The KPZ Equation and its space-time discretization

Giuseppe Cannizzaro
joint project with K. Matetski

University of Warwick

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The KPZ Equation and its Solution

The Kardar-Parisi-Zhang equation (KPZ) is *formally* given by

$$(\partial_t - \Delta)h \quad = \quad (\partial_x h)^2 \quad + \quad \xi \quad , \quad h(0,\cdot) = h_0(\cdot)$$

where $h = h(t,x)$ is our stochastically growing height function, h_0 the initial condition and ξ is space-time white noise.

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Set $u \stackrel{\text{def}}{=} \partial_x h$, then u solves the Stochastic Burgers Equation (SBE)

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Space-Time Discretization

Let $\varepsilon > 0$, $\Lambda_{\varepsilon^2,T} \stackrel{\text{def}}{=} \varepsilon^2\mathbb{Z} \cap (0, T]$ and $\mathbb{T}_\varepsilon \stackrel{\text{def}}{=} \varepsilon\mathbb{Z} \cap \mathbb{T}$. We want to consider

$$(\bar{D}_{t,\varepsilon^2} - \Delta_\varepsilon)u^\varepsilon(z) = D_{x,\varepsilon} B_\varepsilon(u^\varepsilon, u^\varepsilon)(z) + D_{x,\varepsilon}\xi^\varepsilon(z)\,, \quad u^\varepsilon(0,\cdot) = u_0^\varepsilon(\cdot)$$

for $z \in \Lambda_{\varepsilon^2,T} \times \mathbb{T}_\varepsilon$, where $\{\xi^\varepsilon(z)\}_z$ is a family of i.i.d. mean zero normal random variables with variance ε^{-3}

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$$B_\varepsilon(f,g)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} f(x+\varepsilon y_1)g(x+\varepsilon y_2)\mu(\mathrm{d}y_1, \mathrm{d}y_2),$$

where μ is a symmetric measure supported on the integers.

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- $\mu = \frac{1}{3}(\delta_{(0,0)} + \frac{1}{2}\delta_{(0,1)} + \frac{1}{2}\delta_{(1,0)} + \delta_{(1,1)})$, Zabusky/Sasamoto-Spohn

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AIM: Show that, in a suitable sense, $u^\varepsilon \longrightarrow u$ as $\varepsilon \to 0$.

The Derivative of the KPZ Equation

Let h be the solution to the KPZ equation, then formally $u \stackrel{\text{def}}{=} \partial_x h$ satisfies

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Let h be the solution to the KPZ equation, then formally $u \stackrel{\text{def}}{=} \partial_x h$ satisfies

$$u_\varepsilon = P_t u_0 + P' * (u_\varepsilon^2) + P' * \xi_\varepsilon$$

on $[0, T] \times \mathbb{T}$, where u_0 is the initial condition, ξ a space-time white noise , $\xi_\varepsilon \stackrel{\text{def}}{=} \xi * \varrho_\varepsilon$, ϱ_ε a smooth mollifier, P is the heat kernel, $P' \stackrel{\text{def}}{=} \partial_x P$

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$$v_\varepsilon = 4X_\varepsilon^{\vee\!\!\vee\!\!\vee} + 2P' * (v_\varepsilon X_\varepsilon^{\bullet}) + P' * F^\varepsilon_{v_\varepsilon}$$

where $X_\varepsilon^{\vee\!\!\vee\!\!\vee} \sim \frac{1}{2}^{-}$ and $X^{\bullet} \sim -\frac{1}{2}^{-}$.

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IDEAS:

- Look for a solution with the following structure

$$v(\bar z) = v(z) + v'(z)\big(X^{\rceil}(\bar z) - X^{\rceil}(z)\big) + R(z, \bar z)$$

where $X^{\rceil} = P' * X^{\bullet}$ has regularity $\frac{1}{2}^-$ and R has regularity $> \frac{1}{2}$

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  where $X^{\intercal} = P' * X^{\bullet}$ has regularity $\frac{1}{2}^-$ and R has regularity $> \frac{1}{2}$
- Then, the product to be defined is $\tilde{R}^{\intercal\bullet}(z,\bar z) \stackrel{\text{def}}{=} \big(X_\varepsilon^{\intercal}(\bar z) - X_\varepsilon^{\intercal}(z)\big)X_\varepsilon^{\bullet}(z)$.

Making sense of the product and Fixed Point

- We want to make sense of the product $v_\varepsilon X^{\bullet}_{\varepsilon}$
- We made the *ansatz* $\delta_{z,\bar{z}} v = v'(z)\delta_{z,\bar{z}} X^{\mathring{|}} + \mathcal{C}^{\frac{1}{2}+}$, where $X^{\mathring{|}} = P' * X^{\bullet}$
- We need $\tilde{R}^{\mathring{|}\bullet}_{\varepsilon}(x,y) \stackrel{\text{def}}{=} (X^{\mathring{|}}_{\varepsilon}(y) - X^{\mathring{|}}_{\varepsilon}(x)) X^{\bullet}_{\varepsilon}(x)$

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1. The Stochastic term $\tilde{R}^{\dagger\bullet}_\varepsilon$:

$$\tilde{R}^{\dagger\bullet,\varepsilon}(z,\bar{z}) = \big(X^{\dagger}_\varepsilon(\bar{z}) - X^{\dagger}_\varepsilon(z)\big)X^{\bullet}_\varepsilon(z)$$

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Theorem (Hairer '14, Gubinelli-Perkowski '15, C.-Matetski '16)

There exists a unique solution u to SBE. Moreover,

- *the map $\mathcal{S}_{SBE}$ that assigns to $(u_0, \mathbb{X}) \in \mathcal{C}^\eta \times \mathcal{X}$ the solution u is jointly locally Lipschitz continuous.*

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Theorem (Hairer '14, Gubinelli-Perkowski '15, C.-Matetski '16)

There exists a unique solution u to SBE. Moreover,

- *the map $\mathcal{S}_{SBE}$ that assigns to $(u_0, \mathbb{X}) \in \mathcal{C}^\eta \times \mathcal{X}$ the solution u is jointly locally Lipschitz continuous.*
- *for a space-time white noise ξ, $\mathbb{X}(\xi_\varepsilon)$ converges to $\mathbb{X}(\xi)$ in $\mathcal{X}$, in probability.*

Sasamoto-Spohn type models

For $\varepsilon > 0$, the family of discrete models we want to consider is

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- $\bar{D}_{t,\varepsilon^2}$ is the discrete *forward* difference and $D_{x,\varepsilon}, \Delta_\varepsilon$ are discrete operators

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$$B_\varepsilon(f,g)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} f(x+\varepsilon y_1) g(x+\varepsilon y_2) \mu(\mathrm{d}y_1, \mathrm{d}y_2) ,$$

- $\bar{D}_{t,\varepsilon^2}$ is the discrete *forward* difference and $D_{x,\varepsilon}$, Δ_ε are discrete operators

We set P^ε to be the space-time discrete heat kernel .

Sasamoto-Spohn type models

For $\varepsilon > 0$, the family of discrete models we want to consider is

$$u^\varepsilon = P^\varepsilon u_0^\varepsilon + D_{x,\varepsilon} P^\varepsilon *_\varepsilon B^\varepsilon(u^\varepsilon, u^\varepsilon) + X^{\bullet,\varepsilon}$$

where

- $z \in \Lambda_{\varepsilon^2,T} \times \mathbb{T}_\varepsilon$ for $\Lambda_{\varepsilon^2,T} \stackrel{\text{def}}{=} (0,T] \cap (\varepsilon^2\mathbb{Z})$ and $\mathbb{T}_\varepsilon \stackrel{\text{def}}{=} \mathbb{T} \cap (\varepsilon\mathbb{Z})$
- $\{\xi^\varepsilon(z)\}_z$ is a family of i.i.d. centered normal random variables with variance ε^{-3}
- B_ε is a bilinear map defined by

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- $\bar{D}_{t,\varepsilon^2}$ is the discrete *forward* difference and $D_{x,\varepsilon}$, Δ_ε are discrete operators

We set P^ε to be the space-time discrete heat kernel and $X^{\bullet,\varepsilon} \stackrel{\text{def}}{=} D_{x,\varepsilon} P^\varepsilon *_\varepsilon \xi^\varepsilon$.

Expanding u^ε

Expand u^ε around $X^{\bullet,\,\varepsilon} + X^{\vee,\,\varepsilon} + 2X^{\vee\!\!\vee,\,\varepsilon}$:

Expanding u^ε

Expand u^ε around $X^{\bullet,\varepsilon} + X^{\vee,\varepsilon} + 2X^{\overset{\vee}{\vee},\varepsilon}$:set $u^\varepsilon \stackrel{\text{def}}{=} X^{\bullet,\varepsilon} + X^{\vee,\varepsilon} + 2X^{\overset{\vee}{\vee},\varepsilon} + v^\varepsilon$, so that v^ε satisfies

$$v_\varepsilon = 4X^{\overset{\overset{\vee}{\vee}}{\vee},\varepsilon} + 2D_{x,\varepsilon}P^\varepsilon *_\varepsilon (B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})) + D_{x,\varepsilon}P^\varepsilon *_\varepsilon F^\varepsilon_{v^\varepsilon}$$

where

$$B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x) = \int v^\varepsilon(x+\varepsilon y_1)X^{\bullet,\varepsilon}(x+\varepsilon y_2)\mu(\mathrm{d}y_1, \mathrm{d}y_2)$$

Expanding u^ε

Expand u^ε around $X^{\bullet,\varepsilon} + X^{\vee,\varepsilon} + 2X^{\vee\vee,\varepsilon}$:set $u^\varepsilon \stackrel{\text{def}}{=} X^{\bullet,\varepsilon} + X^{\vee,\varepsilon} + 2X^{\vee\vee,\varepsilon} + v^\varepsilon$, so that v^ε satisfies

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where

$$B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x) = \int v^\varepsilon(x+\varepsilon y_1)X^{\bullet,\varepsilon}(x+\varepsilon y_2)\mu(\mathrm{d}y_1,\mathrm{d}y_2)$$

IDEAS

- The discrete controlled structure we can expect is

$$\delta_{z,\bar z}v^\varepsilon = v'^{,\varepsilon}(z)\int (X^{|\bullet,\varepsilon}(\bar z+\varepsilon y_2) - X^{|\bullet,\varepsilon}(z+\varepsilon y_2))\mu(\mathrm{d}y_1,\mathrm{d}y_2) + R^\varepsilon(z,\bar z)$$

where $X^{|\bullet,\varepsilon} \stackrel{\text{def}}{=} D_{x,\varepsilon}P^\varepsilon *_\varepsilon X^{\bullet,\varepsilon}$.

Expanding u^ε

Expand u^ε around $X^{\bullet,\varepsilon}+X^{\vee,\varepsilon}+2X^{\curlyvee,\varepsilon}$:set $u^\varepsilon \stackrel{\text{def}}{=} X^{\bullet,\varepsilon}+X^{\vee,\varepsilon}+2X^{\curlyvee,\varepsilon}+v^\varepsilon$, so that v^ε satisfies

$$v_\varepsilon = 4X^{\curlyvee,\varepsilon} + 2D_{x,\varepsilon}P^\varepsilon *_\varepsilon \left(B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})\right) + D_{x,\varepsilon}P^\varepsilon *_\varepsilon F^\varepsilon_{v^\varepsilon}$$

where

$$B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x) = \int v^\varepsilon(x+\varepsilon y_1)X^{\bullet,\varepsilon}(x+\varepsilon y_2)\mu(\mathrm{d}y_1,\mathrm{d}y_2)$$

IDEAS

- The discrete controlled structure we can expect is

$$\delta_{z,\bar z}v^\varepsilon = v^{\prime,\varepsilon}(z)\int \left(X^{\mathsf{I},\varepsilon}(\bar z+\varepsilon y_2) - X^{\mathsf{I},\varepsilon}(z+\varepsilon y_2)\right)\mu(\mathrm{d}y_1,\mathrm{d}y_2) + R^\varepsilon(z,\bar z)$$

  where $X^{\mathsf{I},\varepsilon} \stackrel{\text{def}}{=} D_{x,\varepsilon}P^\varepsilon *_\varepsilon X^{\bullet,\varepsilon}$.

- The term to define is then

$$\tilde R^{\mathsf{I}\bullet,\varepsilon}(x,y) = \int \left(X^{\mathsf{I},\varepsilon}(y+\varepsilon y_1) - X^{\mathsf{I},\varepsilon}(x+\varepsilon y_1)\right)X^{\bullet,\varepsilon}(y+\varepsilon y_2)\mu(\mathrm{d}y_1,\,\mathrm{d}y_2)$$

Discrete Product and Renormalization

- the product is $B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x)$
- the ansatz $\delta_{z,\bar{z}} v^\varepsilon = v^{\prime,\varepsilon}(z) \int \left(X^{\rceil,\varepsilon}(\bar{z}+\varepsilon y_2) - X^{\rceil,\varepsilon}(z+\varepsilon y_2)\right)\mu(\mathrm{d}y_1, \mathrm{d}y_2) + \dots$
- the “ill-posed” term $\tilde{R}^{\rceil\bullet,\varepsilon}$

Discrete Product and Renormalization

- the product is $B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x)$
- the ansatz $\delta_{z,\bar z} v^\varepsilon = v^{\prime,\varepsilon}(z) \int (X^{\mathbf{I}^\bullet,\varepsilon}(\bar z + \varepsilon y_2) - X^{\mathbf{I}^\bullet,\varepsilon}(z + \varepsilon y_2))\mu(\mathrm{d}y_1, \mathrm{d}y_2) + \dots$
- the "ill-posed" term $\tilde R^{\mathbf{I}^\bullet\bullet,\varepsilon}$

1. The Stochastic term $\tilde R^{\mathbf{I}^\bullet\bullet}_\varepsilon$:

$$\tilde R^{\mathbf{I}^\bullet\bullet,\varepsilon}(x,y) = \int (X^{\mathbf{I}^\bullet,\varepsilon}(y + \varepsilon y_1) - X^{\mathbf{I}^\bullet,\varepsilon}(x + \varepsilon y_1))X^{\bullet,\varepsilon}(y + \varepsilon y_2)\mu(\mathrm{d}y_1, \mathrm{d}y_2)$$

in principle does not converge

Discrete Product and Renormalization

- the product is $B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x)$
- the ansatz $\delta_{z,\bar{z}} v^\varepsilon = v^{\prime,\varepsilon}(z) \int (X^{\dot{\vert},\varepsilon}(\bar{z}+\varepsilon y_2) - X^{\dot{\vert},\varepsilon}(z+\varepsilon y_2))\mu(\mathrm{d}y_1, \mathrm{d}y_2) + \dots$
- the “ill-posed” term $\tilde{R}^{\dot{\vert}\bullet,\varepsilon}$

1. The Stochastic term $\tilde{R}^{\dot{\vert}\bullet}_\varepsilon$:

$$R^{\dot{\vert}\bullet,\varepsilon}(x,y) = \int (X^{\dot{\vert},\varepsilon}(y+\varepsilon y_1) - X^{\dot{\vert},\varepsilon}(x+\varepsilon y_1))X^{\bullet,\varepsilon}(y+\varepsilon y_2)\mu(\mathrm{d}y_1, \mathrm{d}y_2) - \mathbf{C}^\varepsilon$$

in principle does not converge $\Longrightarrow$ **RENORMALIZATION**

Discrete Product and Renormalization

- the product is $B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x)$
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- the "ill-posed" term $\tilde R^{\mathsf{I}\bullet,\varepsilon}$

1. The Stochastic term $\tilde R^{\mathsf{I}\bullet}_\varepsilon$:

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in principle does not converge $\Longrightarrow$**RENORMALIZATION**

2. The product $v_\varepsilon X^\bullet_\varepsilon$:

$$\mathcal{R}^\varepsilon_t(\mathbf{V}\bullet)^\varepsilon(x) = B^\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x)$$

where $B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x) = \int v^\varepsilon(x+\varepsilon y_1)X^{\bullet,\varepsilon}(x+\varepsilon y_2)\mu(\mathrm{d}y_1,\mathrm{d}y_2)$.

Discrete Product and Renormalization

- the product is $B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x)$
- the ansatz $\delta_{z,\bar z} v^\varepsilon = v^{\prime,\varepsilon}(z) \int (X^{\dot{|},\varepsilon}(\bar z + \varepsilon y_2) - X^{\dot{|},\varepsilon}(z + \varepsilon y_2))\mu(\mathrm{d}y_1, \mathrm{d}y_2) + \dots$
- the "ill-posed" term $\tilde R^{\dot{|}\bullet,\varepsilon}$

1. The Stochastic term $\tilde R^{\dot{|}\bullet}_\varepsilon$:

$$R^{\dot{|}\bullet,\varepsilon}(x,y) = \int (X^{\dot{|},\varepsilon}(y+\varepsilon y_1) - X^{\dot{|},\varepsilon}(x+\varepsilon y_1))X^{\bullet,\varepsilon}(y+\varepsilon y_2)\mu(\mathrm{d}y_1,\,\mathrm{d}y_2) - \mathbf{C}^\varepsilon$$

in principle does not converge $\Longrightarrow$ **RENORMALIZATION**

2. The product $v_\varepsilon X^{\bullet}_\varepsilon$:

$$\mathcal{R}^\varepsilon_t(\mathbf{V}\bullet)^\varepsilon(x) = B^\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x) - \mathbf{C}^\varepsilon\, v^{\prime,\varepsilon}(t,x)$$

where $B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x) = \int v^\varepsilon(x+\varepsilon y_1)X^{\bullet,\varepsilon}(x+\varepsilon y_2)\mu(\mathrm{d}y_1,\mathrm{d}y_2)$.

Discrete Product and Renormalization

- the product is $B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x)$
- the ansatz $\delta_{z,\bar z} v^\varepsilon = v'^{,\varepsilon}(z) \int (X^{\dot{\mathsf{I}},\varepsilon}(\bar z+\varepsilon y_2) - X^{\dot{\mathsf{I}},\varepsilon}(z+\varepsilon y_2))\mu(\mathrm{d}y_1,\mathrm{d}y_2) + \dots$
- the “ill-posed” term $\tilde R^{\dot{\mathsf{I}}\bullet,\varepsilon}$

1. The Stochastic term $\tilde R_\varepsilon^{\dot{\mathsf{I}}\bullet}$:

$$\tilde R^{\dot{\mathsf{I}}\bullet,\varepsilon}(x,y) = \int (X^{\dot{\mathsf{I}},\varepsilon}(y+\varepsilon y_1) - X^{\dot{\mathsf{I}},\varepsilon}(x+\varepsilon y_1))X^{\bullet,\varepsilon}(y+\varepsilon y_2)\mu(\mathrm{d}y_1,\,\mathrm{d}y_2) - \mathbf{C}$$

in principle does not converge $\Longrightarrow$ **RENORMALIZATION**

2. The product $v_\varepsilon X_\varepsilon^{\bullet}$:

$$\mathcal{R}_t^\varepsilon(\mathbf{V}\bullet)^\varepsilon(x) = B^\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x) - \mathbf{C}\, v'^{,\varepsilon}(t,x)$$

where $B_\varepsilon(v^\varepsilon, X^{\bullet,\varepsilon})(x) = \int v^\varepsilon(x+\varepsilon y_1)X^{\bullet,\varepsilon}(x+\varepsilon y_2)\mu(\mathrm{d}y_1,\mathrm{d}y_2)$.

Convergence

Theorem (C.-Matetski '16)

Let ξ be a space-white noise and $\{\xi^\varepsilon(z)\}_z$ be a family of independent rescaled normal random variable converging to ξ. Let u^ε be the solution to

$$\bar{D}_{t,\varepsilon^2} u^\varepsilon(z) = \Delta_\varepsilon u^\varepsilon(z) + D_{x,\varepsilon} B_\varepsilon(u^\varepsilon, u^\varepsilon)(z) - \mathbf{C} D_{x,\varepsilon} u^\varepsilon + D_{x,\varepsilon}\xi^\varepsilon(z)\,, \qquad u^\varepsilon(0,\cdot) = u_0^\varepsilon(\cdot)$$

and u be the solution to

$$\partial_t u = \Delta u + \partial_x u^2 - \mathbf{C}\partial_x u + \partial_x \xi\,, \qquad u(0,\cdot) = u_0(\cdot)$$

then if u_0^ε converges to u_0 a.s. in $\mathcal{C}^\eta$, then u^ε converges to u in probability in $\mathcal{C}^{\alpha_\star - 1}$.

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Thank you for the attention!