Periods of meromorphic quadratic differentials and Goldman bracket

Dmitry Korotkin

Concordia University, Montreal

Geometric and Algebraic aspects of integrability, August 05, 2016
References

- D.Korotkin, *Periods of meromorphic quadratic differentials and Goldman bracket*, to appear
Main equation

- \( C_g \) - Riemann surface of genus \( g \).
- "Schrödinger equation" on \( C_g \):
  \[
  \varphi'' - u\varphi = 0
  \]
  where \( \varphi \) is a \((-1/2)\)-differential (locally), and \(-2u\) - meromorphic projective connection on \( C_g \) with \( n \) simple poles.
- Parametrization of space of all "potentials":
  \[
  \varphi'' + \left( \frac{1}{2}S_0 + Q \right)\varphi = 0
  \]
  \( S_0 \) - "base" projective connection, \( Q \) - meromorphic quadratic differential with \( n \) simple poles.
Canonical symplectic structure on $\mathcal{T}^* \mathcal{M}_g$

- Moduli space of pairs $(C_g, Q)$ is $\mathcal{Q}_{g,n} = \mathcal{T}^* \mathcal{M}_{g,n}$; $\dim \mathcal{Q}_{g,n} = 6g - 6 + 2n$

- $\{q_i\}_{i=1}^{3g-3+n}$ - any complex coordinates on $\mathcal{M}_{g,n}$ (say, $3g - 3$ entries of period matrix and $(v_1/v_2)(y_k)$); $\{\nu_i\}$ - normalized abelian differentials. $\{d\, q_i\}_{i=1}^{3g-3+n}$ - basis in cotangent space; $\{p_i\}_{i=1}^{3g-3+n}$ - coordinates of cotangent vector in this basis.

- Symplectic structure and symplectic potential:

$$\omega_{\text{can}} = \sum_{i=1}^{3g-3+n} dp_i \wedge dq_i \quad \theta_{\text{can}} = \sum_{i=1}^{3g-3+n} p_i \, dq_i$$
Homological Darboux coordinates

Let all zeros of $Q$ be simple: $\{x_i\}_{i=1}^{4g-4+n}$. Canonical cover (spectral, Hitchin, Seiberg-Witten...) $\hat{C}$:

$$v^2 = Q$$

in $T^* C_g$; $4g - 4 + 2n$ branch points at $\{x_i, y_i\}$; genus $4g - 3 + n$; involution $\mu : \hat{C} \rightarrow \hat{C}$

Decomposition of $H_1(\hat{C}, \mathbb{Z})$ into even and odd parts:

$$H_1(\hat{C}, \mathbb{Z}) = H_- \oplus H_+$$

where $\dim H_+ = 2g$, $\dim H_- = 6g - 6 + 2n$. Generators of $H_- : \{a_i^-, b_i^-\}_{i=1}^{3g-3+n}$; intersection $a_i^- \circ b_j^- = \delta_{ij}$.

Homological coordinates $A_i = \int_{a_i^-} v$, $B_i = \int_{b_i^-} v$. 
Canonical cover

Figure: Canonical basis of cycles on the canonical cover $\hat{C}$
Homological and canonical symplectic structures

- Homological symplectic structure on $T^*\mathcal{M}_g$:

  $$\omega_{\text{hom}} = \sum_{i=1}^{3g-3+n} dA_i \wedge dB_i$$

- Theorem 1.

  $$\omega_{\text{hom}} = \omega_{\text{can}}$$

  Thus $(A_i, B_i)$ are Darboux coordinates for $\omega_{\text{can}}$ on the main stratum of $T^*\mathcal{M}_g$ (all zeros of $Q$ are simple).
Symplectic structure on the space of projective connections

- Space $\mathcal{S}_g$: pairs $(C_g, S)$, $S$ holomorphic projective connection on $C_g$. Affine bundle over $\mathcal{M}_g$.
- Given the "base" projective connection $S_0$ on $C_g$ which holomorphically depends on moduli of $C_g$, write any $S$ as $S = S_0 + 2Q$, for some holomorphic quadratic diff. $Q$.
- The map $F^{S_0} : Q_g \to \mathcal{S}_g$ is used to induce symplectic structure on $\mathcal{S}_g$ from $\omega_{\text{can}}$.
- Equivalence: $S_0 \equiv S_1$ if corresponding symplectic structures on $\mathcal{S}_g$ coincide. Generating function $G_{01}$:

$$
\delta_\mu G = \int_{C_g} \mu(S_1 - S_0)
$$
Equivalent projective connections

- Schottky projective connection $S_{Sch}(\cdot) = \{w, \cdot\}$, where $w$ is the Schottky uniformization coordinate; $\{\cdot, \cdot\}$- Schwarzian derivative.

- main example: Bergman projective connection $S_B$. Canonical bimeromorphic differential $B(x, y)$ on $\mathbb{C}_g$:
  \[ \oint_{a_\alpha} B(\cdot, y) = 0, \]

  \[ B(x, y) = \left( \frac{1}{(\xi(x) - \xi(y))^2} + \frac{1}{6} S_B(\xi(x)) + \ldots \right) d\xi(x)d\xi(y) \]

  $B$ depends on Torelli marking (choice of canonical basis in homologies on $\mathcal{C}$)

- Generating function from $S_{Sch}$ to $S_B$: Zograf’s $F$-function $F = \mathcal{Z}_B'(1); \mathcal{Z}_B$- Bowen’s zeta-function of Schottky group.

- Generating function corresponding to change of Torelli marking defining $S_B$ is given by $\det(C\Omega + D)$ (cocycle of determinant of Hodge vector bundle).
Main tool: Variational formulas

For any $s_i \in H_-$ define $s_i^* \in H_-$ ($s_i \circ s_j^* = \delta_{ij}$); $P_i = \int_{s_i} v$. Then

$$\frac{\partial B(x, y)}{\partial P_i} = \frac{1}{2} \int_{t \in s_i^*} \frac{B(x, t)(B(t, y))}{v(t)}$$

where $z(x)$ and $z(y)$ are kept constant.

$$\frac{\partial v_j(x)}{\partial P_i} = \frac{1}{2} \int_{t \in s_i^*} \frac{v_j(t)(B(t, x))}{v(t)}$$

$$\frac{\partial \Omega_{jk}}{\partial P_i} = \frac{1}{2} \int_{t \in s_i^*} \frac{v_j v_k}{v}$$
Poisson bracket for potential \( u(z) \)

- Let \( S_0 = S_B; \psi = \phi \sqrt{v}; z(x) = \int_{x_0}^{x} v \) - "flat" coordinate on \( C_g \) and \( \hat{C} \).
- Main equation: \( \psi_{zz} - u(z) \psi = 0 \) where

\[
u(z) = -1 - \frac{1}{2} \frac{S_B - S_v}{Q}\]

and \( S_v(\cdot) = \{ \int^{x} v, \cdot \} \)

- Invariant matrix form (on \( \hat{C} \)): \( d\psi = \begin{pmatrix} 0 & v \\ uv & 0 \end{pmatrix} \psi \)

- Define \( h(x, y) = \frac{B^2(x,y)}{Q(x)Q(y)} \)
Poisson bracket for potential $u(z)$ (continued)

\[
\frac{4\pi i}{3} \{u(z), u(\zeta)\} = \mathcal{L}_z h^\zeta(z) - \mathcal{L}_\zeta h^z(\zeta)
\]

where

\[
\mathcal{L}_z = \frac{1}{2} \partial_z^3 - 2u(z)\partial_z - u_z(z)
\]

is known as "Lenard" operator in KdV theory;

\[
h^{(y)}(x) = \int_{x_1}^{x} h(y, \cdot) \, v(\cdot)
\]

The first example of holomorphic Poisson bracket on a Riemann surface (get as a Dirac bracket from Atiyah-Bott symplectic structure??).
Monodromy representation and Goldman bracket

- Fundamental group: \(\pi_1(C_g \setminus \{y_i\}_{i=1}^n, x_0)\) with generators \((\gamma_i, \alpha_j, \beta_j)\) and relation \(\prod_{j=1}^g \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} \prod_{i=1}^n \gamma_i = id\)

- Monodromy matrices: \(M_\alpha, M_\beta, M_y\) with relation

\[
\prod_{i=1}^n M_y \prod_{j=1}^g M_\beta^{-1} M_{\alpha_j}^{-1} M_\beta M_{\alpha_j} = I
\]

- Goldman’s bracket on character variety \(V_{g,n}\):

\[
\{\text{tr} M_\gamma, \text{tr} M_{\tilde{\gamma}}\} = \frac{1}{2} \sum_{p \in \gamma \cap \tilde{\gamma}} (\text{tr} M_{\gamma_p \tilde{\gamma}} - \text{tr} M_{\gamma_p \tilde{\gamma}^{-1}})
\]
Relation to results of S. Kawai, Math Ann (1996)

- Kawai: "canonical symplectic structure on $T^* \mathcal{M}_g$ implies Goldman bracket if $S_0 = S_{Bers}$".

- Together with our results: $S_B$ and $S_{Bers}$ are in the same equivalence class; implies existence of generating function.

- Conjecture:

  \[
  G = -6\pi i \log \frac{\mathcal{Z}'[\Gamma_{C_0,\eta}] (1)}{\det (\Omega - \bar{\Omega}_0)}
  \]

  where $\mathcal{Z}$ is the Selberg zeta-function
  \[
  \mathcal{Z}(s) = \prod_{\gamma} \prod_{m=0}^{\infty} (1 - q^s \gamma^m)
  \]
  corresponding to quasi-fuchsian group $\Gamma_{C_0,\eta}$; $\Omega$ and $\Omega_0$ are period matrices of $C_0$ and $C$. 
Tau-function of Schrödinger equation

- Motivation: Jimbo-Miwa tau-function for Schlesinger system:
  \[ \frac{d\psi}{dx} = \sum_{i=1}^{N} \frac{A_i}{x-x_i} \psi : \]
  \[ \frac{\partial \log \tau_{JM}}{\partial x_i} = \frac{1}{2} \text{res}_{x_i} tr(d\psi \psi^{-1})^2 \]

- A straightforward analog of this definition in the case of Schrödinger equation (no isomonodromy!)
  \[ \frac{\partial \log \tau}{\partial \mathcal{P}_{s_i}} := \frac{1}{4\pi i} \int_{s_i^*} \left( \frac{tr(d\psi \psi^{-1})^2}{v} + 2v \right) \]
  where \( \mathcal{P}_{s_i} = \int_{s_i} v; \) this gives rise to Bergman tau-function
  \[ \frac{\partial \log \tau}{\partial \mathcal{P}_{s_i}} = \frac{1}{4\pi i} \int_{s_i^*} \frac{S_B - S_v}{v} \]
  \[ \tau^\sigma = \det^6 (C\Omega + D) \tau \quad \tau(\epsilon Q) = \epsilon^{1/6(5g-5+n)} \tau(Q) \]
Open: "Yang-Yang" function

- $(\varphi_i, l_i)$ - complexified Fenchel-Nielsen Darboux coordinates on character variety $V_{g,n}$.

\[
\omega_{\text{can}} = \sum_i d l_i \wedge d \varphi_i = \sum_i dp_i \wedge dq_i
\]

\[
dG_{YY} = \sum_i l_i d \varphi_i - \sum_i p_i dq_i
\]

(Nekrasov-Rosly-Shatashvili); $G_{YY}$ - "Yang-Yang" function (depends on pants decomposition on the Character variety side; transforms with dilogarithms; depends also on Torelli marking; transforms as a section of Hodge line bundle).
Simplest example: genus 0 with 4 simple poles

Poles: 0, 1, \( t \), \( \infty \); \( B(x, y) = \frac{dx \, dt}{(x-t)^2} \), \( S_B = 0 \);

\[
Q = \frac{\mu}{x(x - 1)(x - t)} (dx)^2
\]

Poisson structure:

\[
\{\mu, t\} = \frac{t(1 - t)}{4\pi i}
\]

Equation (Heun):

\[
\varphi'' + \frac{\mu}{x(x - 1)(x - t)} \varphi = 0
\]

Homological coordinates:

\[
\sqrt{\mu} \int_{a,b} \frac{dx}{\sqrt{x(x - 1)(x - t)}}
\]