A limit theorem for the moments in space of Browian local time increments

Simon Campese

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UNIVERSITÉ DU LUXEMBOURG

Motivation

Theorem (Marcus, Rosen, 2006)

As $h \to 0$,

$$\int_{-\infty}^{\infty} \left| \frac{L_t^{x+h} - L_t^x}{\sqrt{h}} \right|^q \mathrm{d}x \xrightarrow{a.s.} C_q \int_{-\infty}^{\infty} (L_t^x)^{q/2} \mathrm{d}x$$

CLT for second moment in space

Theorem (Chen, Li, Marcus, Rosen, 2010)

For $h \to 0$,

$$\frac{1}{h^{3/2}}\left(\int_{-\infty}^{\infty}\left(L_t^{x+h}-L_t^x\right)^2 \mathrm{d}x - 4ht\right) \xrightarrow{d} c_2\sqrt{\int_{-\infty}^{\infty}(L_t^x)^2\,\mathrm{d}x}\;Z,$$

where $Z \sim \mathcal{N}(0,1)$, independent of $(L_t^x)_{x\in\mathbb{R}}$.

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- Alternative proof by Hu/Nualart (2009) using Malliavin calculus

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- Proof by Method of Moments
- Alternative proof by Hu/Nualart (2009) using Malliavin calculus
- Another proof by Rosen (2011) using self-intersection local times

CLT for third moment in space

Theorem (Rosen, 2011)

For $h \to 0$,

$$\frac{1}{h^2}\int_{-\infty}^{\infty}\left(L_t^{x+h} - L_t^x\right)^3 \,\mathrm{d}x \xrightarrow{d} c_3\sqrt{\int_{-\infty}^{\infty}(L_t^x)^3\,\mathrm{d}x}\; Z,$$

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- Again, proof by Method of Moments
- Malliavin calculus proof by Hu/Nualart (2010)

Conjecture for fourth moment in space

Conjecture (Rosen, 2011)

For $h \to 0$,

$$\frac{1}{h^{5/2}}\left(\int_{-\infty}^{\infty} (\Delta_h L_t^x)^4 \,\mathrm{d}x \right.$$
$$\left. - 24h \int_{-\infty}^{\infty} (\Delta_h L_t^x)^2 L_t^x \,\mathrm{d}x + 48h^2 \int_{-\infty}^{\infty} (L_t^x)^2 \,\mathrm{d}x \right)$$
$$\xrightarrow{d} c_4 \sqrt{\int_{-\infty}^{\infty} (L_t^x)^4 \,\mathrm{d}x}\; Z,$$

where $Z \sim \mathcal{N}(0,1)$, independent of $(L_t^x)_{x\in\mathbb{R}}$.

Main result: Limit theorem for any moment

Theorem (C., 2016)

For $h \to 0$,

$$\frac{1}{h^{(q+1)/2}}\left(\int_{-\infty}^{\infty}(\Delta_h L_t^x)^q \,\mathrm{d}x + \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} h^k a_{q,k}\int_{-\infty}^{\infty}(\Delta_h L_t^x)^{q-2k}(4L_t^x)^k \,\mathrm{d}x\right) \xrightarrow{d} c_q\sqrt{\int_{-\infty}^{\infty}(L_t^x)^q \,\mathrm{d}x}\; Z,$$

where $Z \sim \mathcal{N}(0,1)$, independent of $(L_t^x)_{x\in\mathbb{R}}$.

Key ingredient: Kailath-Segall identity

For continuous L^2-martingale $(M_x)_{x\in\mathbb{R}}$, define iterated integrals $I_0(x) = 1$,

$$I_1(x) = \int_{-\infty}^{x} \mathrm{d}M_u \qquad \text{and} \qquad I_q(x) = \int_{-\infty}^{x} I_{q-1}(u)\,\mathrm{d}M_u.$$

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Theorem (Kailath, Segall, 1976)

$$q\, I_q(x) = I_{q-1}(x)\, M_x - I_{q-2}(x)\, \langle M, M\rangle_x$$

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Theorem (Kailath, Segall, 1976)

$$q\, I_q(x) = I_{q-1}(x)\, M_x - I_{q-2}(x)\, \langle M, M\rangle_x$$

Recursively,

$$q!\, I_q(x) = \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} a_{q,k}\, M_x^{q-2k} \langle M, M\rangle_x^k$$

Isolation of dominating term

- Perkins, 1981: $(L_t^x)_{x\in\mathbb{R}}$ is continuous semimartingale with quadratic variation $4\int_{-\infty}^{x} L_t^u \,\mathrm{d}u$

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- Binomial theorem and stochastic analysis:

$$\int_{-\infty}^{\infty} (\Delta_h L_t^x)^q \ \mathrm{d}x = \int_{-\infty}^{\infty} (\Delta_h M_x)^q \ \mathrm{d}x + o\left(h^{(q+1)/2}\right)$$

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- Kailath-Segall:

$$(\quad M_x)^{q} = q! \quad I_q(x) - \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} (\quad M_x)^{q-2k} (\quad \langle M, M\rangle_x)^{k}$$

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- Kailath-Segall: *(consider martingale $\left(\int_x^y \mathrm{d}M_u\right)_{y\geq x}$ at $y = x+h$)*

$$(\Delta_h M_x)^q = q!\Delta_h I_q(x) - \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} (\Delta_h M_x)^{q-2k} (\Delta_h \langle M, M\rangle_x)^k$$

CLT for dominating term

Theorem (C., 2016)

For $h \to 0$,

$$\frac{1}{h^{(q+1)/2}} \int_{-\infty}^{\infty} q! \Delta_h I_q(x)\, \mathrm{d}x \xrightarrow{d} c_q \sqrt{\int_{-\infty}^{\infty} (L_t^x)^q \; \mathrm{d}x}\; Z,$$

where $Z \sim \mathcal{N}(0,1)$, independent of $(L_t^x)_{x \in \mathbb{R}}$.

Key ingredient: asymptotic Ray-Knight theorem

- (M_x^h), (N_x^h) sequences of continuous L^2-martingales
- (β_x^h), (γ_x^h) Dambis-Dubins-Schwarz Brownian motions

Theorem (Revuz, Yor, 1999)

If

$$\sup_{x\in[-\infty,a]} \left| \left\langle M^h, N^h \right\rangle_x \right| \xrightarrow{prob.} 0$$

as $h \to 0$, then (β_x^h, γ_x^h) converges in distribution to a two-dimensional standard Brownian motion.

Proof of CLT for dominating term

- Iterated stochastic Fubini:

$$\begin{aligned}
\int_{-\infty}^{\infty} \Delta_h I_q(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \int_{x}^{x+h} \int_{-\infty}^{u_1} \cdots \int_{-\infty}^{u_{q-1}} \mathrm{d}M_{u_q} \ldots \mathrm{d}M_{u_2}\,\mathrm{d}M_{u_1}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \int_{u_1-h}^{u_1} \int_{u_1-h}^{u_2} \cdots \int_{u_1-h}^{u_q} \mathrm{d}x\,\mathrm{d}M_{u_q} \ldots \mathrm{d}M_{u_2}\,\mathrm{d}M_{u_1} \\
&= \int_{-\infty}^{\infty} K_{q,h}(u_1)\,\mathrm{d}M_{u_1}
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- Define martingale $(\widetilde{M}_x^h)_{x\geq-\infty}$ by

$$\widetilde{M}_x^h = \frac{q!}{h^{(q+1)/2}}\int_{-\infty}^{x} K_{q,h}(u_1)\,\mathrm{d}M_{u_1}$$

Proof of CLT for dominating term

- Use Burkholder-Davis-Gundy-type arguments to show that

$$\sup_{x \in (-\infty, x_0]} \left| \left\langle \widetilde{M}^h, M \right\rangle_x \right| \xrightarrow{L^1} 0$$

for $x_0 \geq -\infty$ and

$$\left\langle \widetilde{M}^h, \widetilde{M}^h \right\rangle_x \xrightarrow{L^1} c_q^2 \int_{-\infty}^{x} (L_t^u)^q \,\mathrm{d}u$$

for $x \in \mathbb{R} \cup \{-\infty, \infty\}$.

Proof of CLT for dominating term

- Asymptotic Ray-Knight:

$$\left(\beta, \beta^h, \left\langle \widetilde{M}^h, \widetilde{M}^h \right\rangle\right) \xrightarrow{d} \left(\beta, \widetilde{\beta}, c_q^2 \int_{-\infty}^{\cdot} (L_t^u)^q \, \mathrm{d}u\right),$$

where β, β^h DDS-Brownian motions of M and $\widetilde{M}^h$, respectively

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- Consequently:

$$\widetilde{M}_x^h = \beta^h_{\left\langle \widetilde{M}^h, \widetilde{M}^h \right\rangle_x} \xrightarrow{d} \widetilde{\beta}_{c_q^2 \int_{-\infty}^{x} (L_t^u)^q \,\mathrm{d}u}.$$

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- Letting $x \to \infty$ finishes proof

Comparison with literature

- For $q = 2$ and $q = 3$: new proofs of CLT by Rosen et al.

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- For $q = 4$, conjecture by Rosen confirmed

Generalizations

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- In principle, Brownian motion should be replaceable with more general square-integrable martingale
- analogously, one expects to be able to prove analogous results for more general functions than powers

Generalizations (cont.)

Theorem (C., 2016)

$$\frac{1}{\sqrt{h}}\left(\int_{-\infty}^{\infty}\left(\frac{\Delta_h L_t^x}{\sqrt{h}}\right)^q Y_{t,x}\,\mathrm{d}x \right.$$
$$\left. + \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} \int_{-\infty}^{\infty}\left(\frac{\Delta_h L_t^x}{\sqrt{h}}\right)^{q-2k} (4\, L_t^x)^k\, Y_{t,x}\,\mathrm{d}x\right)$$
$$\xrightarrow{d} c_q \sqrt{\int_{-\infty}^{\infty} (L_t^x)^q\, Y_{t,x}\,\mathrm{d}x}\, Z,$$

$(Y_{t,x})_{x\in\mathbb{R}}$ *non-negative and nice.*

Generalizations (cont.)

In particular, for $n \in \mathbb{N}_0$:

$$\frac{1}{\sqrt{h}}\left(\int_{-\infty}^{\infty} \left(\frac{\Delta_h L_t^x}{\sqrt{h}} \right)^q (L_t^x)^n \,\mathrm{d}x \right.$$
$$\left. + \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} \int_{-\infty}^{\infty} \left(\frac{\Delta_h L_t^x}{\sqrt{h}} \right)^{q-2k} (4L_t^x)^k (L_t^x)^n \,\mathrm{d}x \right)$$
$$\xrightarrow{d} c_q \sqrt{\int_{-\infty}^{\infty} (L_t^x)^{q+n} \,\mathrm{d}x}\, Z,$$

Question

Recursive argument should yield:

$$\frac{1}{\sqrt{h}}\left(\int_{-\infty}^{\infty}\left(\frac{\Delta_h L_t^x}{\sqrt{h}}\right)^q \mathrm{d}x - C_q\int_{-\infty}^{\infty}(L_t^x)^{q/2}\,\mathrm{d}x\right) \xrightarrow{d} \widetilde{C}_q\sqrt{\int_{-\infty}^{\infty}(L_t^x)^q\,\mathrm{d}x}\,Z,$$

with C_{2p} given by Marcus/Rosen (2006).

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with C_{2p} given by Marcus/Rosen (2006).

Unfortunately, didn't succeed to make this rigorous using this approach.

small/big blocks approach (w/ M. Podolskij)

Theorem (C., Podolskij, 2017+)

For f with polynomial growth,

$$\int_{-\infty}^{\infty} f\left(\frac{\Delta_h L_t^x}{\sqrt{h}}\right)\,\mathrm{d}x \xrightarrow{P} \int_{-\infty}^{\infty} \mathrm{E}\left[f\left(2\sqrt{L_t^x}\,N\right)\right]\,\mathrm{d}x,$$

where $N \sim \mathcal{N}(0,1)$*, independent of* $(L_t^x)_{x\in\mathbb{R}}$.

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- Heuristic for proof as before:

$$\frac{\Delta_h L_t^x}{\sqrt{h}} \approx 2L_t^x \frac{\Delta_h B_t^x}{\sqrt{h}}$$

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- small/big blocks technique to break asymptotic dependence of increments

small/big blocks approach (w/ M. Podolskij)

In particular:

$$\int_{-\infty}^{\infty} \left(\frac{\Delta_h L_t^x}{\sqrt{h}} \right)^q \, \mathrm{d}x \xrightarrow{P} C_q \int_{-\infty}^{\infty} (L_t^x)^{q/2} \, \mathrm{d}x,$$

with $C_{2p+1} = 0$.

small/big blocks approach (w/ M. Podolskij)

Theorem (C., Podolskij, 2017+)

For $f \in C^1(\mathbb{R})$ such that f, f' have polynomial growth,

$$\frac{1}{\sqrt{h}} \left(\int_{-\infty}^{\infty} f\left(\frac{\Delta_h L_t^x}{\sqrt{h}}\right) \mathrm{d}x - \int_{-\infty}^{\infty} \mathrm{E}\left[f\left(2\sqrt{L_t^x}\, N\right) \,\middle|\, L_t^x \right] \mathrm{d}x \right)$$

$$\xrightarrow{d} \sqrt{\int_{-\infty}^{\infty} U_f\left(2\sqrt{L_t^x}\right) \mathrm{d}x}\, Z$$

where $Z, N \sim \mathcal{N}(0,1)$, independent of $(L_t^x)_{x\in\mathbb{R}}$.

small/big blocks approach (w/ M. Podolskij)

In particular:

$$\frac{1}{\sqrt{h}}\left(\int_{-\infty}^{\infty}\left(\frac{\Delta_h L_t^x}{\sqrt{h}}\right)^q \,\mathrm{d}x - C_q\int_{-\infty}^{\infty}(L_t^x)^{q/2}\,\mathrm{d}x\right)\xrightarrow{d}\widetilde{C}_q\sqrt{\int_{-\infty}^{\infty}(L_t^x)^q\,\mathrm{d}x}\,Z.$$