Navier-Stokes equations with constrained L^2 energy of the solution

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joint works with Mauro Mariani (Roma 1) and Gaurav Dhariwal (York)

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Motivation

- Caglioti *et.al* [5] studied 2D NSEs in $\mathbb{R}^2$ with constraints

$$E(\omega) = \int_{\mathbb{R}^2} \psi(x)\,\omega(x)\,dx = \int_{\mathbb{R}^2} |u(x)|^2\,dx = a,$$
$$I(\omega) = \int_{\mathbb{R}^2} |x|^2 \omega(x)\,dx = b,$$

where

$$\omega = \operatorname{curl} \mathrm{u}, \qquad \psi = -(\Delta)^{-1}\omega.$$

- They proved that for a certain stationary solution ω_{MF} of the Euler equation (in the vorticity form) with constraints a, b, for every initial data ω_0 "close enough" ω_{MF} with the same constraints a, b;

$$\omega(t) \to \omega_{MF}, \text{ as } t \to \infty,$$

where $\omega(t)$ is the solution of the NSEs (in the vorticity form) with inital data ω_0 and the same constraints.

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- Rybka [7] and Caffarelli & Lin [4] studied heat equation with constraint

$$|u|_{L^2} = 1. \tag{1}$$

- The heat equation is given by

$$\frac{\partial u}{\partial t} = -A\,u, \tag{2}$$

where $A\,u = -\Delta\,u$ is a self adjoint operator on H.

- We define a Hilbert manifold

$$\mathcal{M} = \{u \in \mathrm{H} : |u|_{\mathrm{H}} = 1\}. \tag{3}$$

- Note that $A\,u \notin T_u\mathcal{M}$ for $u \in \mathcal{M}$ but $\Pi_u(-A\,u) \in T_u\mathcal{M}$ for every $u \in \mathcal{M}$, where

$$\Pi_u : \mathrm{H} \ni \mathrm{x} \mapsto \mathrm{x} - \langle \mathrm{x}, \mathrm{u}\rangle_{\mathrm{H}}\,\mathrm{u} \in \mathrm{T_u}\mathcal{M} = \{\mathrm{y} \in \mathrm{H} : \langle \mathrm{u}, \mathrm{y}\rangle_{\mathrm{H}} = 0\} \tag{4}$$

is the orthogonal projection.

- Since $\Pi_u(-A\,u) = -A\,u + |A^{1/2}u|^2_{\mathrm{H}}u$, we get

$$\frac{\partial u}{\partial t} = -A\,u + |A^{1/2}|^2_{\mathrm{H}}u. \tag{5}$$

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Constrained Heat equation

- A special case of heat equation with Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + |\nabla u|^2_{L^2} u \\ u(0) = u_0 \end{cases} \tag{6}$$

- Note that the heat equation (2) can be seen as an L^2-gradient flow of energy

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathcal{O}} |\nabla u(x)|^2 \, dx, \tag{7}$$

as formally $$-\nabla_{L^2} \mathcal{E}(u) = \Delta u.$$

- Similarly, the constrained heat equation (6) can be seen as the gradient flow of $\mathcal{E}$ restricted to the manifold $\mathcal{M}$ with L^2-metric on the "tangent bundle".
  In fact one can prove that the solution of (6) with $u_0 \in H^1_0(\mathcal{O}) \cap \mathcal{M}$ satisfies

$$\mathcal{E}(u(t)) + \int_0^t |\Delta u(s) + |\nabla u|^2_{L^2} u(s)|^2_{L^2} \, ds = \mathcal{E}(u(0)) \tag{8}$$

from which one can deduce the global existence.

- An essential step in proving the global existence is to establish the invariance of $\mathcal{M}$.

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$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta\, u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} \mathrm{u} = 0, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

Here

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where $\Pi : L^2(\mathcal{O}) \to \mathrm{H}$ is the orthogonal projection.

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- The projected version of (9) can be found in a similar way as before.
- Note

$$\Pi_u(B(u,u)) = B(u,u) - \underbrace{\langle B(u,u), u\rangle_{\mathrm{H}}}_{=0} u = B(u,u). \tag{12}$$

So we get

$$\begin{cases} \dfrac{\partial u}{\partial t} + A\,u + B(u,u) = |\nabla u|^2_{L^2} u \\ u(0) = u_0 \in \mathrm{V} \cap \mathcal{M}, \end{cases} \tag{13}$$

where $\mathrm{V} = \mathrm{H}^{1,2}_0 \cap \mathcal{M}$ or $H^{1,2} \cap \mathcal{M}$.

- We can show existence of a local maximal solution $u(t), t < \tau$ which lies on $\mathcal{M}$.
- However to prove the global existence one needs to assume that we deal with periodic boundary conditions (or torus), because then

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and the L^2-norm of $u(t)$ doesn't explode. In order to show that $\|u(t)\|_{\mathrm{V}}^2$ doesn't explode, it suffices to show that $|\nabla u(t)|_{L^2}$ neither does.

- Formally, we have

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- Hence $|A\,u|^2_{L^2} - |\nabla u|^4_{L^2} \geq 0$ and

$$\frac{1}{2}|\nabla u(t)|^2_{L^2} + \int_0^t |\nabla_{\mathcal{M}} \mathcal{E}(u(s))|^2_{L^2}\,ds = \frac{1}{2}|\nabla u_0|^2_{L^2}, \qquad t \in [0,T). \tag{17}$$

Thus we can summarise our results in the following theorem :

Theorem 1

For every $u_0 \in \mathrm{V} \cap \mathcal{M}$ there exists a unique global solution u of the constrained NSEs (13) *such that $u \in X_T$ for all $T > 0$.*

Here $X_T = \mathcal{C}([0,T];\mathrm{V}) \cap \mathrm{L}^2(0,\mathrm{T};\mathrm{D}(\mathrm{A}))$.

Stochastic Constrained NSEs

- We assume that $W = (W_1, \cdots, W_m)$ is $\mathbb{R}^m-$valued Wiener process , $c_1 \cdots, c_m$ and $\hat{C}_1, \cdots, \hat{C}_m$ are respectively vector fields and assosciated linear operators given by

$$\hat{C}_j u = c_j(x) \cdot \nabla u, : \quad \mathrm{div}\, \mathrm{c_j} = 0, \quad \mathrm{j} \in \{1, \cdots, \mathrm{m}\}.$$

- Since

$$C_j u = \Pi \hat{C}_j u, \qquad j \in \{1, \cdots, m\},$$

is skew symmetric in H, these operators don't produce any correction term when projected on $T_u\mathcal{M}$.

- Thus the stochastic NSE

$$du + [A\, u + B(u,u)]\; dt = \sum_{j=1}^m C_j u \circ dW_j = \underbrace{\sum_{j=1}^m C_j u\, dW_j + \frac{1}{2}\sum_{j=1}^m C_j^2 u\, dt}_{\text{Stratonovich} \,=\, \text{Itô} \,+\, \text{correction}}$$

under the constraint is given by

$$du + \left[A\, u + B(u,u) - |\nabla u|^2_{L^2}\, u\right] dt = \frac{1}{2}\sum_{j=1}^m C_j^2 u\, dt + \sum_{j=1}^m C_j u\, dW_j. \qquad (18)$$

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Martingale solution

Definition 2

We say that there exists a **martingale solution** of (18) iff there exist

- a stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ with filtration $\hat{\mathbb{F}} = \left\{\hat{\mathcal{F}}_t\right\}_{t\geq 0}$,
- an $\mathbb{R}^m$-valued $\hat{\mathbb{F}}$-Wiener process $\hat{W}$,
- and an $\hat{\mathbb{F}}$-progressively measurable process $u : [0,T] \times \hat{\Omega} \to \mathrm{V} \cap \mathcal{M}$ with $\hat{\mathbb{P}}$-a.e. paths

$$u(\cdot,\omega) \in \mathcal{C}([0,T];\mathrm{V_w}) \cap \mathrm{L}^2(0,\mathrm{T};\mathrm{D(A)}),$$

such that for all $t \in [0,T]$ and all $\mathrm{v} \in \mathrm{D(A)}$:

$$\langle u(t),\mathrm{v}\rangle + \int_0^{\mathrm{t}} \langle \mathrm{Au(s)},\mathrm{v}\rangle\,\mathrm{ds} + \int_0^{\mathrm{t}} \langle \mathrm{B(u(s))},\mathrm{v}\rangle\,\mathrm{ds} = \langle \mathrm{u_0},\mathrm{v}\rangle$$
$$+ \int_0^t |\nabla u(s)|^2_{L^2} \langle u(s),\mathrm{v}\rangle\,\mathrm{ds} + \frac{1}{2}\int_0^{\mathrm{t}} \sum_{\mathrm{j=1}}^{\mathrm{m}} \langle \mathrm{C_j^2 u(s)},\mathrm{v}\rangle\,\mathrm{ds} + \int_0^{\mathrm{t}} \sum_{\mathrm{j=1}}^{\mathrm{m}} \langle \mathrm{C_j u(s)},\mathrm{v}\rangle \mathrm{dW_j}, \tag{19}$$

the identity hold $\hat{\mathbb{P}}$-a.s.

Theorem 3 (Assume that our domain is the 2-d torus)

Then for every $u_0 \in \mathrm{V} \cap \mathcal{M}$, there exists a martingale solution to the stochastic constrained NSEs (18).

Sketch of the proof : Galerkin approximation :
Let $\{e_j\}$ be ONB of H and eigenvectors of A.

$$\mathrm{H}_n := \mathrm{lin}\{\mathrm{e}_1, \cdots, \mathrm{e_n}\} \text{ is the finite dimensional Hilbert space}$$

$$P_n : \mathrm{H} \to \mathrm{H}_n \text{ be the orthogonal projection operator given by } P_n u = \sum_{i=1}^{n} \langle u, e_i \rangle \, e_i.$$

We consider the following "projection" of onto H_n:

$$\begin{cases} du_n & = -\left[P_n A u_n + P_n B(u_n) - |\nabla u_n|^2_{L^2} u_n\right] dt + \sum_{j=1}^{m} P_n C_j u_n \circ dW_j, \quad t \geq 0, \\ u_n(0) & = \frac{P_n u_0}{|P_n u_0|_{L^2}}, \qquad \text{for } n \text{ large enough} \end{cases} \tag{20}$$

We fix $T > 0$. Equation (20) is a stochastic ODE on a finite dimensional compact manifold $\mathcal{M}_n = \{u \in \mathrm{H}_n : |u|_{L^2} = 1\}$.
Hence it has a unique $\mathcal{M}$-valued solution (with continuous paths). Moreover, $\forall q \geq 2$

$$\mathbb{E} \int_0^T |u_n(t)|^q_{\mathrm{H}} \, dt < \infty.$$

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These depend deeply on the property that

$$\langle B(u), Au\rangle_{\mathrm{H}} = 0, \quad u \in \mathrm{D(A)}.$$

and a very specific assumption

- We assume $c_1 \cdots , c_m$ are constant vector fields.

Let $K_c = \max_{j\in 1,\cdots,m} |c_j|_{\mathbb{R}^2}$.

Lemma 4

Let $p \in \left[1, 1+\frac{1}{K_c^2}\right)$ and $\rho > 0$. Then there exist positive constants $C_1(p,\rho)$, $C_2(p,\rho)$ and $C_3(\rho)$ such that if $\|u_0\|_{\mathrm{V}} \le \rho$, then

$$\sup_{n\ge 1} \mathbb{E}\left(\sup_{r\in[0,T]} \|u_n(r)\|_{\mathrm{V}}^{2p}\right) \le C_1(p,\rho), \tag{21}$$

$$\sup_{n\ge 1} \mathbb{E}\int_0^T \|u_n(s)\|_{\mathrm{V}}^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2 \, ds \le C_2(p,\rho)\,, \tag{22}$$

and

$$\sup_{n\ge 1} \mathbb{E}\int_0^T |u_n(s)|_{\mathrm{D(A)}}^2 \, ds \le C_2(1) + C_1(2)T =: C_3(\rho). \tag{23}$$

A'priori estimates

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Aldous condition

We put

$$\mathcal{Z}_T = \mathcal{C}([0,T];\mathrm{H}) \cap \mathrm{L}^2_{\mathrm{w}}(0,\mathrm{T};\mathrm{D(A)}) \cap \mathrm{L}^2(0,\mathrm{T};\mathrm{V}) \cap \mathcal{C}([0,\mathrm{T}];\mathrm{V}_{\mathrm{w}}),$$

and $\mathcal{T}_T$ the corresponding topology.
In order to prove that the laws of u_n are tight on $\mathcal{Z}_T$. Apart from a'priori estimates we also need one additional property to be satisfied :

Lemma 5 (Aldous condition in H)

$\forall\,\varepsilon > 0$, $\forall\,\eta > 0\ \exists\,\delta > 0$: *for every stopping time* $\tau_n : \Omega \to [0,T]$

$$\sup_{n\in\mathbb{N}} \sup_{0\le\theta\le\delta} \mathbb{P}\left(|u_n(\tau_n+\theta) - u_n(\tau_n)|_{\mathrm{H}} \ge \eta\right) < \varepsilon. \tag{24}$$

Lemma 5 can be proved by applying Lemma 4 to equations (20).

Corollary 6

The laws of (u_n) *are tight on* $\mathcal{Z}_T$, *i.e.* $\forall\,\varepsilon > 0\ \exists\,K_\varepsilon \subset \mathcal{Z}_T$ *compact, such that*

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By the application of the Prokhorov and the Jakubowski-Skorokhod Theorems (since $\mathcal{Z}_T$ is not a Polish space, we need Jakubowski) we deduce that there exists a subsequence, a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, $\mathcal{Z}_T-$valued random variables $\tilde{u}_n$ such that

$$\mathrm{Law}(\tilde{\mathrm{u}}_\mathrm{n}) = \mathrm{Law}(\mathrm{u}_\mathrm{n}),$$

and there exists $\tilde{u} \colon \hat{\Omega} \to \mathcal{Z}_T$ random variable such that

$$\tilde{u}_n \to \tilde{u} \quad \text{in, } \hat{\mathbb{P}} - a.s.$$

Then, using Kuratowski Theorem, we can deduce that the sequence $\tilde{u}_n$ satisfies the same a'priori estimates as u_n. In particular $\forall\, p \in [1, 1 + \frac{1}{K_c^2})$

$$\sup_{n\geq 1} \mathbb{E}\left(\sup_{r\in[0,T]} \|\tilde{u}_n(r)\|_{\mathrm{V}}^{2p} \right) \leq C_1(p), \tag{25}$$

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The choice of $\mathcal{Z}_T$ allows to deduce that $\forall\,\psi \in \mathrm{H}(or\mathrm{V})$ and $s, t \in [0, T]$:

(a) $\lim_{n\to\infty} \langle \tilde{u}_n(t), P_n\psi\rangle = \langle \tilde{u}(t), \psi\rangle$, $\tilde{\mathbb{P}}$-a.s.,

(b) $\lim_{n\to\infty} \int_s^t \langle A\tilde{u}_n(\sigma), P_n\psi\rangle \, d\sigma = \int_s^t \langle A\tilde{u}(\sigma), \psi\rangle \, d\sigma$, $\tilde{\mathbb{P}}$-a.s.,

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(b) $\lim_{n\to\infty} \int_s^t \langle A\tilde{u}_n(\sigma), P_n\psi\rangle \, d\sigma = \int_s^t \langle A\tilde{u}(\sigma), \psi\rangle \, d\sigma$, $\tilde{\mathbb{P}}$-a.s.,

(c) $\lim_{n\to\infty} \int_s^t \langle B(\tilde{u}_n(\sigma), \tilde{u}_n(\sigma)), P_n\psi\rangle \, d\sigma = \int_s^t \langle B(\tilde{u}(\sigma), \tilde{u}(\sigma)), \psi\rangle \, d\sigma$, $\tilde{\mathbb{P}}$-a.s.,

(d) $\lim_{n\to\infty} \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n\psi\rangle \, d\sigma = \int_s^t |\nabla \tilde{u}(\sigma)|_{L^2}^2 \langle \tilde{u}(\sigma), \psi\rangle \, d\sigma$, $\tilde{\mathbb{P}}$-a.s.

(e) $\lim_{n\to\infty} \int_s^t \langle C_j^2\, \tilde{u}_n(\sigma), P_n\psi\rangle \, d\sigma = \int_s^t \langle C_j^2 \tilde{u}(\sigma), \psi\rangle \, d\sigma$, $\tilde{\mathbb{P}}$-a.s.

Since $\tilde{u}_n \to \tilde{u}$ in $C([0, T]; \mathrm{H})$ and $u_n(t) \in \mathcal{M}$ for every $t \in [0, T]$, we infer that

$$\tilde{u}(t) \in \mathcal{M}, \quad t \in [0, T]. \tag{27}$$

We are close to conclude the proof of Theorem 3. We are just left to deal with the Itô integral.

Define

$$M_n(t) = \sum_{j=1}^{m} \int_0^t P_n C_j u_n(s)\, dW_j(s).$$

M_n is a martingale on $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover

$$\begin{aligned} M_n(t) = u_n(t) - P_n u_n(0) + \int_0^t P_n A u_n(s)\, ds + \int_0^t P_n B(u_n(s))\, ds \\ - \int_0^t |\nabla u_n(s)|_{L^2}^2 u_n(s)\, ds - \frac{1}{2} \sum_{j=1}^{m} \int_0^t (P_n C_j)^2 u_n(s)\, ds \end{aligned} \tag{28}$$

The equation (28) can also be used on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ to define a process $\tilde{M}_n$, i.e.

$$\begin{aligned} \tilde{M}_n(t) = \tilde{u}_n(t) - P_n \tilde{u}_n(0) + \int_0^t P_n A \tilde{u}_n(s)\, ds + \int_0^t P_n B(\tilde{u}_n(s))\, ds \\ - \int_0^t |\nabla \tilde{u}_n(s)|_{L^2}^2 \tilde{u}_n(s)\, ds - \frac{1}{2} \sum_{j=1}^{m} \int_0^t (P_n C_j)^2 \tilde{u}_n(s)\, ds \end{aligned} \tag{29}$$

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The equation (28) can also be used on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ to define a process $\tilde{M}_n$, i.e.

$$\begin{aligned} \tilde{M}_n(t) = \tilde{u}_n(t) - P_n \tilde{u}_n(0) + \int_0^t P_n A \tilde{u}_n(s)\, ds + \int_0^t P_n B(\tilde{u}_n(s))\, ds \\ - \int_0^t |\nabla \tilde{u}_n(s)|^2_{L^2} \tilde{u}_n(s)\, ds - \frac{1}{2} \sum_{j=1}^{m} \int_0^t (P_n C_j)^2 \tilde{u}_n(s)\, ds \end{aligned} \tag{29}$$

Using the earlier convergence results and a *priori* estimates (25), (26), we can prove that

$$\tilde{M}_n(t) \to \tilde{M}(t) := \tilde{u}(t) - \tilde{u}(0) + \int_0^t A\tilde{u}(s)\,ds + \int_0^t B(\tilde{u}(s))\,ds - \int_0^t |\nabla \tilde{u}(s)|^2_{L^2}\tilde{u}(s)\,ds - \frac{1}{2}\sum_{j=1}^m \int_0^t C_j^2\tilde{u}(s)\,ds. \tag{30}$$

From equality (30) one can deduce that

(i) $\tilde{M}$ is $\tilde{\mathbb{F}}-$martingale.

(ii) $\mathrm{Cov}(\tilde{\mathrm{M}}_\mathrm{n}) \to \mathrm{Cov}(\tilde{\mathrm{M}}) = \sum_{\mathrm{j}=1}^{\mathrm{m}} \int_0^{\mathrm{t}} \mathrm{C_j}\tilde{\mathrm{u}}(\mathrm{s})\,(\mathrm{C_j}\tilde{\mathrm{u}}(\mathrm{s}))^*\ \mathrm{ds}.$

This allows to use the martingale representation theorem to deduce that there exists a bigger probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ and a Wiener process $\bar{W}$ on the same probability space such that

$$\bar{M}(t) = \int_0^t \sum_{j=1}^m C_j\bar{u}(s)\,d\bar{W}_j(s).$$

Hence we proved Theorem 3.

Using the earlier convergence results and a *priori* estimates (25), (26), we can prove that

$$\tilde{M}_n(t) \to \tilde{M}(t) := \tilde{u}(t) - \tilde{u}(0) + \int_0^t A\tilde{u}(s)\,ds + \int_0^t B(\tilde{u}(s))\,ds$$
$$- \int_0^t |\nabla \tilde{u}(s)|^2_{L^2} \tilde{u}(s)\,ds - \frac{1}{2}\sum_{j=1}^m \int_0^t C_j^2 \tilde{u}(s)\,ds. \tag{30}$$

From equality (30) one can deduce that

(i) $\tilde{M}$ is $\tilde{\mathbb{F}}-$martingale.

(ii) $\mathrm{Cov}(\tilde{M}_n) \to \mathrm{Cov}(\tilde{M}) = \sum_{j=1}^m \int_0^t C_j\tilde{u}(s)\,(C_j\tilde{u}(s))^*\ ds.$

This allows to use the martingale representation theorem to deduce that there exists a bigger probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ and a Wiener process $\bar{W}$ on the same probability space such that

$$\bar{M}(t) = \int_0^t \sum_{j=1}^m C_j \bar{u}(s)\,d\bar{W}_j(s).$$

Hence we proved Theorem 3.

Martingale representation theorem

Using the earlier convergence results and a *priori* estimates (25), (26), we can prove that

$$\tilde{M}_n(t) \to \tilde{M}(t) := \tilde{u}(t) - \tilde{u}(0) + \int_0^t A\tilde{u}(s)\,ds + \int_0^t B(\tilde{u}(s))\,ds$$
$$- \int_0^t |\nabla \tilde{u}(s)|^2_{L^2} \tilde{u}(s)\,ds - \frac{1}{2}\sum_{j=1}^m \int_0^t C_j^2 \tilde{u}(s)\,ds. \tag{30}$$

From equality (30) one can deduce that

(i) $\tilde{M}$ is $\tilde{\mathbb{F}}-$martingale.

(ii) $\mathrm{Cov}(\tilde{\mathrm{M}}_\mathrm{n}) \to \mathrm{Cov}(\tilde{\mathrm{M}}) = \sum_{\mathrm{j}=1}^{\mathrm{m}} \int_0^{\mathrm{t}} \mathrm{C_j}\tilde{\mathrm{u}}(\mathrm{s})\,(\mathrm{C_j}\tilde{\mathrm{u}}(\mathrm{s}))^*\ \mathrm{ds}$.

This allows to use the martingale representation theorem to deduce that there exists a bigger probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ and a Wiener process $\bar{W}$ on the same probability space such that

$$\bar{M}(t) = \int_0^t \sum_{j=1}^m C_j \bar{u}(s)\,d\bar{W}_j(s).$$

Hence we proved Theorem 3.

Theorem 7

Pathwise Uniqueness holds for the the stochastic constrained NSEs (18).

Theorem 8

The stochastic constrained NSEs (18) *have a unique strong solution for each* $u_0 \in \mathrm{V} \cap \mathcal{M}$. *Moreover, the paths of this solution belong to the space* X_T *for all* $T > 0$. *In particular, the paths are* V*-valued continuous (strongly and not only weakly).*

- Z. Brzeźniak, G. Dhariwal and M. Mariani, *2D Constrained Navier-Stokes equations,* arXiv:1606.08360v2 (2016) (Submitted).
- Z. Brzeźniak and G. Dhariwal, *Stochastic Constrained Navier-Stokes equations on* $\mathbb{T}^2$, arXiv:1701.01385 (2017)
- Z. Brzeźniak and E. Motyl, *Existence of a martingale solution of the stochastic Navier-Stokes equations in unbounded 2D and 3D-domains*, J. Differential Equations, **254**(4), 1627-1685 (2013).
- L. Caffarelli and F. Lin, *Nonlocal heat flows preserving the* L^2 *energy,* Discrete and Continuous Dynamical Systems, **23**($1\&2$), 49-64 (2009).
- E. Caglioti, M. Pulvirenti and F. Rousset, *On a constrained 2D Navier-Stokes Equation,* Communications in Mathematical Physics, **290**, 651-677 (2009).
- A. Jakubowski, *The almost sure Skorokhod representation for subsequences in nonmetric spaces*, Teor. Veroyatn. Primen. **42**(1), 209-216 (1997); translation in Theory Probab. Appl. **42**(1), 167-174 (1998).
- P. Rybka, *Convergence of a heat flow on a Hilbert manifold,* Proceedings of the Royal Society of Edinburgh, **136A**, 851-862 (2006).