Simulation methods based on the parametrix

A second order method

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Ritsumeikan University
LMS-EPSRC Durham Symposium-Stochastic Analysis

July, 2017

*Supported by KAKENHI 24340022, 16H03642, 16K05215

Outline

Goal: Unbiased simulatable formula for $\mathbb{E}[f(X_T)]$

Methods for obtaining the unbiased formula

The probabilistic parametrix method

A general methodology

Creating a second order method

Simulation experiments

Some Conclusions

Goal: Obtain formulas of the type

$$\mathbb{E}\left[f(X_T)\right] = \mathbb{E}\left[f(\bar{X}^{\pi}_T)Z_T\right],$$

$$Z_T = \mathbf{1}_{N_T=0} + \mathbf{1}_{N_T>0}\prod_{i=0}^{N_T-1}\theta_{\tau_{i+1}-\tau_i}(\bar{X}^{\pi}_{\tau_i}, \bar{X}^{\pi}_{\tau_{i+1}}).$$

1. $f \in \mathcal{B}_b(\mathbb{R}^d) \equiv \{f : \mathbb{R}^d \to \mathbb{R} \text{ is a bounded measurable function}\}$.
2. $\bar{X}^{\pi}$: approximation process for X defined for any partition π of $[0, T]$.
3. N is a Poisson process with jump times $\{\tau_i\}_i$ independent of $\{\bar{X}^{\pi}; \pi\}$.
4. In the above formula, we abuse the notation letting $\pi := \{\tau_i\}_i$.

Ways of reading the formula:

1. Girsanov Theorem
2. Feynman-Kac formula
3. Multi-level Monte Carlo

Ways of obtaining this formula

1. Taylor expansion
2. Malliavin Calculus
3. Probabilistic Parametrix method

Expand $f(x) - f(x_0)$. Define $g(\alpha) = f(x_0 + \alpha(x - x_0))$ then by mean value theorem

$$f(x)-f(x_0) = g(1)-g(0) = \int_0^1 g'(\alpha)d\alpha = \int_0^1 f'(x_0+\alpha(x-x_0))d\alpha(x-x_0).$$

Notice the relation *Functional Distance* = *Derivative* × *Distance between arguments*. The Taylor formula for a analytic function f can be rewritten as

$$\boxed{f(x) = e^{\lambda}\mathbb{E}\left[f^N(x_0)(\lambda^{-1}(x - x_0))^N\right]}.$$

We intend to show one method to do this in infinite dimensions.

The probabilistic parametrix method The goal is to repeat this argument for $\mathbb{E}[f(X_T)]$ for

$$X_t = x_0 + \int_0^t \sigma(X_s)dW_s{}^{\dagger}.$$

The approximation is

$$\bar{X}_t = x_0 + \sigma(x_0)W_t.$$

Goal: Find an expansion for $\mathbb{E}[f(X_T)] - \mathbb{E}[f(\bar{X}_T)]$ in powers of T for a class of functions f.

Let $P_t f(x) = \mathbb{E}[f(X_t^x)]$ with generator L. By Itô's formula and IBP:

$$P_{T-r}f(\bar{X}_r) \overset{\mathbb{E}}{=} P_T f(x) + \int_0^r (\bar{L} - L)P_{T-t}f(\bar{X}_t)dt,$$

$$f(X_T) \overset{\mathbb{E}}{=} f(\bar{X}_T) + \int_0^T ds P_s f(\bar{X}_{T-s})\theta_{T-s}(x, \bar{X}_{T-s}).$$

Here for any $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,$a \equiv \sigma\sigma^* \in C_b^2$ and uniformly elliptic

$$\theta_t(x, y) = 2^{-1}\mathbb{E}\left[H^{i,j}(\bar{X}_t, a^{i,j}(\bar{X}_t) - a^{i,j}(x))\middle|\bar{X}_t = y\right].$$

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†When we would like to emphasize the initial point x_0, we will use $X_t^{x_0}$ instead of X_t.

$$\theta_t(x,y) = 2^{-1}\mathbb{E}\left[H^{i,j}(\bar{X}_t, a^{i,j}(\bar{X}_t) - a^{i,j}(x))\middle|\bar{X}_t = y\right].$$

The rate of degeneration of $\theta_t(x, \bar{X}_t) = O(t^{-1/2})$. Therefore, there exists a constant C which depends on $||f||_\infty$, $||a||_{2,\infty}$ and T such that

$$\sup_x \boxed{\left|\mathbb{E}\left[f(X_T^x)\right] - \mathbb{E}\left[f(\bar{X}_T^x)\right]\right| \le CT^{1/2}}.$$

Here $H^i(\bar{X}_T, Y)$ denotes the IBP weight with respect to $\bar{X}_T$ (Gaussian) in the sense that $\mathbb{E}[\partial_i f(\bar{X}_T)Y] = \mathbb{E}[f(\bar{X}_T)H^i(\bar{X}_T, Y)]$. This is a scheme of order one.

Now we build the unbiased scheme by randomization of time.

$$\mathbb{E}\left[\int_0^T \theta_s(x, \bar{X}_s)P_{T-s}f(\bar{X}_s)ds\right] = T\mathbb{E}\left[\theta_U(x, \bar{X}_U)P_{T-U}f(\bar{X}_U)\right].$$

Here U is a uniform random variable on $[0, T]$ independent of W.

Repeating the argument and remembering that condition on N_T the jump times of the Poisson process are distributed according to the order statistics, one obtains the final formula.
Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$, $\sigma \in C_b^\infty$ and uniformly elliptic. Define

$$Z_t := e^{\lambda t} \prod_{i=0}^{N_t - 1} \lambda^{-1} \theta_{\tau_{i+1} - \tau_i}(\bar{X}^\pi_{\tau_i}, \bar{X}^\pi_{\tau_{i+1}}).$$

Then

$$\mathbb{E}\left[f(X_T)\right] = \mathbb{E}\left[f(\bar{X}^\pi_T) Z_T\right].$$

But this formula due to the degeneration of θ_t has infinite variance in most cases. Importance sampling in time is one solution.

Theorem Fix $\mu \geq 0$, $q > 0$. Suppose that there exists $\mathbb{R}$- valued measurable functions $\eta_t(x, y)$, $\theta_t(x, y)$, $\theta_t^\eta(x, y)$, $0 < t \leq T$, $x, y \in \mathbb{R}^d$ s.t. they satisfy the following integrability estimate. There exists a constant C

- $\sup_x \mathbb{E}[|\theta_t(x, \bar{X}_t^x)(1 + \eta_t(x, \bar{X}_t^x)) + \theta_t^\eta(x, \bar{X}_t^x)|] \leq Ct^{(q-2)/2}$,
- $\sup_x \sup_{0 \leq t \leq T} \mathbb{E}[|\eta_t(x, \bar{X}_t^x)|] \leq C$.

Furthermore assume that the following first order expansion formula is valid for $f \in C_c^\infty(\mathbb{R}^d)$ and $0 < t \leq r \leq T$

$$\mathbb{E}\left[e^{\mu T^{q/2}} f(X_T)\right] - \mathbb{E}\left[f(\bar{X}_T)(1 + \eta_T(x, \bar{X}_T))\right] = \mathbb{E}\left[\int_0^T ds e^{\mu s^{q/2}} P_s f(\bar{X}_{T-s}) \Theta_{T-s}^T(x, \bar{X}_{T-s})\right],$$

$$\Theta_t^r(x, y) := 2^{-1}\mu q(r-t)^{(q-2)/2}(1 + \eta_t(x, y)) + \theta_t(x, y)(1 + \eta_t(x, y)) + \theta_t^\eta(x, y).$$

Then one has the following error estimate. There exists a constant C which depends on $||f||_\infty$, $||a||_{2,\infty}$ and T such that

$$\sup_x \boxed{\left|\mathbb{E}\left[e^{\mu T^{q/2}} f(X_T^x)\right] - \mathbb{E}\left[f(\bar{X}_T^x)(1 + \eta_T(x, \bar{X}_T^x))\right]\right| \leq CT^{q/2}}.$$

Then

$$\mathbb{E}\left[f(X_T)\right] = \mathbb{E}\left[f(\bar{X}_T^\pi)Z_T\right],$$

$$Z_T := e^{\lambda T - \mu T^{q/2}}(1 + \eta_{T-\tau_{N_T}}(\bar{X}^\pi_{\tau_{N_T}}, \bar{X}^\pi_T)) \prod_{i=0}^{N_T - 1} \lambda^{-1}\Theta^{T-\tau_i}_{\tau_{i+1}-\tau_i}(\bar{X}^\pi_{\tau_i}, \bar{X}^\pi_{\tau_{i+1}}).$$

Moreover, suppose that $q \geq 2$ and if for fixed $p > 0$, one has that there exists a constant C such that

- $\sup_x \mathbb{E}[|\theta_t(x, \bar{X}^x_t)(1 + \eta_t(x, \bar{X}^x_t)) + \theta^\eta_t(x, \bar{X}^x_t)|^p] \leq Ct^{(q-2)p/2}$,
- $\sup_x \sup_{0\leq t\leq T} \mathbb{E}[|\eta_t(x, \bar{X}^x_t)|^p] \leq C$.

Then $\boxed{\mathbb{E}\left[|Z_T|^p\right] < \infty}$.

Exponential scaling: $e^{-\mu T^{q/2}}$. Poisson sampling: $e^{\lambda T}$

Next: How to obtain a second order method.

Studying the residue for a second order method (but it has to be simple and iterative)

$$
\begin{aligned}
\mathbb{E}\left[f(\bar{X}_t)\theta_t(x,\bar{X}_t)\right] &= \mathbb{E}\left[f(\bar{X}_t)2^{-1}H^{i,j}(\bar{X}_t, a^{i,j}(\bar{X}_t) - a^{i,j}(x))\right] \\
&= 2^{-1}\partial_m a^{i,j}(x)\mathbb{E}\left[\partial_{i,j} f(\bar{X}_t)(\bar{X}_t - x)^m\right] + O(1) \\
&= 2^{-1}a^{k,l}\partial_l a^{i,j}(x)\mathbb{E}\left[\partial_{i,j} f(\bar{X}_t)a^{-1}_{k,m}(\bar{X}_t - x)^m\right] + O(1) \\
&= 2^{-1}t a^{k,l}\partial_l a^{i,j}(x)\mathbb{E}\left[\partial_{i,j,k} f(\bar{X}_t)\right] + O(1) \\
&= 2^{-1}t a^{k,l}\partial_l a^{i,j}(x)\mathbb{E}\left[f(\bar{X}_T)H^{i,j,k}(\bar{X}_T, 1)\right] + O(1).
\end{aligned}
$$

The (explicit) correction term is then

$$
\eta_t(x,\bar{X}_t) := 4^{-1}t^2 a^{k,l}\partial_l a^{i,j}(x)H^{i,j,k}(\bar{X}_t, 1),
$$

$$
\mathbb{E}\left[e^{\mu T}f(X_T)\right] - \mathbb{E}\left[f(\bar{X}_T)(1+\eta_T(x,\bar{X}_T))\right] = \mathbb{E}\left[\int_0^T ds e^{\mu s}P_s f(\bar{X}_s)\Theta_{T-s}(x,\bar{X}_{T-s})\right]^{\ddagger}.
$$

$$
\sup_x \left|\mathbb{E}\left[e^{\mu T}f(X_T^x)\right] - \mathbb{E}\left[f(\bar{X}_T^x)(1+\eta_T(x,\bar{X}_T^x))\right]\right| \le CT.
$$

‡ In this case Θ does not depend on r, we will shorten the notation $\Theta_t(x,y) \equiv \Theta_t^r(x,y)$.

$$\boxed{\sup_x \left|\mathbb{E}\left[e^{\mu T} f(X_T^x)\right] - \mathbb{E}\left[f(\bar{X}_T^x)(1+\eta_T(x,\bar{X}_T^x))\right]\right| \leq CT.}$$

Therefore this gives a method of order two and then the probabilistic representation is valid for

$$Z_T := e^{(\lambda-\mu)T}(1+\eta_{T-\tau_{N_T}}(\bar{X}^\pi_{\tau_{N_T}}, \bar{X}^\pi_T)) \prod_{i=0}^{N_T-1} \lambda^{-1}\Theta_{\tau_{i+1}-\tau_i}(\bar{X}^\pi_{\tau_i}, \bar{X}^\pi_{\tau_{i+1}}).$$

The p-moments of Z_T are finite.

$$\begin{aligned}
\Theta_t(x,\bar{X}_t) &= \mu(1+\eta_t(x,\bar{X}_t)) + \theta_t(x,\bar{X}_t)(1+\eta_t(x,\bar{X}_t)) + \theta^\eta_t(x,\bar{X}_t),\\
\theta^\eta_t(x,\bar{X}_t) &= 2^{-1}H^{i,j}(\bar{X}_t, \eta_t(x,\bar{X}_t)(a^{i,j}(\bar{X}_t) - a^{i,j}(x)))\\
&\quad - \theta_t(x,\bar{X}_t)\eta_t(x,\bar{X}_t) - 2^{-1} t a^{k,l}\partial_l a^{i,j}(x) H^{i,j,k}(\bar{X}_t, 1).
\end{aligned}$$

Recall

$$\theta_t(x,\bar{X}_t) = 2^{-1}H^{i,j}(\bar{X}_t, a^{i,j}(\bar{X}_t) - a^{i,j}(x)).$$

Experiment set-up: $\sigma(x) = \sigma(\sin(\omega x) + 2)$ and $b(x) = -\frac{x}{x^2+\frac{c_1}{3c_3}}\sigma^2(x)$. We consider the payoff function $f(x) = c_3x^3 + c_1x + c_0$. Then since,

$$b(x)f'(x) + 2^{-1}\sigma^2(x)f''(x) = 0,$$

$f(X_t)$ is a martingale and $\mathbb{E}[f(X_T)] = f(X_0)$.
In the experiment we choose the parameters given in Table 1.

c_0	c_1	c_3	X_0
0	1	1	1

Table: Parameters in experiment

Here $f(X_0) = 2$. The interest is in large parameters for σ and ω.
First, we discuss about the choice of Poisson sampling by taking the general set-up to optimize the variance over $\{p_n; n \in \mathbb{N}\}$

$$\mathbb{E}[f(X_T)] = \sum_{n=0}^{\infty} \mathbb{E}[J_n] = \mathbb{E}\left[\frac{J_N}{p_N}\right].$$

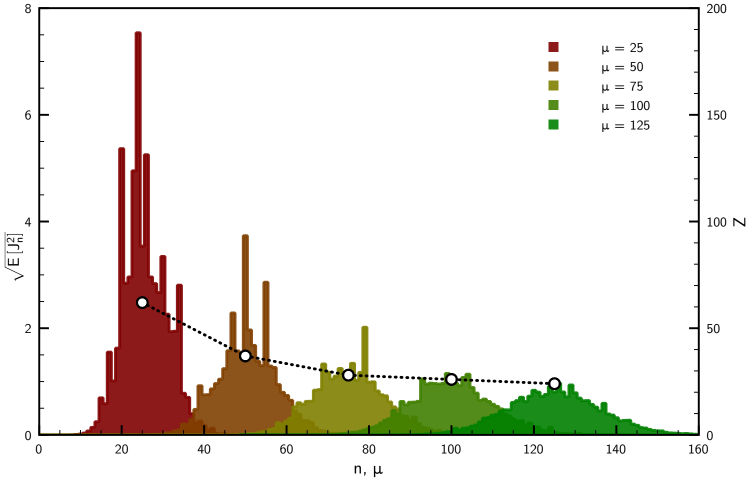


Figure: $\boldsymbol{\sigma = \omega = 0.9}$.

Variances at each level depending on the value of μ. No λ component is used. Optimization criteria is:

$$\sum_{n=0}^{\infty} \mathbb{E} p_n^{-1} \left[J_n^2 \right].$$

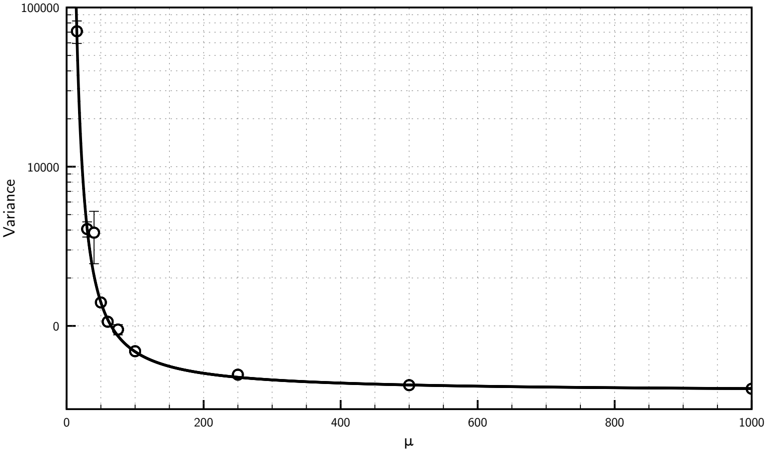


Figure: $\sigma = \omega = 0.9$.

Simulated variance, varying $\lambda = \mu$, and fitted curve.

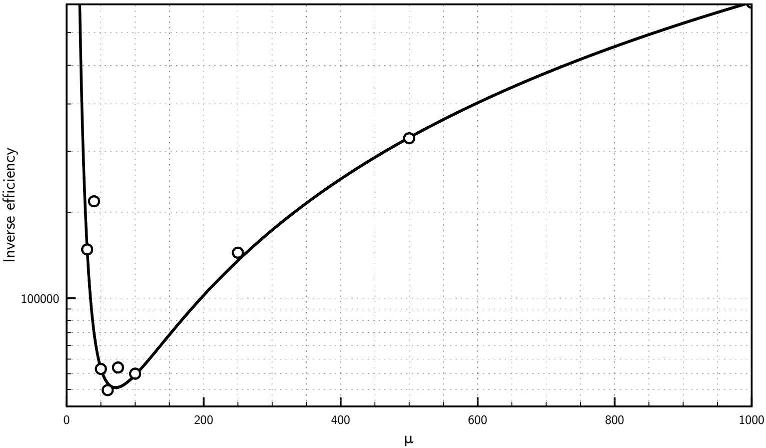
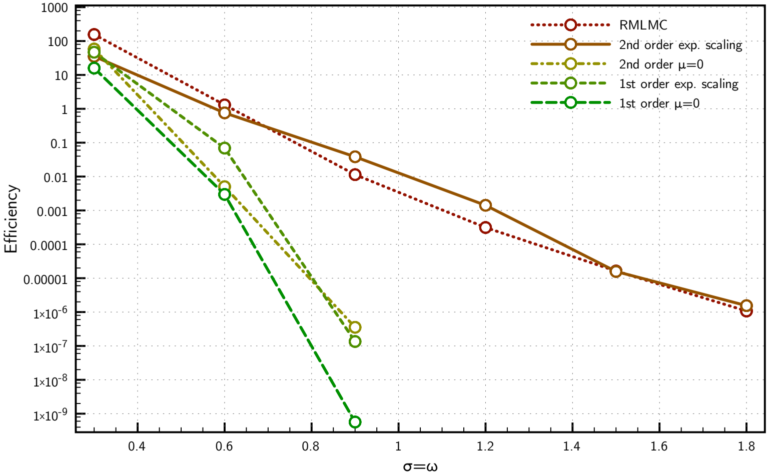


Figure: $\sigma = \omega = 1.0$.

Simulated inverse efficiency, varying $\lambda = \mu$, and fitted curve.
$\sigma = \omega = 0.9$.



Simulated efficiencies with optimal simulation parameters, when varying $\sigma = \omega$. Observe improvement from first order to second order scheme. It is comparable to RMLMC. $\mu = 0$ indicates no exponential rescaling.

1. We have presented a general set-up for higher order methods based on the parametrix approach
2. The order of the method is tied with the required accuracy and parameter values
3. One advantage is the fact that there is only one parameter to tune: the frequency of the Poisson process N
4. Preparations are on the way for other situations. Eg.: Stopping times, local times, jumps, non-bounded coefficients (the extension to uniformly elliptic linearly growing smooth coefficients is clear after [7]), etc.

Thanks with some references

[1] P. Andersson and A. Kohatsu-Higa, Unbiased simulation of stochastic differential equations using parametrix expansions, Bernoulli, 2017, 23, 3, 2028-2057,

[2] Bally, V. and Kohatsu-Higa, A. : A probabilistic interpretation of the parametrix method. Ann. Appl. Probab., 2015, 3095-3138.

[3] Friedman, A. : Partial Differential Equations of Parabolic Type. Dover Publications, Inc., (1964)

[4] Beskos, A. and Papaspiliopoulos, O. and Roberts, G., Retrospective exact simulation of diffusion sample paths with applications, Bernoulli, 12, 2006.

[5] Giles, M., Multilevel Monte Carlo path simulation, Oper. Res. 56, 2008.

[6] Wagner, W. Unbiased Monte Carlo estimators for functionals of weak solutions of stochastic differential equations, Stochastics Stochastics Rep. 28, 1989.

[7] P. Henry-Labordére, X. Tan and N. Touzi, Unbiased simulation of stochastic differential equations. To appear in Annals of Applied Probability.