Perturbation to Conservation Laws and Averaging on Manifolds

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Objects

- Motivating examples, Singular perturbation, $\frac{\partial}{\partial t} = \frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1$ on a space N.
- Reduction to slow-fast systems on product spaces $N \times G$.
- slow-fast systems:

$$\frac{\partial f(x,y)}{\partial t} = \frac{1}{\epsilon}\mathcal{L}^x f(x,y) + \mathcal{L}_1^y f(x,y).$$

$$\begin{cases} dx_t^\epsilon = \sum_{k=1}^{m_1} X_k(x_t^\epsilon, y_t^\epsilon) \circ dB_t^k + X_0(x_t^\epsilon, y_t^\epsilon)\, dt, \\ dy_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{m_2} Y_k(x_t^\epsilon, y_t^\epsilon) \circ dW_t^k + \frac{1}{\epsilon} Y_0(x_t^\epsilon, y_t^\epsilon)\, dt. \end{cases}$$

$\mathcal{L}_x = \frac{1}{2}\sum_{i=1}^m Y_i^2(x,\cdot) + Y_0(x,\cdot)$ differentiates G directions, $\mathcal{L}_1^y$ in N directions.

Birkhoff's Ergodic Theorem

Suppose that (y_t) is an ergodic stationary stochastic process with one-time marginal μ.

Theorem (Birkhoff's Ergodic Theorem)

Then for any $f \in L^1$,

$$\frac{1}{t}\int_0^t f(y_r)dr \xrightarrow{(t\to\infty)} \bar{f} = \int f d\mu, \qquad (a.e.)$$

If (y_t) is a Markov process with y_0 a point, we need to assume that y_t convergence to equilibrium μ reasonably fast.

Denote by $\mathcal{L}$ the generator, then μ is typically an invariant probability measure solving $\mathcal{L}^* p = 0$.

Time averaging

$\dot{x}_t^\epsilon = b(x_t^\epsilon, y_{\frac{t}{\epsilon}})$. Stratonovich, Khasminskii, Wentzell, Freidlin, Papanicolaou, Varadhan, Keller, Kurtz, Kipnis,

$$\begin{aligned} x_t^\epsilon &= x_0 + \int_0^t b(x_s^\epsilon, y_{s/\epsilon})ds \\ &= x_0 + \epsilon \int_0^{t/\epsilon} b(x_{r\epsilon}^\epsilon, y_r)dr \\ &= x_0 + \sum_i \Delta t_i \frac{\epsilon}{\Delta t_i} \int_{t_i/\epsilon}^{t_{i+1}/\epsilon} b(x_{r\epsilon}^\epsilon, y_r)dr \\ &\approx x_0 + \sum_i \Delta t_i \frac{\epsilon}{\Delta t_i} \int_{t_i/\epsilon}^{t_{i+1}/\epsilon} b(x_{t_i}^\epsilon, y_r)dr \\ &\approx x_0 + \sum_i \Delta t_i \int_{t_i/\epsilon}^{t_{i+1}/\epsilon} b(x_{t_i}^\epsilon, y)\mu(dy) \qquad \approx x_0 + \int_0^t \bar{b}(x_s^\epsilon)ds. \end{aligned}$$

If $\bar{b} = 0$, we investigate the limit on $[0, \frac{1}{\epsilon}]$ (diffusion creation).

SDEs with Hörmander's conditions

- Suppose that $f^{(k)} \neq 0$ on $Z = \{f(y_1, y_2) = 0\}$.

$$dy_t^1 = dt, \quad dy_t^2 = f(y_t^1, y_t^2) dB_t,$$

- Let $x \in \mathbf{R}$ be fixed,

$$dy_t^x = \frac{1}{\sqrt{\epsilon}} \sin(x + y_t^x) dB_t + \frac{1}{\epsilon} \cos(x + y_t^x) dt.$$

- $SU(2)$, Pauli matrices

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

$$dy_t = X_1(y_t) \circ dB_t^1 + X_2(y_t) \circ dB_t^2.$$

$$dy_t^g = \alpha(g) X_1(y_t^g) \circ dB_t^1 + \alpha(g) X_2(y_t^g)\, dt.$$

- G Lie group. If $\{A_k\}$ generating the Lie algebra $\mathfrak{g}$

$$dg_t = \sum A_k(g_t) \circ dB_t^k.$$

Hörmander's conditions

If $\mathcal{L}$ satisfies Hörmander's conditions, so does $\mathcal{L}^*$. Existence of an invariant prob. measure $\mu(dy)$ is easy (compact, Lyapunov function), or Krylov-Bogoliubov).
Suppose the state space is compact.

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  Then $\mathcal{L}_0$ is Fredholm, with index zero. The set of g s.t. $\mathcal{L}f = g$ is solvable iff $\langle g, \pi \rangle = 0$, $\pi \in \ker(\mathcal{L}^*)$. Invariant measures have densities, smooth in y (not necessarily strictly positive).
- Hörmander (Acta 68, Thm. 1.1): $\mathcal{L}$ is hypo-elliptic. There exists $\delta > 0$. For all $u \in C_K^\infty(M)$:
  $$\|u\|_{s+\delta} \leq c_0(\|\mathcal{L}u\|_s + \|u\|_s).$$
- Sub-elliptic estimates leads to Birkhoff 's type LLN, with rate $C(\delta, c_0)\frac{1}{\sqrt{t}}$ on $[0, t]$. [PTRF2016]

Locally Uniform Law of Large Numbers

$Y_i \in BC^\infty$, VF on G, compact. $x \in N$. Suppose

$$\mathcal{L}_x = \frac{1}{2}\sum_{i=1}^m Y_i^2(x,\cdot) + Y_0(x,\cdot)$$

satisfies Hörmander's conditions and has a unique invariant probability measure μ_x. Denote by y^x an $\mathcal{L}_x$ diffusion.

Theorem [arxiv 2017] We conclude that

(a) Also $x \mapsto \mu_x$ is locally Lipschitz continuous in the total variation norm. $\mathcal{L}_x^* q = 0$ implies q is smooth in x, (regularity in y follows from hypo-ellipticity)

$$\|q\|_{s+\delta} \le c_0(\|\mathcal{L}_x^* q\|_s + \|q\|_s).$$

(b) For every $s > 1 + \frac{\dim(G)}{2}$ there exists $C(x)$, depending continuously in x, such that for f smooth,

$$\left|\frac{1}{T}\int_t^{t+T} f(y_r^x)\, dr - \int_G f(y)\mu_x(dy)\right|_{L_2(\Omega)} \le C(x)\|f\|_s \frac{1}{\sqrt{T}}.$$

Small/Large perturbations

We may want to consider a small perturbation to a dynamical system with a conservation law. Or we want to approximate a model by one with many degrees of symmetries.

- Small perturbations ignore factor that are small.
- Large perturbations ignore large influences that are oscillatory.
  - The oscillation is captured in Birkhoff's ergodic theorem with rate (LLN).
- Conservation laws or symmetries are used to separate slow and fast variables.
  A reduction procedure leads to a slow-fast systems on the orbit manifold N, typically we have a principal bundle $\pi : P \to N$ with G a group describing the symmetry.

A slow-fast systems of SDEs

$$\begin{cases} dx_t^\epsilon = \sum_{k=1}^{m_1} X_k(x_t^\epsilon, y_t^\epsilon) \circ dB_t^k + X_0(x_t^\epsilon, y_t^\epsilon)\, dt, \\ dy_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{m_2} Y_k(x_t^\epsilon, y_t^\epsilon) \circ dW_t^k + \frac{1}{\epsilon} Y_0(x_t^\epsilon, y_t^\epsilon)\, dt. \end{cases}$$

On $\mathbf{R}^n$: Khasminskii, Freidlin, Veretennikov, Also related to homogenisation of parabolic and elliptic pdes: Otto, Sougnidis, Lions, Pardoux, ...
Olla-Liverani.
In action angle coordinates, when $X_1 = X_2 = \cdots = 0$, L.08.
Ruffino et al for foliated manifolds, convergence in probability. (Method is essentially Euclidean...) Random ODE on manifolds (PTRF2016)

A slow-fast systems of SDEs

$x \in N$, non-compact, $y \in G$, compact.

$$\begin{cases} dx_t^\epsilon = \sum_{k=1}^{m_1} X_k(x_t^\epsilon, y_t^\epsilon) \circ dB_t^k + X_0(x_t^\epsilon, y_t^\epsilon)\, dt, \\ dy_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{m_2} Y_k(x_t^\epsilon, y_t^\epsilon) \circ dW_t^k + \frac{1}{\epsilon} Y_0(x_t^\epsilon, y_t^\epsilon)\, dt. \end{cases}$$

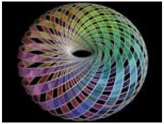
Theorem (arxiv 2017) If $\mathcal{L}_x = \frac{1}{2}\sum Y_i^2(x,\cdot) + Y_0(x,\cdot)$ satisfies Hörmander's conditions+ growth restrictions. As $\epsilon \to 0$, x_t^ϵ converges weekly on $C([0,1], N)$. Limit is:

$$\bar{\mathcal{L}} f(x) = \int_G \left(\frac{1}{2} \sum_{i=1}^{m_1} X_i^2(\cdot, y) f + X_0(\cdot, y) f \right)(x) \mu^x(dy).$$

Kifer, Ikeda, Ogura, L., Liverani-Olla, Gonzales-Ruffino, Hoegele-Ruffino, (foliated manifolds).

Collapsing of manifolds

$$S^3 = \left\{ \begin{pmatrix} z & w \\ -\bar{w}, & \bar{z} \end{pmatrix} \right\}, \mathfrak{g} = \langle X_1, X_2, X_3 \rangle,$$



Making $\{\frac{1}{\sqrt{\epsilon}} X_1, X_2, X_3\}$ an orthonormal frame defines Berger's metrics g^ϵ, $(S^3, g^\epsilon) \xrightarrow{\epsilon \to 0} S^2$, curvature bounded (J. Cheeger).

- Convergence of spectra. All operators below commute:

$$\Delta^\epsilon_{S^3} = \frac{1}{\epsilon}(X_1)^2 + (X_2)^2 + (X_3)^2 = \frac{1}{\epsilon}\Delta_{S^1} + \Delta^H.$$

$\lambda_3(\Delta^\epsilon_{S^3}) = \frac{1}{\epsilon}\lambda_1(\Delta_{S^1}) + \lambda_2(\Delta^H)$. Non-zero eigenvalues of Δ_{S^1} flies away. Eigenfunctions of $\lambda_1 = 0$ are fucntions on S^2. L. Bérard-Bergery, J.-P. Bourguignon, Urakawa, Tanno (first eigenvalues), Fukaya, Kasue-Kumura.

Dynamical models

1. $dy_t^\epsilon = \frac{1}{\epsilon} X_1(y_t^\epsilon) \circ dB_t^1 + X_2(y_t^\epsilon) \circ dB_t^2 + X_3(y_t^\epsilon) \circ dB_t^3$.
2. $Y_0 = aX_2 + bX_3$, (arxiv2012)
$$dy_t^\epsilon = \frac{1}{\epsilon} X_1(y_t^\epsilon) \circ dB_t + Y_0(y_t^\epsilon)\, dt.$$
Convergence of slow variables on $[0, \frac{1}{\epsilon}]$ +their horizontal lifts (e.g. Heisenberg group). See also Friz-Lyons-2014, Baillul-Gubinelli (rough paths)
3. This extends to inhomogenously scaled Riemannian metric on $\pi :\to G/H$. $\mathfrak{g} = (\frac{1}{\epsilon})\mathfrak{h} \oplus (\mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_l)$. (To appear: J. Math. Soc. Japan)
$$dy_t^\epsilon = \frac{1}{\epsilon} \sum_{k=1}^p A_k(y_t^\epsilon) \circ dB_t^k + Y_0(y_t^\epsilon) dt.$$
$\{A_1, \ldots, A_p\}$ generates the Lie algebra $\mathfrak{h}$ of H. Convergence on $[0, \frac{1}{\epsilon}]$ (diffusion creation).

A dynamical description for Brownian motions

Einstein's atom theory (1905) leads to the formulation for BM: $\frac{\partial}{\partial t} = D\Delta$, $D = \frac{kT}{m\beta}$, $m\beta = 6\pi\eta a$. J. Perrin (1926 Nobel): $k = 10^{-23}Jk^{-1}$. Smoluchowski: BM in a force field.

- Langevin, Ornstein-Uhlenbeck (1930): $\frac{1}{\beta}$ small:

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = -\beta v(t)dt + \sqrt{2D}\beta dB_t. \end{cases}$$

  $x(t)$ is approximately $N(x_0, 2Dt)$-distributed. Kramers (1940), Nelson (1967).

PDEs, multi-scale

Does the solutions f^ϵ converges? where

$$\frac{\partial f^\epsilon}{\partial t} = (\frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1) f^\epsilon.$$

1. In O-U model, the slow and fast are separate:

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon}\left(\frac{1}{2}\frac{\partial^2}{\partial v^2} + v\frac{\partial}{\partial v}\right) + v\frac{\partial}{\partial x}.$$

2. Not separate:

$$\frac{\partial f^\epsilon(u)}{\partial t} = \left(\frac{1}{\epsilon}(X_1)^2 + (X_2)^2 + (X_3)^2\right) f^\epsilon(u).$$

Extensions to manifolds

- R.W. Dowell (1980) extended this to manifolds. Bismut and Lebeaux [2005].
- Let $\{A_1, \ldots, A_N\}$ be an o.n.b of $\mathfrak{so}(n)$.

$$du_t^\epsilon = H_{u_t^\epsilon}(e_0)dt + \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{N} A_k^*(u_t^\epsilon) \circ dw_t^k.$$

  Then $\pi(u_{\frac{t}{\epsilon}}^\epsilon, 0 \le t \le T)$ converges to a Brownian motion with generator $\lambda_0 \Delta$ where $\lambda_0 = \frac{4}{n(n-1)}$. Parallel translations also converge. [Ann.Prob. 2016]. As minimiser of energy...
- Using a theorem of [PTRF2016], this can be extend to hypo-elliptic situation.
- Angst-Bailleul-Tardiff (2016), Birrel-Hottovy-Volpe (2017),
- Also, in progress with Xin Chen, Riemannian manifold evolving with curvature flow.

End of the Talk