Regularity for jump equations using an interpolation method

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The Equation :

$$X_t(x) = x + \int_0^t \int_R c(z, X_{s-}(x))\widetilde{N}_\mu(ds, dz)$$

with

$$d\widetilde{N}_\mu = dN_\mu - d\widehat{N}_\mu \quad \widehat{N}_\mu(ds, dz) = ds\mu(dz).$$

The Problem : give sufficient conditions in order to get

$$P(X_t(x) \in dy) = p_t(x, y)dy$$

Three approaches :

1. Jump amplitudes :

$$\mu(dz) = h(z)dz \quad h \in C^\infty(R)$$

Malliavin type calculus based on $h(z)$. Bismut, Leandre, Bichteler Gravereux Jacod, Bouleau Denis ....

2. Jump times : Mall. calc. based on "$T_k - T_{k-1}$" : Carlen Pardoux (B Clément).

3. Mall. Calc. with "Finite differences" **Picard**, Kunita, Ischikawa ...

Sector condition : $\exists\theta$ such that

$$C\varepsilon^{2-\theta} \geq \int_{|z|\leq\varepsilon} |z|^2 \, d\mu(z) \geq \frac{1}{C}\varepsilon^{2-\theta}.$$

One also assumes "ellipticity" and "regularity" for the coefficients and proves "regularity" of the law

$$P(X_t(x) \in dy) = p_t(x,y)dy, \quad p_t \in C^\infty(R \times R),$$

and "short time behaviour"

$$|p_t(x,y)| \leq \frac{C}{t^{d/\theta}}.$$

Heuristics (Picard) "small jumps" produce sufficient noise in order to regularize "as in the CLT"

Our approach :

Step 1. Replace small jumps by a Brownian motion :

$$X_t^\varepsilon(x) = x + \int_0^t \sigma_\varepsilon(X_s^\varepsilon(x) dW_s + \int_0^t \int_{|z|\geq\varepsilon} c(z, X_{s-}^\varepsilon(x)) \widetilde{N}_\mu(ds, dz)$$

$$\sigma_\varepsilon = \sqrt{A_\varepsilon} \quad A_\varepsilon^{i,j}(x) = \int_{|z|\leq\varepsilon} c^i(z,x) c^j(z,x) d\mu(dz)$$

Infinitesimal operators

$$(L - L_\varepsilon) f = \int_{|z|\leq\varepsilon} (f(x+c) - f(x) - \langle f(x), \nabla c \rangle - \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j f(x) c^i c^j) d\mu(z)$$

so that

$$\|(L - L_\varepsilon) f\|_\infty \leq C \|f\|_{3,\infty} \times \sup_x \int_{|z|\leq\varepsilon} |c(z,x)|^3 \, d\mu(z) \quad \to 0.$$

Trotter Kato

$$L - L_\varepsilon \to 0 \quad implies \quad P_t^\varepsilon \to P_t.$$

Step 2. Using standard Malliaiv calculus for W one gets

$$P_t^{\varepsilon}(x,dy) = p_t^{\varepsilon}(x,y)dy$$

Step 3

$$P_t(x,dy) - P_t^{\varepsilon}(x,dy) \to 0, \quad p_t^{\varepsilon}(x,y) \to \infty$$

One looks for "equilibrium" in order to "save" some regularity. (FournierPrintemps, Debouche Romito, BCaramellino (interpolation) **Use it for Markv semigroups.**

HYPOTHESIS

1. Regularity for c and $(I + \nabla_x c)(I + \nabla_x c)^* > 0$.

2. Ellipticity

$$\frac{1}{\int_{|z|\leq\varepsilon} |z|^2 \, d\mu(z)} \times \int_{|z|\leq\varepsilon} c^i(z,x)c^j(z,x)d\mu(dz) \geq \lambda > 0$$

3. **"noise"** $\exists \delta > 0$ such that

$$\overline{\lim_{\varepsilon \to 0}} \frac{(\int_{|z| \leq \varepsilon} |z|^3 \, d\mu(z))^{\frac{1}{3}}}{(\int_{|z| \leq \varepsilon} |z|^2 \, d\mu(z))^{\frac{1}{2}+\delta}} < \infty$$

3 BIS. **"generalized sector condion"** There exists

$$0 < \theta_* < 2 \quad and \quad \theta^* \in [\theta_*, \frac{3}{2}\theta_*)$$

such that

$$C\varepsilon^{2-\theta^*} \geq \int_{|z| \leq \varepsilon} |z|^2 \, d\mu(z) \geq \frac{1}{C}\varepsilon^{2-\theta_*}.$$

Theorem

$$P(X_t(x) \in dy) = p_t(x, y)dy, \quad p_t \in C^\infty(R \times R),$$

and "short time behaviour"

$$|p_t(x, y)| \leq Ct^{-d \times \frac{1+\theta_* - \theta^*}{3\theta_* - 2\theta^*} \times c(\delta)}.$$

Idea of the Proof.

Abstract criterion. μ, ν probability measures on R^d.

$$d_k(\mu,\nu) = \sup\{\left|\int f d\mu - \int f d\nu\right| : \|f\|_{k,\infty} = \sum_{|\alpha|\le k} \|\partial^\alpha f\|_\infty \le 1\}$$

We consider a sequence of measures

$$\mu_n(dx) = f_n(x)dx \quad with \quad f_n \in C^\infty(R^d)$$

Equilibrium between

$$\mu_n \to \mu \quad and \quad f_n \to \infty$$

Theorem Assume that there exists $p \ge 1$ and $h \in N$ such that

$$\overline{\lim_n} d_k(\mu, \mu_n) \times \|f_n\|_{h,p}^{\frac{k+d/p_*}{h}} < \infty$$

Then

$$\mu(dx) = f(x)dx \quad and \quad f \in L^p(R^d).$$

Transfer of regularity for Markov Semigroups

We consider P_t with generator L and P_t^n with generator L_n and we assume : for every $q \in N, p \geq 1$

Error (Infinitesimal operators) There exists $a \in N$ such that

$$\begin{array}{ll} A_1 & \|(L-L_n)f\|_{q,\infty} \leq \varepsilon_n \|f\|_{q+a,\infty} \\ A_1^* & \|(L-L_n)^* f\|_{q,p} \leq \varepsilon_n \|f\|_{q+a,p} \end{array}$$

In our case $a = 3$.

Propagation of regularity

$$\begin{array}{ll} A_2 & \|P_t^n f\|_{q,\infty} \leq C \|f\|_{q,\infty} \\ A_2^* & \left\|P_t^{n,*} f\right\|_{q,p} \leq C \|f\|_{q,p} \end{array}$$

Regularization One has

$$P_t^n(x,dy) = p_t^n(x,y) \quad with \quad p_t^n \in C^\infty(R^d \times R^d)$$

and

$$A_3 \quad \left|\partial_x^\alpha \partial_y^\beta p_t^n(x,y)\right| \leq \frac{C}{(\lambda_n t)^{\theta(|\alpha|+|\beta|+d)}}$$

In our case $\theta = \frac{1}{2}$ and λ_n is the lower eigenvalue of

$$A_n^{i,j}(x) = \int_{|z|\leq\varepsilon_n} c^i(z,x)c^j(z,x)d\mu(dz).$$

Equilibriu condition $\exists\delta > 0$ such that

$$E \quad \overline{\lim_n} \frac{\varepsilon_n}{\lambda_n^{\theta(a+\delta)}} < \infty.$$

Theorem Under the above hypothesis

$$P_t(x,dy) = p_t(x,y) \quad with \quad p_t \in C^\infty(R^d \times R^d)$$

Sckach of the proof.

Step 1 (Lindemberg) Let $\Delta_n = L - L_n$.

$$P_t f(x) - P_t^n f(x) = \int_0^t \partial_s P_{t-s} P_s^n f(x) ds = \int_0^t P_{t-s} \Delta_n P_s^n f(x) ds$$
$$= \sum_{m=1}^{m_0} I_m^n f(x) + R_{m_0}^n f(x)$$

with

$$I_m^n f(x) = \int_0^t dt_1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} \prod_{i=1}^{m} (P_{t_i - t_{i+1}}^n \Delta_n) P_{t_m}^n f(x) dt_m.$$

Step 2

$$(P_{t_i - t_{i+1}}^n \Delta_n) P_{t_m}^n (x, dy) = p_{t_1, \ldots, t_m}^n (x, y) dy \quad with \quad p_{t_1, \ldots, t_m}^n \in C^\infty(R^d \times R^d)$$

Step 3 Regularity of $p_{t_1, \ldots, t_m}^n$

Step 4 Regularity of

$$P_t^n(x, dy) + \sum_{m=1}^{m_0} I_m^n(x, dy)$$

Step 5 Use the abstract criterion.