Automorphic Lie Algebras and Root System Cohomology

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Automorphic Lie Algebras and Cohomology of Root Systems
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V. Knibbeler, S. Lombardo, and J.A. Sanders
Higher dimensional Automorphic Lie Algebras
Journal of Foundations of Computational Mathematics, DOI: 10.1007/s10208-016-9312-1, 1–49, 2016.

S. Lombardo, J.A. Sanders
On the Classification of Automorphic Lie Algebras
Communications in Mathematical Physics, 299: 793–824, 2010.

S. Lombardo, A.V. Mikhailov
Automorphic Lie Algebras and the Reduction Group
Communications in Mathematical Physics, 258(1): 179–202, 2005.

What is an Automorphic Lie Algebra?

Automorphic Lie Algebras (ALiAs)

An ALiA is the space of invariants

$$\left(\mathfrak{g} \otimes \mathcal{M}(\overline{\mathbb{C}})\right)^G = \left\{a \in \mathfrak{g} \otimes \mathcal{M}(\overline{\mathbb{C}}) \mid ga = a,\ \forall g \in G\right\}$$

It is obtained by imposing a discrete group symmetry on a current algebra of Krichever-Novikov (KN) type

$$\mathfrak{g} \otimes \mathcal{M}(\overline{\mathbb{C}})$$

i.e. current algebra with $\mathcal{M}(\overline{\mathbb{C}})$-linear Lie bracket.

Automorphic Lie Algebras in a nutshell

$G \subset \mathrm{Aut}(\overline{\mathbb{C}})$	is a finite group of FLTs (Möbius transformations); $g \in G$ acts on a complex parameter λ by $g(\lambda) = \frac{a\lambda+b}{c\lambda+d}$, $a, b, c, d \in \mathbb{C}$.
$\Gamma \subset \overline{\mathbb{C}}$	is an *exceptional* orbit of the G-action on $\overline{\mathbb{C}}$ ($\|\Gamma\| < \|G\|$).
$\mathcal{M}(\overline{\mathbb{C}})$	the field of rational functions on $\overline{\mathbb{C}}$ and $\mathcal{M}(\overline{\mathbb{C}})_\Gamma$ is the ring of functions in $\mathcal{M}(\overline{\mathbb{C}})$ with poles in a G-orbit Γ.
$\mathfrak{g}$	a (simple) Lie algebra with a G-action preserving the Lie bracket; we assume the G-action to be *fixed-point-free*, i.e. $\mathfrak{g}^G = 0$.

Automorphic Lie Algebras

$$\left(\mathfrak{g} \otimes \mathcal{M}(\overline{\mathbb{C}})\right)_\Gamma^G = \left\{a \in \mathfrak{g} \otimes \mathcal{M}(\overline{\mathbb{C}})_\Gamma \mid ga = a,\ \forall g \in G\right\}$$

Inner and outer group actions on Lie algebras

Let G be a finite group, let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\rho : G \to \mathrm{Aut}(\mathfrak{sl}_{n+1})$ a homomorphism of groups.

- If $n = 1$ the image of ρ is contained in $\mathrm{Int}(\mathfrak{sl}_{n+1})$ (inner automorphisms).
- If $n > 1$ then $\mathrm{Out}(\mathfrak{sl}_{n+1}) = \mathrm{Aut}(\mathfrak{sl}_{n+1})/\mathrm{Int}(\mathfrak{sl}_{n+1}) \cong \mathbb{Z}/2$.
  Then $\rho(G) \cap \mathrm{Int}(\mathfrak{sl}_{n+1})$ is a normal subgroup of $\rho(G)$ of index 1 or 2.

  If $\rho(G)$ has not an index 2 normal subgroup, the action is inner: finite groups for which this is the case are

  - the tetrahedral
  - the icosahedral groups and
  - the cyclic groups of odd order.

  Finite groups that do have a normal subgroup of index 2 are

  - cyclic groups of even order,
  - dihedral groups and
  - the octahedral group.

We consider here G-actions which are inner.

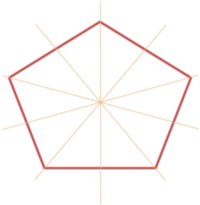
Automorphic Functions $\mathcal{M}(\overline{\mathbb{C}})^G_\Gamma$: G-action on $\mathbb{C}$

$$\mathcal{M}(\overline{\mathbb{C}})^G_\Gamma = \left\{ f \in \mathcal{M}(\overline{\mathbb{C}})_\Gamma \mid g \cdot f = f, \; \forall g \in G \right\}$$

Example

If a dihedral group $\mathbb{D}_N = \langle r, s \mid r^N = s^2 = (rs)^2 = 1 \rangle$ acts on $\lambda \in \overline{\mathbb{C}}$ by

$$r \cdot \lambda = \omega \lambda \,, \quad \omega^N = 1 \,; \qquad s \cdot \lambda = \frac{1}{\lambda}$$



and if one considers the $\mathbb{D}_N$-orbit $\Gamma = \{0, \infty\} \subset \overline{\mathbb{C}}$, then all automorphic functions are polynomials in

$$\mathbb{I} = \lambda^N + 2 + \lambda^{-N}$$

i.e.

$$\mathcal{M}(\overline{\mathbb{C}})^{\mathbb{D}_N}_\Gamma = \mathbb{C}[\mathbb{I}].$$

G-action on $\mathfrak{g}$

Let V be a G-module. A G-action on V induces a G-action on $V \otimes V^* \cong \mathrm{End}(V)$ corresponding to conjugation.

Example

If V is a 2-dimensional $\mathbb{D}_N$-representation having a basis such that

$$\tau(r) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \qquad \tau(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then the action on $\mathrm{End}(V)$ reads

$$r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \tau(r) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau(r)^{-1} = \begin{pmatrix} a & \omega^2 b \\ \omega^{-2} c & d \end{pmatrix}, \qquad s \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Notice that

$$\mathfrak{gl}_2^{\mathbb{D}_N} = \Bbbk \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \mathfrak{sl}_2^{\mathbb{D}_N} = 0, \quad \omega^2 \neq 1.$$

Classification problem

The goal is the classification of Lie algebras $\left(\mathfrak{g} \otimes \mathcal{M}(\overline{\mathbb{C}})\right)_{\Gamma}^{G}$ where

$$G < \mathrm{Aut}(\overline{\mathbb{C}}), \quad \Gamma \in \overline{\mathbb{C}}/G, \quad \mathfrak{g} < \mathfrak{gl}(V) \text{ simple.}$$

This relies on the classical classifications of

Finite subgroups of $\mathrm{Aut}(\overline{\mathbb{C}})$

$$\mathbb{Z}/N, \quad \mathbb{D}_N, \quad \mathbb{T}, \quad \mathbb{O}, \quad \mathbb{Y}.$$

Related to each group there is a finite list of orbits Γ_i and fixed-point free G-action on $\mathfrak{g}$.

Root Systems - $\mathbb{TOY}$

$$A_1 - A_5, \quad B_2 - B_7, \quad C_2 - C_7, \quad D_3 - D_6, D_8, \quad E_6 - E_8, \quad F_4, \quad G_2$$

History and Motivation

Zakharov, Shabat, '74	Reductions of *Lax pairs*
Mikhailov, '79-'81	*Reduction Group* in the classification of integrable systems in $1+1$ dim.
Drinfeld, Sokolov, '85	Lie Algebras and equations of KdV type.
S L, Mikhailov, '04-'05	*Automorphic Lie Algebras* (ALiAs); first examples and related PDEs.
S L, Sanders, '10	Invariant theory $\lambda = \frac{X}{Y}$; classification of $\mathfrak{sl}_2(\mathbb{C})$-based ALiAs using *Chevalley normal forms*.
Bury, Mikhailov, '10	Classification of $\mathfrak{sl}_2(\mathbb{C})$-based ALiAs integrable PDEs (coupled systems).
Chopp, Schlichenmaier, '11	start to replace $\overline{\mathbb{C}}$ by arbitrary compact Riemann surface.
Knibbeler, S L, Sanders, '14	ALiAs with $\mathbb{D}_N$ symmetry in normal form. Method to treat all poles at once.
Knibbeler, '14	*Invariants of ALiAs*.
Knibbeler, S L, Sanders, '15	Classification of ALiAs for inner auts ($\mathfrak{g} = \mathfrak{sl}_n$)
Knibbeler, S L, Sanders, '15	ALiAs & Root System Cohomology

ALiAs classification

V. Knibbeler, S. Lombardo, and J.A. Sanders
Higher dimensional Automorphic Lie Algebras
JoFoCM, 1–49, 2016.

Presentation of the Reduction Group

$$G = \langle g_v, g_f, g_e \mid {g_v}^{n_v} = {g_f}^{n_f} = {g_e}^{n_e} = g_v g_f g_e = 1 \rangle$$
$$\mathbb{O} = \langle g_v, g_f, g_e \mid {g_v}^{4} = {g_f}^{3} = {g_e}^{2} = g_v g_f g_e = 1 \rangle$$

$\Gamma_v = \{\text{vertices}\} = \{\text{red dots}\}$,
$\Gamma_f = \{\text{mid's of faces}\} = \{\text{green dots}\}$,
$\Gamma_e = \{\text{mid's of edges}\} = \{\text{blue dots}\}$,
$|\mathbb{O}| = 24,\ |\Gamma_v| = 6,\ |\Gamma_f| = 8\ |\Gamma_e| = 12$,

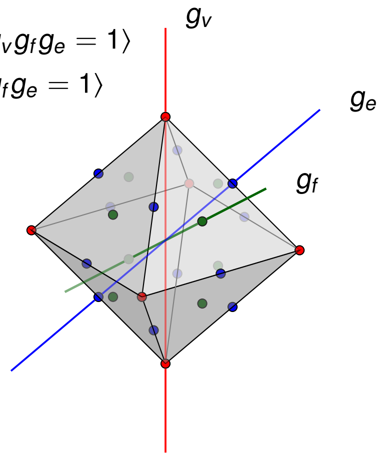


Figure: Octahedron

The Automorphic functions $\mathbb{I}_i = F_{\Gamma_i}^{\nu_{\Gamma_i}} / F_{\Gamma}^{\nu_{\Gamma}}$

$$\mathbb{I}_v, \mathbb{I}_e, \mathbb{I}_f \in \mathcal{M}(\overline{\mathbb{C}})^G_\Gamma$$

are defined by $\mathbb{I}_i = 1$ if $\Gamma = \Gamma_i$ and by

$$\mathbb{I}_i(\lambda) = 0 \Leftrightarrow \lambda \in \Gamma_i \neq \Gamma$$

Invariants of Automorphic Lie Algebras

1. Number of matrices is dim $\mathfrak{g}$.
2. Power of $\mathbb{I}_i$ in each matrix *and* in each structure constant is 0 or 1.
3. Total number of $\mathbb{I}_i$ appearing in the matrices of invariants is in the Table.

Φ	$\mathbf{A_1}$	A_2	B_2	A_3	C_3	A_4	A_5
κ_v	**1**	3	4	6	8	10	14
κ_f	**1**	3	3	5	7	8	12
κ_e	**1**	2	3	4	6	6	9
Σ	**3**	8	10	15	21	24	35

$$A_\ell - \mathfrak{sl}_{\ell+1}(\mathbb{C})$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & \mathbb{I}_e\mathbb{I}_f \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ \mathbb{I}_v & 0 \end{pmatrix}.$$

V. Knibbeler
Invariants of Automorphic Lie Algebras
http://arxiv.org/abs/1504.03616, 2014.

The integers $1/2 \operatorname{codim} \mathfrak{g}^{\langle g_i \rangle}$, $i \in \{v, e, f\}$, by the root system Φ of $\mathfrak{g}$.

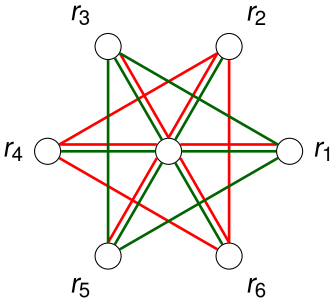
Φ	A_1	A_2	A_3	A_4	A_5	B_2/C_2	B_3/C_3	B_4/C_4	B_5/C_5	B_6/C_6	B_7/C_7
κ_v	1	3	6	10	14	4	8	14	22	31	42
κ_f	1	3	5	8	12	3	7	12	18	26	35
κ_e	1	2	4	6	9	3	6	10	15	21	28
Σ	3	8	15	24	35	10	21	36	55	78	105

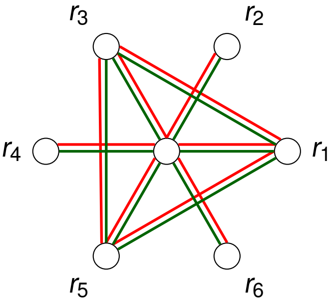
Φ	D_3	D_4	D_5	D_6	D_8
κ_v	6	11	18	26	48
κ_f	5	9	15	22	40
κ_e	4	8	12	18	32
Σ	15	28	45	66	120

Φ	E_6	E_7	E_8	F_4	G_2
κ_v	31	53	100	20	5
κ_f	27	45	84	18	5
κ_e	20	35	64	14	4
$\Sigma/\dim \mathfrak{g}$	78	133	248	52	14

The last table suggests the existence of a fixed-point-free G-action by inner automorphisms of $\mathfrak{g}$, where G is one of the $\mathbb{TOY}$ groups.

Root System Cohomology





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arXiv:1512.07020, 2015.

Root System Cohomology

Let $q \in \mathbb{N}$; let Φ be a RS of rank ℓ of a simple Lie algebra $\mathfrak{g}$, and $\Phi_0 = \Phi \cup \{0\}$.

1-chains: $C_1(\Phi) = \mathbb{Z}\langle \Phi_0 \rangle$

2-chains: $C_2(\Phi) = \mathbb{Z}\langle (\alpha, \beta) \in \Phi_0^2 \mid \alpha + \beta \in \Phi_0 \rangle\,, \quad \Phi_0^m = \Phi^m \cup \{0\}$

$\dots$

$C_m(\Phi) = \mathbb{Z}\langle (\alpha_1, \dots, \alpha_m) \in \Phi_0^m \mid (\alpha_1, \dots, \alpha_j + \alpha_{j+1}, \dots, \alpha_m) \in C_{m-1}(\Phi), 1 \leq j < m \rangle$

Dually, we define *m-cochains* by

$$C^m(\Phi, \mathbb{Z}^q) = \mathrm{Hom}(C_m(\Phi), \mathbb{Z}^q).$$

One can then define $\mathrm{d}^m : C^m(\Phi, \mathbb{Z}^q) \to C^{m+1}(\Phi, \mathbb{Z}^q)$ in the usual manner

$$\mathrm{d}^0 \omega^0(\alpha_0) = 0$$
$$\mathrm{d}^1 \omega^1(\alpha_0, \alpha_1) = \omega^1(\alpha_1) - \omega^1(\alpha_0 + \alpha_1) + \omega^1(\alpha_0)$$
$$\mathrm{d}^2 \omega^2(\alpha_0, \alpha_1, \alpha_2) = \omega^2(\alpha_1, \alpha_2) - \omega^2(\alpha_0 + \alpha_1, \alpha_2) + \omega^2(\alpha_0, \alpha_1 + \alpha_2) - \omega^2(\alpha_0, \alpha_1)$$
$\dots$
$$\mathrm{d}^m \omega^i(\alpha_0, \dots, \alpha_m) = \omega^m(\alpha_1, \dots, \alpha_m) + \sum_{j=1}^m (-1)^j \omega^m(\alpha_0, \dots, \alpha_{j-1} + \alpha_j, \cdots, \alpha_m) + -(-1)^m \omega^m(\alpha_0, \dots, \alpha_{m-1}).$$

Application of Root System Cohomology

Theorem

Let Φ *be a root system with basis* Δ*. If* $\omega_+^2 \in C^2(\Phi, \mathbb{N}_0^q)$ *satisfies*

$$d^2\omega_+^2 = 0, \qquad \omega_+^2(\alpha,\beta) = \omega_+^2(\beta,\alpha),$$

then the free $\mathbb{C}[\mathbb{I}_1, .., \mathbb{I}_q]$*-module with generators* $\{h_r, e_\alpha \mid 1 \le r \le \ell,\ \alpha \in \Phi,\}$ *and* $\mathbb{C}[\mathbb{I}_1, .., \mathbb{I}_q]$*-linear Lie bracket*

$$\begin{aligned} [h_r, h_s] &= 0 && \text{if } h_r, h_s \in \mathfrak{h} \\ [h_r, e_\alpha] &= \alpha(h_r)e_\alpha && \text{if } \in \mathfrak{h} \end{aligned}$$

$$[e_\alpha, e_\beta] = \begin{cases} \epsilon(\alpha,\beta)\mathbb{I}^{\omega_+^2(\alpha,\beta)} e_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \\ \epsilon(\alpha,-\alpha)\mathbb{I}^{\omega_+^2(\alpha,-\alpha)} h_\alpha & \text{if } \alpha+\beta = 0, \\ 0 & \text{if } \alpha+\beta \notin \Phi_0 \end{cases}$$

where ϵ *is an antisymmetric 2-form, is a Lie algebra, denoted as* $\mathcal{L}_{d^1\omega^1}(\Phi)$*.*

We use a multi-index notation $\mathbb{I}^{\omega_+^2(\alpha,\beta)} = \prod_{i\in\{v,e,f\}} \mathbb{I}_i^{\omega_+^2(\alpha,\beta)_i}$.
The Jacobi identity is equivalent to $d^2\omega_+^2 = 0$.

Root System Cohomology

The previous theorem essentially states that any symmetric 2-cocycle ω^2 determines a Lie algebra with monomial coefficients.

Moreover, a 1-cochain ω^1 determines a representation.

Indeed consider generators of the form $\mathbb{I}^{\omega^1(\alpha)} e_\alpha$. Then

$$[\mathbb{I}^{\omega^1(\alpha)} e_\alpha, \mathbb{I}^{\omega^1(\beta)} e_\beta] = \mathbb{I}^{\omega^1(\alpha)} \mathbb{I}^{\omega^1(\beta)} [e_\alpha, e_\beta] = \mathbb{I}^{\omega^1(\alpha)+\omega^1(\beta)-\omega^1(\alpha+\beta)} (\mathbb{I}^{\omega^1(\alpha+\beta)} e_{\alpha+\beta}) =$$

$$= \mathbb{I}^{\mathrm{d}^1\omega^1(\alpha,\beta)} (\mathbb{I}^{\omega^1(\alpha+\beta)} e_{\alpha+\beta}).$$

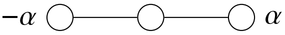
Let $\omega^1 \in C^1(\Phi, \mathbb{Z}^q)$ and let

$$\omega_+^2(\alpha,\beta) = \mathrm{d}^1\omega^1(\alpha,\beta) = \omega^1(\alpha) + \omega^1(\beta) - \omega^1(\alpha+\beta)\,.$$

We say that ω^1 is a *model* for ω_+^2.

The root system A_1

The root system A_1



Δ	$\mathbf{A_1}$	A_2	B_2	A_3	C_3	A_4	A_5
κ_v	**1**	3	4	6	8	10	14
κ_f	**1**	3	3	5	7	8	12
κ_e	**1**	2	3	4	6	6	9
Σ	**3**	8	10	15	21	24	35

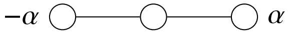
Invariant 1 and 2:

$$\omega^1 : A_1 \to \{0, 1\}$$

Invariant 3:

$$\sum_{\alpha \in A_1} \omega^1(\alpha) = 1$$

The root system A_1



There are only two maps ω^1 satisfying conditions

$$\omega^1(0) = 0 \quad \text{and} \quad \omega^1(-\alpha) + \omega^1(0) + \omega^1(\alpha) = 1\,.$$

Either $\omega^1(-\alpha) = 1$ or $\omega^1(\alpha) = 1$ and the other values are zero. Both of them map to the same 2-coboundary

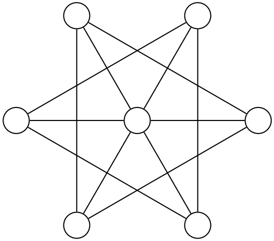
$$\mathrm{d}^1\omega^1(-\alpha, \alpha) = 1$$

Hence, if $\dim V = 2$, then

$$\begin{aligned} \overline{\mathfrak{sl}_{2}}_{\Gamma}^{G} &\cong \mathbb{C}[\mathbb{I}]\,(h, e_+, e_-) \\ [h, e_\pm] &= \pm 2e_\pm \\ [e_+, e_-] &= \mathbb{I}_v\mathbb{I}_f\mathbb{I}_e h \end{aligned}$$

The root system A_2

Basis of 2-chains $\{(\alpha, \beta) \in A_2 \mid \alpha + \beta \in A_2 \cup \{0\}\} \subset C_2(A_2)$.



Δ	A_1	$\mathbf{A_2}$	B_2	A_3	C_3	A_4	A_5
κ_v	1	**3**	4	6	8	10	14
κ_f	1	**3**	3	5	7	8	12
κ_e	1	**2**	3	4	6	6	9
Σ	3	**8**	10	15	21	24	35

The 2-coboundaries $d^1\omega^1$ on A_2 satisfying the invariants

Invariant 1 and 2:

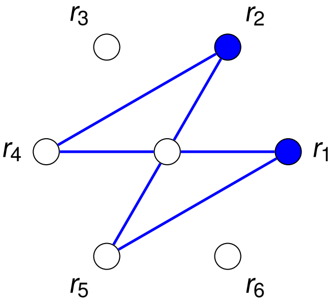
$$\omega^1 : A_2 \to \{0, 1\}$$

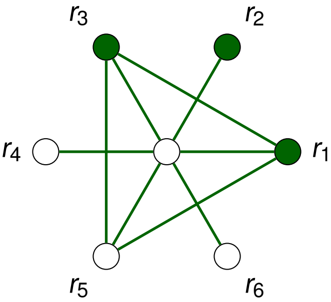
Invariant 3:

$$\sum_{\alpha \in A_2} \omega^1(\alpha) = 2 \text{ or } 3$$

Lie algebra:

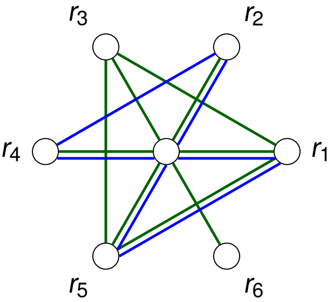
$$d^1\omega^1(\alpha,\beta) = \omega^1(\alpha) - \omega^1(\alpha+\beta) + \omega^1(\beta) \geq 0.$$





The two smallest pole-orbits

The 2-coboundary $\mathrm{d}^1\omega^1 \in B^2(A_2, \mathbb{N}_0^2)$ where $\sum_{A_2} \omega^1(\alpha) = (3, 2)$.

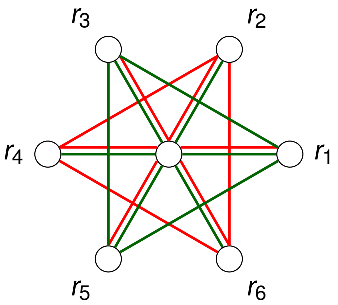


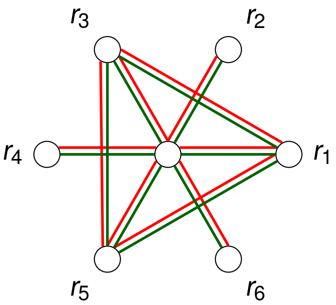
Any $\mathfrak{sl}_3(\mathbb{C})$-based Automorphic Lie Algebra with poles at one of the two smallest orbits, Γ_v or Γ_f, is isomorphic to $\mathcal{L}_{\mathrm{d}^1\omega^1}(A_2)$, where $\mathrm{d}^1\omega^1$ is as depicted.

The largest exceptional orbit

Δ	A_1	$\mathbf{A_2}$	B_2	A_3	C_3	A_4	A_5
κ_v	1	**3**	4	6	8	10	14
κ_f	1	**3**	3	5	7	8	12
κ_e	1	**2**	3	4	6	6	9
Σ	3	**8**	10	15	21	24	35

The two 2-coboundaries $d^1\omega^1 \in B^2(A_2, \mathbb{N}_0^2)$ where $\sum_{A_2} \omega^1(\alpha) = (3,3)$.





Second cohomology group $H^2_+(\Phi, \mathbb{Z}^q)$

One of the fundamental questions is thus whether there is always a model.

This is equivalent to the question whether the second cohomology group $H^2_+(\Phi, \mathbb{Z}^q)$ is trivial.

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The second cohomology group has an obvious interpretation in terms of Lie algebras over graded rings and their representations: it measures the amount of such Lie algebras that do not allow a representation given by a 1-cochain in the canonical way described.

The proof that $H^2_+(\Phi, \mathbb{Z}^q)$ is trivial is entirely constructive, so it also provides an integration procedure, allowing one to find a model from the given ALiA.

Root System Cohomology & Real Lie Algebras

Theorem (Kac (1969), as in Fuchs and Schweigert,1997)

"The finite-dimensional semisimple real Lie algebras are in one-to-one correspondence with the pairs ($\mathfrak{g}$, ω^1), where $\mathfrak{g}$, is a finite-dimensional semisimple complex Lie algebra and ω^1 an involutive automorphism of $\mathfrak{g}$".

We observed that $\max_{\omega^1 \in Z^1(\Phi, \mathbb{Z}/\nu)} \mathrm{codim} \ \ker(\omega^1) = 2\kappa_\nu(\Phi)$; this implies that there is a functional ω^1 related to real Lie algebras with a parabolic part of dimension $2\kappa_\nu(\Phi)$.

The integers $1/2 \, \mathrm{codim} \, \mathfrak{g}^{\langle g_i \rangle}$, $i \in \{v, e, f\}$, by the root system Φ of $\mathfrak{g}$.

Φ	A_1	A_2	A_3	A_4	A_5	B_2/C_2	B_3/C_3	B_4/C_4	B_5/C_5	B_6/C_6	B_7/C_7	$\cdots$	G_2
κ_v	1	3	6	10	14	4	8	14	22	31	42	$\cdots$	5
κ_f	1	3	5	8	12	3	7	12	18	26	35	$\cdots$	5
κ_e	1	2	4	6	9	3	6	10	15	21	28	$\cdots$	4
Σ	3	8	15	24	35	10	21	36	55	78	105	$\cdots$	14

Real Lie Algebras $\mathfrak{g}_2^\star = G_{2(2)}$

From the Table one observes that $\kappa_e = 4$, so the dimension of the parabolic part of this Lie algebra is 8. The compact complement has dimension $6 = 14 - 8$ and it is equal to $\mathfrak{su}_2 \oplus \mathfrak{su}_2$.

ALiAs & real Lie algebras

Let the model for $(\mathfrak{sl}_6 \otimes \mathcal{M}(\overline{\mathbb{C}}))^G_{\mathfrak{a}}$ be

$$\|A_5^{(12,9)}\| = \begin{bmatrix} 0 & 1 & \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} \\ 1 & 0 & \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} \\ 1 & 1 & 0 & 1 & \mathbb{I} & \mathbb{I} \\ \mathbb{J} & \mathbb{J} & \mathbb{J} & 0 & \mathbb{I} & \mathbb{I} \\ \mathbb{J} & \mathbb{J} & \mathbb{J} & 1 & 0 & 1 \\ \mathbb{J} & \mathbb{J} & \mathbb{J} & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{K}(\mathfrak{sl}_6)_{\mathfrak{a}} = 2 + 4\mathbb{I} + \mathbb{J} + 8\mathbb{I}\mathbb{J}\,.$$

real Lie algebras $A_{5(1)}$

From the Table one has that $\kappa_e = 9$, so the dimension of the parabolic part of this Lie algebra is 18. The compact complement has dimension $17 = 35 - 18$ and it is equal to $\mathfrak{su}_3 \oplus \mathbb{C} \oplus \mathfrak{su}_3$:

$$\begin{bmatrix} 0 & 1 & 1 & \mathbb{J} & \mathbb{J} & \mathbb{J} \\ 1 & 0 & 1 & \mathbb{J} & \mathbb{J} & \mathbb{J} \\ 1 & 1 & & \mathbb{J} & \mathbb{J} & \mathbb{J} \\ \mathbb{J} & \mathbb{J} & \mathbb{J} & 0 & 1 & 1 \\ \mathbb{J} & \mathbb{J} & \mathbb{J} & 1 & 0 & 1 \\ \mathbb{J} & \mathbb{J} & \mathbb{J} & 1 & 1 & 0 \end{bmatrix}$$

$$\mathfrak{su}(3,3) = \mathfrak{t} \oplus \mathfrak{p} = \mathfrak{su}_3 \oplus \mathbb{C} \oplus \mathfrak{su}_3 \oplus \mathfrak{p}\,.$$

Platonic Lie Algebras (work in progress)

In the ALiAs case $\dim\mathfrak{g} = \kappa_V + \kappa_f + \kappa_e$; inspired by this, we say that $\mathfrak{g}$ is a Platonic Lie algebra iff

$$\sum_{\nu=2,3,5} \max_{\omega^1 \in Z^1(\Phi, \mathbb{Z}/\nu)} \mathrm{codim}\,\ker(\omega^1) = \dim\mathfrak{g}\,.$$

Conjecture

If $\mathfrak{g}$ is a *Platonic Lie algebra* there is at least one *fixed-point-free* action of one or more of the $\mathbb{TOY}$ groups.

The conjecture holds true if $\mathfrak{g}$ is a classical Lie algebra.

Platonic Root Systems

$\mathbf{A_1} - \mathbf{A_5}$, $B_2 - B_7$, $C_2 - C_7$, $D_3 - D_6, D_8$, $E_6 - E_8$, F_4, G_2

Summary

- ALiAs with $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $n = 1, 2, 3, 4, 5$ and where the G-action is realised by inner automorphisms using irreducible G-representations are completely classified and written in Chevalley normal forms.
  In all cases, there exists a CSA where all elements have constant eigenvalues. We conjecture the existence of such a CSA in the inner case. This is not the case in the outer case.
- We are extending the $\mathfrak{sl}_{n+1}$ classification replacing the irreducibility with the fixed-point-free action requirement; we aim to classify all ALiAs based on

$$\underbrace{\mathbf{A_1} - \mathbf{A_5}}_{\text{classified}}, \quad \mathbf{B_2} - B_7, \quad \mathbf{C_2}, \mathbf{C_3} - C_7, \quad D_3 - D_6, D_8, \quad E_6 - E_8, \; F_4, \; G_2$$

  where the G-action is realised by inner automorphisms (in progress).
- ALiAs invariants (see V Knibbeler, 2014) leads to a formulation of ALiAs in terms of Root System Cohomology. More generally, this theory might provide an interesting way to study Lie algebras over graded rings.

Summary

- The result by Kac (1969) on the classification of real Lie algebras can be reformulated in terms of Root System Cohomology over $\mathbb{Z}/2$ as follows:

  for every $\omega^1 \in Z^1(\Phi, \mathbb{Z}/2)$ one can find a real Lie algebra and for every non-split real Lie algebra one can find at least one ω^1.

- Given an ALiAs "model" (that is, given a Chevalley normal form after generalised Weyl transformations) where the poles are at either one of the two smallest exceptional G-orbits, one can construct a non-split real Lie algebra with maximal parabolic part (see the A_5 example).

Thank you!

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