Algebraic structures in SPDEs

Yvain Bruned
University of Warwick

15 July 2017, Durham

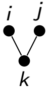
RDEs

Let consider the following RDEs:

$$dY_t = \sum_{i=1}^{d} f_i\left(Y_t\right) dX_t^i \equiv f\left(Y_t\right) dX_t$$

where $X : [0, T] \to \mathbb{R}^d$ is a continuous path of low regularity. The elements needed for the resolution of this equation using Branched rough paths/Regularity structrure are trees of the form:

$$\langle \mathbf{X}, \tau \rangle = \int X^i X^j dX^k \text{ with } \tau =$$



$$\in \mathcal{B}.$$

The translation map M_v^*

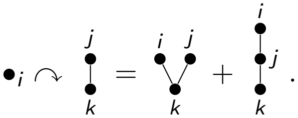
Let $v \in \mathcal{B}^*$ we define $M_v^* : \mathbb{R}^{d+1} \mapsto \mathcal{B}^*$ by:

$$M_v^*(\bullet_0) = \bullet_0 + v, \quad M_v^*(\bullet_i) = \bullet_i, \; i = 1, ..., d.$$

By an universality result given in [CL01], M_v^* extends uniquely to a pre-Lie algebra morphism:

$$M_v^*(\tau_1 \curvearrowright \tau_2) = (M_v^* \tau_1) \curvearrowright (M_v^* \tau_2).$$

where $\curvearrowright$ is the grafting operator. One example is given by:



Action of M_v^* on the RDE

Theorem (B.,Chevyrev,Friz, Preiss, 2017)

Let $\alpha \in (0,1]$ and $\mathbf{X}$ a α-Hölder branched rough path over $\mathbb{R}^{1+d}$ and $v \in \mathcal{B}$. Then Y is an RDE solution flow to

$$dY = f(Y)\, d\left(M_v^* \mathbf{X}\right)$$

if and only if Y is an RDE solution flow to

$$dY = f(Y)\, d\mathbf{X} + f_v(Y)\, dX^0.$$

For $v = \bullet_1 \curvearrowright \bullet_2$, we have the vector field

$$f_{\bullet_1 \curvearrowright \bullet_2} = f_{\bullet_1} \triangleright f_{\bullet_2}$$

where in coordinates $\left(f^i \partial_i\right) \triangleright \left(g^j \partial_j\right) \equiv \left(f^i \partial_i g^j\right) \partial_j$.

SPDEs

Let consider the following system of SPDEs:

$$(\partial_t - \Delta)\,\varphi_j = \sum_{i=1}^{d} f_i^j\,(\varphi, \nabla\varphi)\,\xi_i,$$

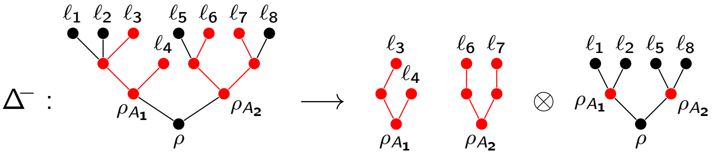
where the ξ_i are space-time noises. One main example is given by the generalised KPZ, the most natural stochastic evolution on loop space. The system of equations in local coordinates is given by

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\xi_i\ .$$

where the ξ_i are independent space-times white noises and the $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the underlying manifold.

Negative renormalisation

In [BHZ16], the renormalisation map M_ℓ is described through the action of a group of character $\mathcal{G}_-$: $M_\ell = (\ell \otimes \mathrm{id})\,\Delta^-$.



Theorem (B., Hairer, Zambotti, 2016)

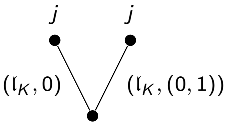
For every $g \in \mathcal{G}_-$ the renormalised model $\mathcal{Z}(\mathbf{\Pi} M_g) = (\Pi^g, \Gamma^g)$ is described by:

$$\Pi^g_z = \Pi_z M_g, \quad \gamma^g_{z\bar{z}} = \gamma_{z\bar{z}} M_g \ .$$

Concrete computation

Let consider the tree $\mathcal{I}_{(\mathfrak{l}_K,0)}(\Xi_j)\mathcal{I}_{(\mathfrak{l}_K,(0,1))}(\Xi_j)$. Graphically this tree is given by

$$\mathcal{I}_{(\mathfrak{l}_K,0)}(\Xi_j)\mathcal{I}_{(\mathfrak{l}_K,(0,1))}(\Xi_j) \longleftrightarrow$$



Then one has:

$$\ell_{\rho,\varepsilon}\left(\mathcal{I}_{(\mathfrak{l}_K,0)}(\Xi_j)\mathcal{I}_{(\mathfrak{l}_K,(0,1))}(\Xi_j)\right) = (\partial_x K_{\varepsilon,\varrho} * K_{\varepsilon,\varrho})(0), \quad K_{\varepsilon,\varrho} = \varrho_\varepsilon * K.$$

The translation map M^*_ℓ

Let $\ell \in \mathcal{G}_-$ we define M^*_ℓ by:

$$M^*_\ell(\bullet^n_\mathfrak{l}) = \bullet^n_\mathfrak{l} + \sum_\tau \ell(\tau)\tau$$

where $\bullet^n_\mathfrak{l}$ is associated to the generator $\Xi_\mathfrak{l} X^n$. Now, we want to apply the previous construction and extend M^*_ℓ uniquely to a pre-Lie algebra morphism. We will not recover the map define in [BHZ16]. We have to find a new grafting operator $\hat{\curvearrowright}$ such that:

$$M^*_\ell(\tau_1 \hat{\curvearrowright} \tau_2) = (M^*_\ell \tau_1) \hat{\curvearrowright} (M^*_\ell \tau_2).$$

A new grafting operator for the Taylor expansion

The usual grafting operator is given by

$$\bullet \curvearrowright_{(f,p)} \bullet^{k} = \begin{array}{l}\bullet \\ \Big|\,{\scriptstyle (f,p)} \\ \bullet_{k}\end{array} \longleftrightarrow y^{k} f^{(p)}(y-x).$$

The one with Taylor expansions is given by

$$\bullet \hat{\curvearrowright}_{(f,p)} \bullet^{k} \longleftrightarrow \sum_{i=0}^{k \wedge p} \frac{1}{(p-i)!} y^{k-i} f^{(p-i)}(y-x)$$

$$\bullet \curvearrowright_{(f,p)} \bullet^{k} = p! \left(\bullet \hat{\curvearrowright}_{(f,p)} \bullet^{k} - \bullet \hat{\curvearrowright}_{(f,p-1)} \bullet^{k-1} \right).$$

Proposition (B.,Chandra, Chevyrev, Hairer, 2017)

The space $\mathcal{B}$ is freely generated by the family $(\hat{\curvearrowright}_{(\mathfrak{l},p)})_{(\mathfrak{l},p) \in \mathcal{O}}$ with generators $\Xi_{\mathfrak{l}} X^{k}$, $k \in \mathbb{N}^{d}$ and $\mathfrak{l} \in \mathcal{D}$.