V-systems

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Plan of the talk

1. V-systems; equivalent formulations
2. Operations with V-systems
3. Examples
4. Harmonic V-systems

Let $V \cong \mathbb{C}^n$. Let $\mathcal{A} \subset V^*$ be a finite set of non-collinear covectors. Define B a bilinear form on V by

$$B(u,v) = \sum_{\alpha\in\mathcal{A}} \alpha(u)\alpha(v)$$

We assume B is non-degenerate.
Then $V \cong V^*$: $\alpha \in V^*$ corresponds to $\alpha^\vee \in V$ s.t. $B(\alpha^\vee, u) = \alpha(u)$ for any $u \in V$.

Definition (Veselov'99)

$\mathcal{A}$ is a $\vee$-system if for any $\alpha \in \mathcal{A}$, $\pi \subset V^*$, $\dim \pi = 2$

$$\sum_{\beta\in\mathcal{A}\cap\pi} \beta(\alpha^\vee)\beta = \nu\alpha$$

for some $\nu = \nu(\alpha, \pi) \in \mathbb{C}$.

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Equivalently,

- if $\pi \cap \mathcal{A} = \{\alpha, \beta\}$ then $B(\alpha^\vee, \beta^\vee) = 0$
- if $|\pi \cap \mathcal{A}| > 2$ then $B_\pi|_{\pi^\vee \times V} = \nu B|_{\pi^\vee \times V}$, where $B_\pi(u, v) = \sum_{\beta\in\mathcal{A}\cap\pi} \beta(u)\beta(v)$, $\nu = \nu(\pi)$.

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Witten–Dijkgraaf–Verlinde–Verlinde equations

Theorem (Veselov'99,01; FV'08)

$\mathcal{A}$ is a $\vee$-system if and only if

$$\mathcal{F}(x) = \sum_{\alpha \in \mathcal{A}} \alpha(x)^2 \log \alpha(x), \quad x \in V$$

satisfies WDVV equations

$$\mathcal{F}_i G^{-1} \mathcal{F}_j = \mathcal{F}_j G^{-1} \mathcal{F}_i$$

for any $i, j = 1, \ldots, n$, where $\mathcal{F}_i$ is $n \times n$ matrix, $(\mathcal{F}_i)_{kl} = \frac{\partial^3 \mathcal{F}}{\partial x_i \partial x_k \partial x_l}$, $G = \sum_{i=1}^n x_i \mathcal{F}_i$.

Associative multipliciation

Let $\Sigma = \cup_{\alpha\in\mathcal{A}}\{x : \alpha(x) = 0\}$.
Let $x \in V_\Sigma := V \setminus \Sigma$. Let $u, v \in T_x V_\Sigma \cong V$. Define

$$u \star v = \sum_{\alpha\in\mathcal{A}} \frac{\alpha(u)\alpha(v)}{\alpha(x)}\alpha^\vee.$$

Theorem (FV'08)

$\mathcal{A}$ is a $\vee$-system if and only if $\star$ is associative.

Flat connection

Define connection ∇ on TV_Σ by

$$\nabla_\xi = \partial_\xi - \kappa \sum_{\alpha \in \mathcal{A}} \frac{\alpha(\xi)}{\alpha(x)} \alpha \otimes \alpha^\vee,$$

where $\xi \in V$, $\kappa \in \mathbb{C}^*$.

Theorem (Veselov'01; Arsie, Lorenzoni'14, FV'14)

∇ is flat if and only if $\mathcal{A}$ is a $\vee$-system.

Example (Veselov'99)

Let R be a Coxeter root system in $\mathbb{R}^n$. That is

- $s_\alpha R = R$ for any $\alpha \in R$, where s_α is orthogonal reflection about the hyperplane $(\alpha, x) = 0$.
- If $\alpha, \beta \in R$ are proportional then $\alpha = \pm\beta$.

Then $\mathcal{A} = R_+$ is a $\vee$-system.

Origin and relations

- Generalized Calogero–Moser systems, generalised root systems and their deformations [Chalykh, F, Sergeev, Veselov'98-07]
- Seiberg–Witten theory [Marshakov, Mironov, Morozov '97], [Martini, Gragert'99]
- Dubrovin's almost duality [Dubrovin'03]. For $\mathcal{A} = R$ - a Coxeter root system $\mathcal{F}$ is almost dual prepotential, $\star$ is almost dual product.

Subsystems

Let $\mathcal{A}$ be a $\vee$-system, let $W \subset V^*$ be a linear subspace. Define

$$\mathcal{A}_W = \mathcal{A} \cap W.$$

Assume that $\langle \mathcal{A}_W \rangle = W$. Define bilinear form

$$B_W(u,v) = \sum_{\beta \in \mathcal{A}_w} \beta(u)\beta(v).$$

Theorem (F, Veselov'08)

$\mathcal{A}_W$ is a $\vee$-system if B_W is non-degenerate on $W^\vee \times W^\vee$.

Restrictions

Let $\mathcal{A}$ be a $\vee$-system, $\mathcal{A}_W = \mathcal{A} \cap W$, $W \subset V^*$, $\langle \mathcal{A}_W \rangle = W$. Define

$$\widehat{W} = \{x \in V : \alpha(x) = 0 \, \forall \alpha \in \mathcal{A}_W\}.$$

Theorem (F, Veselov'07,08)

$\mathcal{A} \setminus \mathcal{A}_W \subset \widehat{W}^*$ is a $\vee$-system if B is non-degenerate on $\widehat{W} \times \widehat{W}$.

Classical families [Chalykh, Veselov'01]:

$$\mathcal{A}_n(c) = \{c_i c_j (e_i - e_j) : 1 \le i < j \le n+1\},$$

where $c_1, \dots, c_{n+1} \in \mathbb{C}$;

$$\mathcal{B}_n(c) = \{(c_i c_j)^{1/2}(e_i \pm e_j) : 1 \le i < j \le n\} \cup$$

$$\{(2c_i(c_i + c_0))^{1/2} e_i : 1 \le i \le n\}$$

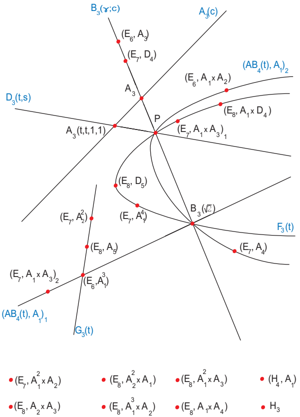
where $c_0, c_1, \dots, c_n \in \mathbb{C}$.

Exceptional families and single systems, e.g.

$$F_3(t) = \{e_i \pm e_j : 1 \le i < j \le 3\} \cup \{(4t^2 + 2)^{1/2} e_i : i = 1, 2, 3\} \cup$$

$$\{t\sqrt{2}(e_1 \pm e_2 \pm e_3)\}$$

Known $\vee$-systems in dimension 3



Theorem (Lechtenfeld, Schwerdtfeger, Thueringen'11)

There are no other 3-dimensional $\vee$-systems with not more than 10 vectors.

Theorem (Schreiber, Veselov'14)

There are no deformations of known isolated 3-dimensional $\vee$-systems preserving the underlying matroid.

Let $\psi(x)$ be a flat section of ∇: for some $\kappa \in \mathbb{C}$ $\nabla_\xi \psi = 0$ for any $\xi \in V$.

Theorem (F, Veselov'14)

Suppose that $\psi(x)$ is polynomial. Then

1. *ψ is gradient, that is $\psi = (dF)^\vee$ for some polynomial $F(x)$.*
2. *ψ is homogeneous of degree κ.*
3. *ψ is a logarithmic vector field that is $\alpha(\psi) = 0$ if $\alpha(x) = 0$ for any $\alpha \in \mathcal{A}$.*

Definition (F, Veselov'14)

A $\vee$-system $\mathcal{A}$ is called *harmonic* if there exist $n = \dim V$ independent (over polynomials) polynomial flat vector fields of degrees $\kappa_1, \ldots, \kappa_n$ such that $\sum_{i=1}^n \kappa_i = |\mathcal{A}|$.

Remark

For any n independent polynomial flat fields $\sum \kappa_i \geq |\mathcal{A}|$.

Remark

Potentials F_i satisfy a system of 2nd order PDEs of Euler–Poisson–Darboux type $\partial_\xi \partial_\eta F_i = \kappa_i \sum_{\alpha \in \mathcal{A}} \frac{\alpha(\xi)\alpha(\eta)}{\alpha(x)} \partial_{\alpha^\vee} F_i$, $\forall \xi, \eta \in V$.

Theorem (F, Veselov'14)

$\mathcal{A} = R_+$ *is harmonic for any Coxeter root system R. If all the roots have the same length then potentials $F_1, \ldots, F_n$ are Saito flat coordinates.*

B^{-1} is invariant with respect to Coxeter group $G = \langle s_\alpha : \alpha \in \mathcal{A} \rangle$, $(SV^*)^G \cong \mathbb{C}[y_1, \ldots, y_n]$, $\deg y_1 \leq \ldots \leq \deg y_n$. Then $\partial_{y_n} B^{-1}$ is flat Saito metric, constant if $y_i = F_i$.

If Coxeter roots have two different lengths then we get explicit one-parameter deformations of Saito polynomials.

Free arrangements of hyperplanes

Let $\Sigma = \cup_{\alpha \in \mathcal{A}}\{\alpha(x) = 0\} \subset V$. Let $Der(\log \Sigma)$ be the space of polynomial logarithmic vector fields v that is $\alpha(v) = 0$ if $\alpha(x) = 0$ for any $\alpha \in \mathcal{A}$. Then $Der(\log \Sigma)$ is a module over SV^*.

Definition (K. Saito'80)

Arrangement Σ is *free* if $Der(\log \Sigma)$ is a free module over SV^*.

Example (Orlik, Terao'93)

Coxeter arrangements and their restrictions are free.

Theorem (Saito criterion)

Arrangement Σ is free if and only if there exist independent over SV^* fields $X_1, \dots, X_n \in Der(\log \Sigma)$ homogeneous of degrees $b_1, \dots, b_n$ such that $\sum b_i = |\Sigma|$.

Conjecture (Terao)

Freeness is a combinatorial property that is it is a property of the lattice of Σ.

Theorem (Terao'81)

Suppose Σ *is free. Then Poincare polynomial* $P_{V\setminus\Sigma}(t) = \sum_{i=0}^{n} \dim H^i(V\setminus\Sigma, \mathbb{C})t^i$ *has the form* $P_{V\setminus\Sigma}(t) = \prod_{i=1}^{n}(1+b_i t)$ *for some* $b_i \in \mathbb{N}$.

Theorem (F,Veselov'14)

If $\vee$*-system* $\mathcal{A}$ *is harmonic then arrangement* Σ *is free. The corresponding flat vector fields* ψ_i *give a free basis in* $Der(\log\Sigma)$.

Remark

All the known $\vee$*-systems have corresponding arrangements linearly equivalent to Coxeter restrictions.*

Potentials for classical families

Theorem (F, Veselov'14)

$A_n(c)$ is harmonic with $F_\kappa(x_1, \ldots, x_{n+1}) = \oint \prod_{i=1}^{n+1} (x - x_i)^{\frac{\kappa c_i}{\sigma}} dx$, $\sigma = \sum c_i, \kappa = 1, 2, \ldots, n$.

$$F_\kappa \sim \det \begin{pmatrix} p_1^\lambda & 1 & 0 & 0 \ldots & 0 \\ p_2^\lambda & p_1^\lambda & 2 & 0 \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_\kappa^\lambda & p_{\kappa-1}^\lambda & p_{\kappa-2}^\lambda & \ldots & \kappa \\ p_{\kappa+1}^\lambda & p_\kappa^\lambda & p_{\kappa-1}^\lambda & \ldots & p_1^\lambda \end{pmatrix},$$

$p_s^\lambda = \sum \lambda_i x_i^s$, $\lambda_i = \frac{\kappa c_i}{\sigma}$.

Theorem (F, Veselov'14)

$B_n(c)$ is harmonic if $c_i + c_0 \neq 0$ for all i with

$$F_k(x_1, \dots, x_n) = \oint \prod_{i=1}^{n+1} (x^2 - x_i^2)^{\frac{(2k-1)c_i}{2\sigma}} x^{\frac{2k-1}{\sigma} c_0} dx,$$

$\sigma = \sum c_i, \kappa = 2k-1, k = 1, 2, \dots, n.$

$$F_k \sim \det \begin{pmatrix} q_1^\lambda & 1 & 0 & 0\dots & 0 \\ q_2^\lambda & q_1^\lambda & 2 & 0\dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_\kappa^\lambda & q_{\kappa-1}^\lambda & q_{\kappa-2}^\lambda & \dots & k-1 \\ q_{\kappa+1}^\lambda & q_\kappa^\lambda & q_{\kappa-1}^\lambda & \dots & q_1^\lambda \end{pmatrix},$$

$q_s^\lambda = \sum \lambda_i x_i^{2s}$, $\lambda_i = \frac{(2k-1)c_i}{2\sigma}$.

Remark

Assumption $c_i + c_0 \neq 0$ *is essential as e.g.* $B_3(-1, 1, 1, 3)$ *is not harmonic.*

Furher questions

- Classification of $\vee$-systems.
- 'More Frobenius manifolds structures' associated with harmonic $\vee$-systems ?
- Relation of generalised Saito polynomials (potentials of harmonic $\vee$-systems) to special representations of rational Cherednik algebras (cf. [F, Silantyev'12]) ?
- Trigonometric [F'08] and Elliptic [Strachan'08] $\vee$-systems.