# Comparison principles for stochastic heat equations

*with coloured noise.*
*(joint with E.Nualart and with M.Joseph and S.T Li)*

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- Consider

$$\frac{\partial}{\partial t} u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + \sigma\big(u_t(x)\big)\dot{F}(t,x) \quad t \geq 0,\, x \in \mathbb{R}^d$$

- The initial condition is a bounded non-negative function $u_0(x)$.
-

$$Cov\big(\dot{F}(t,x),\, \dot{F}(s,y)\big) = \delta_0(t-s) f(x-y),$$

with

$$f(x-y) = |x-y|^{-\beta}, \quad 0 < \beta < d.$$

- We want to extend results about the following Parabolic Anderson Model:

$$\frac{\partial}{\partial t}u_t(x) = \Delta u_t(x) + u_t(x)\dot{W}(t,x) \quad t \geq 0,\, x \in \mathbb{R}^d$$

to the more general equation:

$$\frac{\partial}{\partial t}u_t(x) = -\nu(-\Delta)^{\alpha/2}u_t(x) + \sigma\big(u_t(x)\big)\dot{F}(t,x)$$

# The 'obvious' difficulties:

- The operator is more general.
- The noise is colored in time.
- The equation is non-linear
- The initial condition is not bounded below.

## The general aim

- Existence and uniqueness is not the issue here.
- We are interested in boundedness properties of the solution.
- We will assume that

$$c_1 x \leq \sigma(x) \leq c_2 x \quad \text{for} \quad x \in \mathbb{R}$$

# The mild solution in the sense of Walsh

- We look at the integral equation

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u_s(y))\, F(ds\, dy),$$

  where $p_t(x)$ is the density of the stable process associated with the fractional Laplacian.

- There is no formula for $p_t(x)$. We have

$$c_1\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \leq p_t(x) \leq c_2\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right)$$

# The main results(with E.Nualart).

- Assume that the initial solution is bounded below by a positive constant.
- Roughly speaking for large $R$, we have

$$\sup_{|x|\leq R} u_t(x) \approx e^{\text{const}\cdot(\log R)^{\alpha/(2\alpha-\beta)}}.$$

- If the initial function is not bounded below, then the situation is more complicated.
- $$\lim_{|x|\to\infty} u_0(x) = 0 \quad \text{and} \quad u_0(x) \leq u_0(y) \quad \text{whenever} \quad |x| \geq |y|.$$

  Set
  $$\Lambda := \lim_{|x|\to\infty} \frac{|\log u_0(x)|}{(\log|x|)^{\alpha/(2\alpha-\beta)}}.$$

## Theorem (F.,Nualart(2017+))

*Suppose that the initial function $u_0$ satisfies the above conditions.*

- *If* $0 < \Lambda < \infty$, *there exists a random variable $T$ such that*

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}^d} u_t(x) < \infty, \quad \forall t < T \quad \textit{and} \quad \sup_{x\in\mathbb{R}^d} u_t(x) = \infty, \quad \forall t > T\right) = 1.$$

- *If* $\Lambda = \infty$,

$$\mathbb{P}(\sup_{x\in\mathbb{R}^d} u_t(x) < \infty, \quad \forall t > 0) = 1.$$

- *If* $\Lambda = 0$,

$$\mathbb{P}(\sup_{x\in\mathbb{R}^d} u_t(x) = \infty, \quad \forall t > 0) = 1.$$

## Necessary ingredients

Both results require

- Sharp tail estimates that is bounds on

$$\mathbb{P}(u_t(x) > \lambda)$$

- "Independent quantities".
- When the initial condition is not bounded below, we need an extra argument which we will describe later.

- For the Anderson Model ($\sigma(x) \propto x$), we can sharp estimates for moments via Feynman-Kac formulas. We don't have such formulas here.
- For the independent quantities, we use a localisation procedure to obtain these quantities.

# The localisation procedure: the white noise case

- We look at the truncated equation:

$$U_t^{(n,j)}(x) = (p_t * u_0)(x) + \int_0^t \int_{B(x,\,(nt)^{1/\alpha})} p_{t-s}(x-y)\sigma(U_t^{(n,j-1)}(y))F^{(n)}(ds\,dy)$$

- $\{U^{(n,n)}(x_i)\}_{i=1}^{\infty}$ are independent random variables if $\|x_i - x_j\| \geq 2n^{1+1/\alpha}t^{1/\alpha}$.
- $U_t^{(n,n)}(x)$ approximates the solution $u_t(x)$.

- To obtain sharp tail estimates, we need sharp estimates on $\mathbb{E}|u_t(x)|^k$.
- For the PAM model, that is for $\sigma(u) \propto u$, this has been done.
- We have to find a way to transfer the information about the moments for the PAM model to the more general non-linear model.

# Motivating problem for work with Joseph and Li

- $$\frac{\partial}{\partial t} u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + \sigma_1\big(u_t(x)\big)\dot{F}(t,x)$$
- $$\frac{\partial}{\partial t} u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + \sigma_2\big(u_t(x)\big)\dot{F}(t,x)$$
- Question: Can we compare the moments?

**Theorem (F.,Joseph, Tang-Li(2017+))**

*Let $u$ be the solution to the SPDE and $\bar{u}$ be the solution to the same SPDE but with $\sigma$ replaced by another Lipschitz continuous function $\bar{\sigma}$ such that $\sigma(x) \geq \bar{\sigma}(x) \geq 0$ holds for all $x \in \mathbb{R}_+$. Then for any integer $m \geq 1$*

$$\mathbb{E}\big[u_t(x)^m\big] \geq \mathbb{E}\big[\bar{u}_t(x)^m\big]. \tag{0.1}$$

# Moment bounds.

- Since
$$c_1 x \leq \sigma(x) \leq c_2 x \quad \text{for} \quad x \in \mathbb{R}$$
- There exists a positive constant $A$ such that for $x \in \mathbb{R}^d$, $k \geq 2$ and $t > 0$,
$$\frac{\underline{u}_0^k}{A^k} \exp(\frac{1}{A} k^{(2\alpha-\beta)/\alpha-\beta} t) \leq \mathbb{E}|u_t(x)|^k \leq A^k \bar{u}_0^k \exp(A k^{(2\alpha-\beta)/\alpha-\beta} t),$$
where
$$\underline{u}_0 := \inf_{x \in \mathbb{R}^d} u_0(x) \quad \text{and} \quad \bar{u}_0 := \sup_{x \in \mathbb{R}^d} u_0(x).$$

## Tail estimates

- $$\mathbb{P}(u_t(x) > \lambda) \lesssim \exp\left(-\frac{c_1}{t^{(\alpha-\beta)/\alpha}} \left|\log \frac{\lambda}{A\bar{u}_0}\right|^{(2\alpha-\beta)/\alpha}\right)$$

- $$\mathbb{P}(u_t(x) > \lambda) \gtrsim \exp\left(-\frac{c_1}{t^{(\alpha-\beta)/\alpha}} \left(\log \frac{2\lambda A}{\underline{u}_0}\right)^{(2\alpha-\beta)/\alpha}\right)$$

# General strategy for the proof of the comparison principle.

Consider the following SDEs

$$dX_t = b(X_t)dt + \sigma_1\big(X_t\big)dB_t,$$

and

$$dY_t = b(Y_t)dt + \sigma_2\big(Y_t\big)dB_t,$$

with the same initial condition $x_0$. Set

$$P_t^{\sigma_1}f(x) := \mathbb{E}^x f(X_t) \quad \text{and} \quad P_t^{\sigma_2}f(x) := \mathbb{E}^x f(Y_t),$$

and let $\mathcal{L}^{\sigma_1}$, $\mathcal{L}^{\sigma_2}$ be the generators corresponding to $X_t$ and $Y_t$ respectively.

The idea is to show that

$$P_t^{\sigma_1} f(x) \geq P_t^{\sigma_2} f(x), \tag{0.2}$$

whenever $\sigma_1 \geq \sigma_2$ and $f$ belonging to some appropriate class of functions. By appealing to the following "integration by parts" formula

$$P_t^{\sigma_1} f(x) - P_t^{\sigma_2} f(x) = \int_0^t P_{t-s}^{\sigma_2} (\mathcal{L}^{\sigma_1} - \mathcal{L}^{\sigma_2}) P_t^{\sigma_1} f(x)\, ds,$$

showing (0.2) amounts to proving

$$(\mathcal{L}^{\sigma_1} - \mathcal{L}^{\sigma_2}) P_t^{\sigma_1} f(x) \geq 0.$$

- We approximate the SPDE by a system of interacting SDEs.
- We show that the strategy above works for this system of SDEs as well.
- We can also show that various other comparison principles follow from this approximation.

## The system of SDEs

- We work on discrete space. The stable process is approximated by the random walks which have large jumps. The space-time noise is approximated by Brownian motion.
- We look at a system of the form

$$U_t(x) = \big(P_t * U_0\big)(x) + \int_0^t \sum_{y\in\mathbb{Z}^d} P_{t-s}(x-y)\sigma\big(U_s(y)\big)\, dB_s(y), \quad x \in \mathbb{Z}^d$$

  where

$$\big(P_t * U_0\big)(x) := \sum_{y\in\mathbb{Z}^d} P_t(x-y)U_0(y).$$

  and

$$P_t(x) := \mathbb{P}(X_t = x), \quad x \in \mathbb{Z}^d. \tag{0.3}$$

- Fix $T > 0$. Then, under some assumptions, uniformly for $\epsilon^\alpha \leq t \leq T$ and $|x| > t^{1/\alpha}$, $x \in \epsilon\mathbb{Z}^d$, we have

$$\left| \frac{1}{\epsilon^d} P(\epsilon X_{t/\epsilon^\alpha} = x) - p_t(x) \right| \lesssim \frac{t\epsilon^a}{|x|^{d+\alpha+a}}.$$

# Another comparison principle.

**Theorem (F.,Joseph, Tang-Li(2017+))**

*Let $u$ and $v$ be solutions to the SPDE with initial profiles $u_0$ and $v_0$ respectively, and such that $u_0(x) \leq v_0(x)$ for all $x \in \mathbb{R}^d$. Then*

$$\mathbb{P}\left[u_t(x) \leq v_t(x) \text{ for all } x \in \mathbb{R}^d,\ t \geq 0\right] = 1.$$

- We need a new idea to consider the case when the initial condition is not bounded below.
- The idea is to compare the solution when the initial condition is a constant with the solution when the initial condition is not bounded below.

**Theorem (F., E. Nualart(2017+))**

*Let $a \in \mathbb{R}^d$ and $R > 1$. Let $u$ and $v$ be the solution to our SPDE with respective initial conditions $u_0$ and $v_0$. Suppose that on $B(a, 2R)$, $u_0(x) = v_0(x)$. Then for all $t > 0$ and $k \geq 2$, we have*

$$\sup_{x \in B(a,R)} \mathbb{E}|u_t(x) - v_t(x)|^k \lesssim \frac{1}{R^{\alpha k/2}} e^{c_1 k^{(2\alpha-\beta)/(\alpha-\beta)} t},$$

*where $c_1$ is some positive constant.*

- Suppose that for $R > 0$, the function $f_R(\cdot)$ is a non-negative non-decreasing locally integrable function on $[0,\ T]$ satisfying the following

$$f_R(t) \leq A_R(t) + B \int_0^t \frac{f_{2R}(s)}{(t-s)^\gamma}\ ds,$$

where $A_R(\cdot)$ is also a locally integrable non-decreasing function $[0,\ T]$, $B$ is a positive constant and $\gamma < 1$. If

$$\sup_{n\geq 1} f_{nR}(t) < \infty \quad \text{and} \quad A_{nR}(t) \leq A_{(n-1)R}(t) \quad \text{for} \quad n \geq 1,$$

then there exists a positive constant $c_1$ such that

$$f_R(t) \lesssim A_R(t) e^{c_1 t} \quad \text{for} \quad 0 \leq t \leq T.$$

# Thank You