Renormalisation of singular stochastic PDEs

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Overview

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- $\mathcal{L}$ is an elliptic differential operator;
- ξ_i are "sufficiently nice" noises;
- $F_i(u, \nabla u, \ldots)$ is a smooth function of the jet of u.

Examples

- Dynamical Φ^4_3 model:

$$(\partial_t - \Delta)u = u^3 + \xi,$$

where $u : \mathbb{R}_+ \times \mathbb{T}^3 \to \mathbb{R}$ and ξ is space-time white noise.

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- Generalised parabolic Anderson model (gPAM):

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What we want: a deterministic theory which takes as input ξ^ε and outputs the solution to

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We want to do this in spaces where the noises a.s. take values.

Singular equations

Main difficulty: these equations can be singular.

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Such equations appear in mathematical physics.

- Scaling limits of statistical models (KPZ, PAM, Φ^4_3).
- Quantization of euclidean quantum field theories (Φ^4_3, Φ^p_2, Yang-Mills, sine-Gordon).

Stability in the limit

In fact, stability is impossible even for SDEs.

Theorem (Lyons '91)

There does not exist a Banach space $E \subset C([0,1],\mathbb{R}^2)$ such that

- *E contains all smooth paths,*
- *E contains a.e. sample path of Brownian motion,*
- *the quadratic map*

$$C^\infty([0,1],\mathbb{R}^2) \ni (W_1, W_2) \mapsto \int_0^1 \int_0^t dW_1(s) dW_2(t) dt$$

extends continuously to all of E.

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$$(\partial_t - \Delta)\tilde{u} = \varepsilon \sum_{i,j=1}^{2} f_{i,j}(\varepsilon\tilde{u})\partial_i \tilde{u}\partial_j \tilde{u} + g(\varepsilon\tilde{u})\tilde{\xi}.$$

The non-linearities disappear in the formal limit $\varepsilon \to 0$.

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In QFT, this corresponds to super-renormalisable theories.

Generalised Taylor expansion

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Idea: View u as a function of $(\xi, \xi(G * \xi))$.

Identical to the idea in rough paths to consider iterated integrals.

Extra terms

Hence we instead consider as input $(\xi, \xi(G * \xi))$ to solve PAM.

(For gPAM, one also needs the terms $((\partial_i G * \xi)(\partial_j G * \xi))_{i,j=1}^2$.)

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For a distribution ξ on $\mathbb{R}_+ \times \mathbb{T}^d$:

- These terms together must satisfy certain algebraic constraints.
- They can be given *extrinsically*.
- For smooth ξ^ε, there is a canonical choice for these terms.

Continuity of the solution map

- Given such a collection of terms $\mathbf{\Pi} = (\xi, \xi(G * \xi), \ldots)$ (a model), one can build a solution map

$$\mathcal{S}_A : \mathbf{\Pi} \mapsto U$$

('A' for 'abstract').

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- One then applies a "reconstruction" map

$$\mathcal{R} : U \mapsto u \in \mathcal{S}'(\mathbb{R}_+ \times \mathbb{T}^d).$$

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- Moreover, they commute with the “canonical lift” Ψ

$$\Psi : C^\infty(\mathbb{R}_+ \times \mathbb{T}^d) \ni \xi^\varepsilon \mapsto \mathbf{\Pi} = (\xi^\varepsilon, \xi^\varepsilon(G * \xi^\varepsilon), \dots).$$

and the classical solution map $\mathcal{S}_C$

$$\begin{tikzcd}
\mathbf{\Pi} \arrow[r, "\mathcal{S}_A"] & U \arrow[d, "\mathcal{R}"] \\
\xi^\varepsilon \arrow[u, "\Psi"] \arrow[r, "\mathcal{S}_C"] & u
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What is *not* continuous is Ψ (and thus $\mathcal{S}_C$).

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- Let $\xi^\varepsilon \in C^\infty([0,T],\mathbb{R}^n)$ be a mollification of white noise.
- As $\varepsilon \to 0$, we have convergence a.s.

$$\Psi(\xi^\varepsilon) = \mathbf{\Pi} = (\xi^\varepsilon, \xi^\varepsilon(G * \xi^\varepsilon)) \to \left(\xi, \int_0^\cdot \xi(t)dt \circ \xi(\cdot)\right)$$

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  (G is Green's function of ∂_t).
- Therefore $u^\varepsilon \to u$ a.s. where u solves an SDE.

Multi-dimensional case

In higher dimensions, this typically fails!

Example

For ξ^ε mollification of white noise on $\mathbb{T}^2$ and generic $\psi \in C^\infty(\mathbb{T}^2)$,

$$|\langle \xi^\varepsilon (G * \xi^\varepsilon), \psi \rangle| \to \infty \text{ a.s.}$$

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As a consequence, we have divergence (as distributions) of solutions to PAM

$$(\partial_t - \Delta) u^\varepsilon = u^\varepsilon \xi^\varepsilon.$$

Renormalisation

Remark

Replacing $\xi^\varepsilon(G * \xi^\varepsilon)$ by $\xi^\varepsilon(G * \xi^\varepsilon) - C_\varepsilon$,

$$C_\varepsilon := \mathbb{E}\left[\xi^\varepsilon(G * \xi^\varepsilon)(0)\right],$$

it holds that for all $\psi \in C^\infty(\mathbb{T}^2)$

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In fact, we have convergence in a space of models

$$\hat{\mathbf{\Pi}}^\varepsilon = (\xi^\varepsilon, \xi^\varepsilon(G * \xi^\varepsilon) - C_\varepsilon) \to \mathbf{\Pi} = (\xi, \xi(G * \xi)).$$

Renormalised equations

Recall that a model can be used as input to drive a PDE.

Question

What does it mean to drive the PDE

$$\text{``}(\partial_t - \Delta)u = u\xi\text{''}$$

with the couple $(\xi^\varepsilon, \xi^\varepsilon(G * \xi^\varepsilon) - C_\varepsilon)$?

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Claim: this amounts to solving the classical PDE

$$(\partial_t - \Delta)u^\varepsilon = u^\varepsilon(\xi^\varepsilon - C_\varepsilon).$$

A renormalised equation

Recall the ansatz

$$u^\varepsilon = u_0 + u_1 \, G * \xi^\varepsilon + R.$$

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$$\begin{aligned} u &= u_0 G * \xi^\varepsilon + u_1 G * (\xi^\varepsilon(G * \xi^\varepsilon) - C) + \tilde{R} \\ &= u_0 G * (\xi^\varepsilon - C) + u_1 G * (\xi^\varepsilon(G * \xi^\varepsilon)) + \tilde{R} \quad (\text{using } u_0 = u_1) \\ &= G * (u^\varepsilon(\xi^\varepsilon - C)) \quad (\text{using } u \approx u_0) \end{aligned}$$

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The usual Itô–Stratonovich correction appears for identical reasons.

A general approach

We aim to give a general description of this phenomenon.

Three earlier works are very important

- Bruned–Hairer-Zambotti 2016 (algebraic)
- Chandra–Hairer 2016 (analytic/stochastic)
- Hairer 2014 (core theory)

A general approach

The general theory can be summarised as follows.

$$
\begin{tikzcd}
\mathrm{Eq} \arrow[d, leftrightarrow] & \times & \mathcal{M} \arrow[loop above, "M_\varepsilon \text{ [BHZ16]}"] \arrow[r, "\mathcal{S}_A"] & \mathcal{D}^\gamma \arrow[d, "\mathcal{R}", "\text{[Hai14]}"'] \\
\mathrm{Eq} \arrow[loop below, dashed, "M_\varepsilon \text{ [BCCH]}"] & \times & \mathcal{C} \arrow[u, "\Psi", "\text{[CH16]}"'] \arrow[r, "\mathcal{S}_C"] & \mathcal{C}^\alpha(\mathbb{R}_+ \times \mathbb{T}^d)
\end{tikzcd}
$$

Non-linearities and trees

Consider the equation

$$(\partial_t - \mathcal{L})u = \sum_{i=0}^{n} F_i(u, \nabla u, \dots)\xi_i.$$

For every rooted decorated tree we recursively use $(F_i)_{i=0}^n$ to construct a function of the jet of u.

Non-linearities and trees

Every rooted decorated tree can be uniquely written as either

$$\Xi_j X^p, \quad \text{(no edges)}$$

or

$$\Xi_j X^p \mathcal{I}_{p_1}[\tau_1] \dots \mathcal{I}_{p_k}[\tau_k], \quad (k \text{ edges at the root})$$

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where

- Ξ_j is a noise term, $j \in \{0, \dots, n\}$
- X^p is a polynomial term $p \in \mathbb{N}^{d+1}$
- $\mathcal{I}_{p_i}$ is a convolution with $\partial_{p_i} G$, $p_i \in \mathbb{N}^{d+1}$
- τ_i is another rooted tree.

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$$D_p F$$
is the derivative w.r.t the coordinate p;
- For $j \in \{0, \dots, d\}$
$$\partial^j F = \sum_{p \in \mathbb{N}^{d+1}} Y_{p+j} D_p F$$
where Y_{p+j} is multiplication by the coordinate $(p+j) \in \mathbb{N}^{d+1}$.

Non-linearities and trees

- For the base case $\tau = \Xi_j X^p$ (no edges)

$$F^\tau := \partial^p F_j$$

(∂^p for $p \in \mathbb{N}^{d+1}$ defined via composition).

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- Then inductively for every other tree

$$\tau = \Xi_j X^p \mathcal{I}_{p_1}[\tau_1] \ldots \mathcal{I}_{p_k}[\tau_k],$$

$$F^\tau := (\text{comb. factor}) \left(\prod_{i=1}^k F^{\tau_i} \right) \partial^p \left(\prod_{i=1}^k D_{p_i} \right) F_j .$$

Non-linearities and trees

Final ingredient: every tree τ comes with a degree $|\tau|$ defined inductively in terms of

- Regularising effect of the Green's function G.
- Regularity of the noises $(\xi_i)_{i=0}^n$.

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- Regularity of the noises $(\xi_i)_{i=0}^n$.

Equation is subcritical $\leftrightarrow$ finitely many trees below any degree.

Black-box convergence theorem

Theorem (Renormalised SPDEs)

Consider a subcritical SPDE on $\mathbb{R}_+ \times \mathbb{T}^d$

$$(\partial_t - \mathcal{L})u = \sum_{i=0}^{n} F_i(u, \nabla u, \dots)\xi_i.$$

Suppose that $(\xi_i)_{i=0}^n$ *are "sufficiently nice" stationary noises and* $(\xi_i^\varepsilon)_{i=0}^n$ *are mollifications.*

Black-box convergence theorem

Theorem (Renormalised SPDEs)

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Then there exists a family of constants

$$\{C_{\tau,\varepsilon} \in \mathbb{R} \mid |\tau| < 0, \varepsilon > 0\}$$

such that

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Theorem (Renormalised SPDEs)

the solutions to the classical PDE

$$(\partial_t - \mathcal{L})u^\varepsilon = \sum_{i=0}^{n} F_i(u^\varepsilon, \nabla u^\varepsilon, \dots)\xi_i^\varepsilon + \sum_{|\tau|<0} C_{\tau,\varepsilon} F^\tau(u^\varepsilon, \nabla u^\varepsilon, \dots),$$

$$u^\varepsilon(0,\cdot) = u_0 \in C^\alpha(\mathbb{T}^d),$$

converges in probability to a locally defined distribution on $\mathbb{R}_+ \times \mathbb{T}^d$ *(blow-up is possible) and the limit is a continuous function of* u_0*.*

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Remarks

- The constants $C_{\tau,\varepsilon}$ are *not* unique; a possible choice is given by the BPHZ renormalisation of [BHZ16].
- The limit is used to define a solution of the original SPDE.

Example

Example (gPAM)

Recall gPAM on $\mathbb{R}_+ \times \mathbb{T}^2$:

$$(\partial_t - \Delta)u = \sum_{i,j=1}^{2} f_{i,j}(u)\partial_i u \partial_j u + g(u)\xi.$$

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- Degrees determined by $|\Xi_0| = 0$, $|\Xi| = -1-\kappa$, $|\mathcal{I}| = 2$.
- There are two trees of negative degree (which need renormalisation)

$$\begin{aligned} \tau &= \Xi\mathcal{I}[\Xi], & |\tau| &= -2\kappa, \\ \sigma_{i,j} &= \Xi_0\mathcal{I}_i[\Xi]\mathcal{I}_j[\Xi], & |\sigma_{i,j}| &= -2\kappa. \end{aligned}$$

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The counterterms:

$$F^{\tau} = g(u)g'(u),$$
$$F^{\sigma_{i,j}} = g(u)^2 f_{i,j}(u).$$

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The renormalised SPDE takes the form

$$(\partial_t - \Delta)u^{\varepsilon} = \sum_{i,j=1}^{2} f_{i,j}(u^{\varepsilon})\partial_i u^{\varepsilon}\partial_j u^{\varepsilon} + g(u^{\varepsilon})\xi^{\varepsilon}$$
$$+ C_{\varepsilon} g(u^{\varepsilon})g'(u^{\varepsilon}) + \sum_{i,j=1}^{2} C_{\varepsilon}^{i,j} g(u^{\varepsilon})^2 f_{i,j}(u^{\varepsilon}).$$

Thank you!

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