_{\mathrm{DC}}^{-1}\, X\ , \quad T_2 \mapsto T_{\mathrm{DC}}\ , \quad T_3 \mapsto T_{\mathrm{DL}}\ , \quad T_4 \mapsto X^{-1}\, T_{\mathrm{DL}}^{-1}$$

Another presentation

The Dunkle embedding shows that the algebra $\mathcal{H}_{q,t}$ is also generated by the elements

$$X^{\pm 1}\ , \quad Y := T_{\rm DL} T_{\rm DC}\ , \quad T := T_{\rm DL}$$

For this set of generators, the relations are (Naoumi-Stokman):

$$\begin{aligned} XT &= T^{-1}X^{-1} - \bar{t}_4 \\ T^{-1}Y &= Y^{-1}T + \bar{t}_1 \\ T^2 &= 1 + \bar{t}_3 T \\ TXY &= q^2 T^{-1} YX - q^2 \bar{t}_1 X - q\bar{t}_2 - \bar{t}_4 Y \end{aligned}$$

This presentation shows that $\mathcal{H}_{q,1,1,1,1} = A_q \rtimes \mathbb{Z}_2$ (as subalgebras of D_q).

5. The Main Conjecture and Results

Let $K \subset S^3$ be a knot, and let $\mathcal{M} := \mathcal{O}\mathrm{Char}(\pi(K), G)$. We regard $\mathcal{M}$ as an $\mathcal{A}_{1,1}(W)$-module via the peripheral map

$$\alpha_* : \mathcal{A}_{1,1}(W) \to \mathcal{O}\mathrm{Char}(\pi(K), G)$$

Questions.

1. Is there a *canonical* deformation of $\mathcal{M}$ to a module $\mathcal{M}_{q,t}$ over $\mathcal{A}_{q,t}$?
2. What kind of invariants of K can be extracted from $\mathcal{M}_{q,t}(K)$?

In the case $G = \mathrm{SL}_2$, we have already seen that the KBSM construction produces a natural deformation of $\mathcal{M}(K)$ to a module over $\mathcal{A}_{q,1}(\mathbb{Z}_2) = A_q^{\mathbb{Z}_2}$: namely, the skein module $\mathcal{M}_q(S^3 \backslash K)$. But this module depends *only* on q.

Our main goal is to introduce the *Hecke parameters* t into this story.

First, using the Frohman-Gelca isomorphism $\mathcal{A}_q(T^2) \cong A_q^{\mathbb{Z}_2}$, we define the $nonsymmeteric\ skein\ module$ of K by

$$\tilde{\mathcal{M}}_q(K) := A_q \otimes_{A_q^{\mathbb{Z}_2}} \mathcal{M}_q(S^3 \backslash K)$$

This is a module over $\mathcal{H}_{q,1}(\mathbb{Z}_2) = A_q \rtimes \mathbb{Z}_2$, which (for q not a root of unity) contains $exactly$ the same information as the $A_q^{\mathbb{Z}_2}$-module $\mathcal{M}_q(S^3 \backslash K)$.

Next, we localize the module $\tilde{\mathcal{M}}_q(K)$ by inverting the 'meridians', i.e. nonzero polynomials in X:

$$\tilde{\mathcal{M}}_q^{\mathrm{loc}}(K) := D_q \otimes_{(A_q \rtimes \mathbb{Z}_2)} \tilde{\mathcal{M}}_q(K)$$

This is a D_q-module that comes together with a natural (localization) map

$$\tilde{\mathcal{M}}_q(K) \to \tilde{\mathcal{M}}_q^{\mathrm{loc}}(K)$$

Now, recall the Dunkl embedding $\Theta_{q,t} : \mathcal{H}_{q,t} \hookrightarrow D_q$ that exists for all q, t.

By restriction, $\Theta_{q,t}$ gives the localized nonsymmetric module $\tilde{\mathcal{M}}_q^{\mathrm{loc}}(K)$ the natural structure of a module over $\mathcal{H}_{q,t}$ for *any* value of t.

Conjecture 1. *For all knots K, the localization map $\tilde{\mathcal{M}}_q(K) \to \tilde{\mathcal{M}}_q^{\mathrm{loc}}(K)$ is injective, and its image is preserved under the above action of $\mathcal{H}_{q,t}(\mathbb{Z}_2)$ on $\tilde{\mathcal{M}}_q^{\mathrm{loc}}(K)$ for $t = (t_1, t_2, 1, 1)$.*

Conjecture 1 says that the vector space $\tilde{\mathcal{M}}_q(K)$ carries a *canonical* module structure over the algebra $\mathcal{H}_{q,t_1,t_2}(\mathbb{Z}_2)$ for *all* $(t_1, t_2) \in (\mathbb{C}^*)^2$. We denote this module by $\tilde{\mathcal{M}}_{q,t_1,t_2}(K)$.

It is natural to ask whether Conjecture 1 can be extended to the full DAHA $\mathcal{H}_{q,t_1,t_2,t_3,t_4}$ depending on all five parameters. The simplest example shows that this is not possible: if $t_3 \neq 1$ or $t_4 \neq 1$, the operator T_{DL} does not preserve the skein module of the unknot K_0.

We believe, however, that the skein module of K_0 is the *only* obstruction to a canonical extension of the action of $\mathcal{H}_{q,t_1,t_2}$ on $\tilde{\mathcal{M}}_q(K)$ to all four Hecke parameters.

Conjecture 2. *For any knot K, the skein module $\mathcal{M}_q(K)$ contains a copy of $\mathcal{M}_q(K_0)$ as a submodule. Let $\bar{\mathcal{M}}_q(K) := \tilde{\mathcal{M}}_q(K)/\tilde{\mathcal{M}}_q(K_0)$. Then the action of $\mathcal{H}_{q,t}$ on $\bar{\mathcal{M}}_q^{\mathrm{loc}}(K)$ preserves the subspace $\bar{\mathcal{M}}_q(K) \subset \bar{\mathcal{M}}_q^{\mathrm{loc}}(K)$ for all values $t = (t_1, t_2, t_3, t_4)$.*

We now discuss the evidence for these conjectures and implications.

Results.

Conjectures 1 and 2 have been verified directly in the following cases:

(i) K is the unknot,

(ii) K is any $(2p+1)$-torus knot

(iii) K is the "Figure 8" knot.

Conjecture 1 implies some new algebraic properties of the classical Jones polynomials $J_K(n,q) \in \mathbb{C}[q,q^{-1}]$.

For example, from Conjecture 1 one can easily deduce that the following rational function must be a Laurent polynomial in q for all $n, j \in \mathbb{Z}$:

$$F_K(j;n;q) := \frac{(q^2-1)\,[J_K(n+j,q) + J_K(n-1-j,q)]}{q^{4n-2}-1}$$

Theorem. The rational function $F_K(j; n; q) \in \mathbb{C}(q)$ is a Laurent polynomial for all knots $K \subset S^3$ (independently of Conjecture 1).

We proved the above theorem and a few other similar results, using Habiro's cyclotomic expansion of the Jones polynomial $J_K(n, q)$.

Next, Conjecture 1 makes sense for $q = -1$, and in fact, it is very interesting. We have a lot of evidence that it holds in this case.

Theorem. When $q = -1$, Conjecture 1 follows from (and essentially equivalent to) a known conjecture about the algebraic structure of the peripheral system $(\pi(K), m, l)$, due to G. Brumfiel and H. Hilden (1990).

We have verified the BH conjecture for many classes of knots, including all 2-bridge knots, all torus knots, infinite families of pretzel knots, . . .

More interestingly, we have

Theorem. If Conjecture 1 holds for two knots K and K', then it holds for their connect sum $K\#K'$.

Thus, it suffices to prove Conjecture 1 for prime knots.

The multi-variable Jones polynomials

Recall the topological pairing for the skein module

$$\langle -, - \rangle_K : \ \mathcal{M}_q(D^2 \times S^1) \otimes \mathcal{M}_q(S^3 \backslash K) \to \mathbb{C}$$

and the Kirby-Melvin formula for the Jones polynomial

$$J_K(n; q) = \langle \varnothing, \ S_{n-1}(L) \cdot \varnothing \rangle_K$$

Now, if we deform the module structure on $\mathcal{M}_q(S^3\backslash K)$ we can replace the undeformed Macdonald operator $L = Y + Y^{-1}$ by the t-deformed one

$$L_{t_1,t_1} := Y_{t_1,t_2} + Y_{t_1,t_2}^{-1}$$

which is usually called the *Askey-Wilson operator*. Here, $Y_{t_1,t_2} := s\,T_{\mathrm{DC}}$ is the Dunkl-Cherednik operator which acts on the skein module $\mathcal{M}_q(S^3\backslash K)$ as prescribed in Conjecture 1.

Defintion. If Conjecture 1 holds for K, we define its three-variable colored Jones polynomial by

$$J_K(n; q, t_1, t_2) := \langle \varnothing, S_{n-1}(L_{t_1,t_1}) \cdot \varnothing \rangle_K$$

By the Kirby-Melvin formula, we then have

$$J_K(n; q) = J_K(n; q, t_1 = 1, t_2 = 1)$$

For the unknot, we can actually compute a closed formula for $J_K(n; q, t_1, t_2)$.

Theorem. If K is the unknot, then

$$J_K(n; q, t_1, t_2) = \frac{(t_1^{-1}q^2)^n - (t_1^{-1}q^2)^{-n}}{t_1^{-1}q^2 - (t_1^{-1}q^2)^{-1}}$$

For nontrivial knots, finding such explicit formulas seems to be a hopeless task. Still, one can deduce for $J_K(n; q, t_1, t_2)$ some nice properties. For example, we can prove

Theorem. Let $\bar{K}$ be the mirror image of K, and suppose Conjecture 1 holds for K. Then

$$J_K(n; q, t_1, t_2) = J_{\bar{K}}(n; q^{-1}, t_1^{-1}, t_2^{-1})$$

Examples

1. For the unknot, the module $\mathcal{M}_{q,t_1,t_2}(K)$ is isomorphic to the sign-polynomial representation of the DAHA $\mathcal{H}_{q,t_1,t_2}$, i.e.

$$\mathcal{M}_{q,t_1,t_2}(K) \cong \mathbb{C}[X^{\pm 1}]\,\varnothing$$

with $s \cdot \varnothing = -\varnothing$. The topological pairing coincides the Dunkl-Cherednik pairing on sign-polynomial representation.

2. For the trefoil, the deformed skein module $\mathcal{M}_{q,t_1,t_2}(K)$ is a free module of rank two over $\mathbb{C}[X^{\pm 1}]$

$$\mathcal{M}_{q,t_1,t_2}(K) \cong \mathbb{C}[X^{\pm 1}]\,u \oplus \mathbb{C}[X^{\pm 1}]\,v$$

where $v = \varnothing$ is the empty link and u is a generator of the unknot submodule.

The action of the generators $T = s$ and Y of $\mathcal{H}_{q,t_1,t_2}$ is given explicitly by

$$\begin{aligned} s \cdot u &= -u\,, \quad s \cdot v = v\,, \quad Y \cdot u = -t_1 u \\ Y \cdot v &= [t_1(q^2X^{-1} - q^6X^{-5}) - (q^2\bar{t}_1X^{-2} + q\bar{t}_2X^{-1})(q^4X^{-3} + q^2X^{-1})]u \\ &\quad +[t_1q^6X^{-6} - (q^2\bar{t}_1X^{-2} + q\bar{t}_2X^{-1})(q^4X^{-4} + q^2X^{-2} + 1)]v \end{aligned}$$

Remark. Note that $\mathcal{M}_{q,t_1,t_2}(K)$ admits a decomposition into a *nonsplit* exact sequence

$$0 \to V^- \to \mathcal{M} \to \tau^{-6}(V^+) \to 0$$

where $\tau^N(V^+)$ is a twist of the trivial representation $V^+ \cong \mathbb{C}[X^{\pm 1}]\,u$ and $V^- \cong \mathbb{C}[X^{\pm 1}]\,v$ is the sign representation. If q is not a root of unity, then V^- is the unique nontrivial submodule of $\mathcal{M}$.