Numerical Methods for Highly Nonlinear Stochastic Differential Equations

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1. Introduction
   - Numerical methods
   - Pre 2002
   - 2002
2. Contributions of Higham, Mao and Stuart 2002
   - General result for Euler-Maruyama
   - Convergence Rate
3. SDEs without linear growth condition
   - Post 2002
   - The truncated EM method
     - Local Lipschitz and Khasminskii condition
     - Definition of the truncated EM
     - Convergence
     - Convergence rate
4. Summary

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Consider a d-dimensional SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{1.1}$$

on $t \geq 0$ with the initial value $x(0) = x_0 \in \mathbb{R}^d$, where $B(t)$ is an m-dimensional Brownian motion,

$$f : \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad g : \mathbb{R}^d \to \mathbb{R}^{d \times m}$$

The stochastic theta method: Given a stepsize $\Delta > 0$ and a parameter $\theta \in [0, 1]$, compute approximations $X_\Delta(t_k) \approx x(t_k)$, where $t_k = k\Delta$ ($k = 0, 1, 2, \cdots$), by setting $X_\Delta(0) = x_0$ and forming

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + [\theta f(X_\Delta(t_k)) + (1-\theta) f(X_\Delta(t_{k+1}))]\Delta + g(X_\Delta(t_k))\Delta B_k, \tag{1.2}$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$.

The Euler-Maruyama (EM) method:

$$\text{The stochastic theta method when } \theta = 1. \tag{1.3}$$

The backward EM method:

$$\text{The stochastic theta method when } \theta = 0. \tag{1.4}$$

There are other numerical methods, e.g. the Milstein method, the split step backward Euler.

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Up to 2002, all positive results on the numerical methods for SDEs were based on a much more restrictive global Lipschitz assumption (namely both coefficients f and g satisfy the global Lipschitz condition). See, for example, Kloeden and Platen 1992, Mao 1997, Milstein and Tretyakov 2004. One of the key results is that for any given $T > 0$,

$$\mathbb{E}|X_\Delta(T) - x(T)|^2 = O(\Delta).$$

However, the global Lipschitz assumption rules out most realistic models.

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- *Higham, D.J., Mao, X. and Stuart, A.M., Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM Journal on Numerical Analysis 40(3) (2002), 1041-1063.*
- This was the first to study the strong convergence of numerical solutions of SDEs under a local Lipschitz condition.
- The field of numerical analysis of SDEs now has a very active research profile, much of which builds on the techniques developed in that paper, which has so far attracted 386 Google Scholar Citations.
- In particular, the theory developed there has formed the foundation for several recent very popular methods, including tamed Euler-Maruyama method and truncated Euler-Maruyama.

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Let $X_\Delta(t_k)$ be the discrete-time EM approximation (1.3). The continuous-time approximation are defined by

$$\bar{X}_\Delta(t) := x_0 + \int_0^t f(X_\Delta(s))ds + \int_0^t g(X_\Delta(s))dB(s), \tag{2.1}$$

where

$$X_\Delta(t) := X_\Delta(t_k) \quad \text{for } t \in [t_k, t_{k+1}). \tag{2.2}$$

Note

$$\bar{X}_\Delta(t_k) = X_\Delta(t_k) \quad \forall k \geq 0.$$

Assumption 1

For each $R > 0$ there exists a constant C_R, depending only on R, such that

$$|f(a) - f(b)|^2 \vee |g(a) - g(b)|^2 \leq C_R|a - b|^2, \tag{2.3}$$

for $\forall a, b \in \mathbb{R}^d$ with $|a| \vee |b| \leq R$. Moreover, for some $p > 2$ there is a constant A such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\bar{X}_\Delta(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |x(t)|^p\right] \leq A. \tag{2.4}$$

Theorem 2

Under Assumption 1, the Euler–Maruyama solution (1.3) with continuous-time extension (2.1) satisfies

$$\lim_{\Delta \to 0} \mathbb{E}\left[\sup_{0 \le t \le T} |\bar{X}_\Delta(t) - x(t)|^2\right] = 0. \tag{2.5}$$

Outlined Proof

Define

$$\tau_R := \inf\{t \geq 0 : |\bar{X}_\Delta(t)| \geq R\}, \quad \rho_R := \inf\{t \geq 0 : |x(t)| \geq R\},$$

$$\theta_R := \tau_R \wedge \rho_R \quad e(t) := \bar{X}_\Delta(t) - x(t).$$

By the Young inequality, show that for any $\delta > 0$

$$\begin{aligned} & \mathbb{E}\left[\sup_{0\leq t\leq T} |e(t)|^2\right] \\ \leq \quad & \mathbb{E}\left[\sup_{0\leq t\leq T} |e(t\wedge\theta_R)|^2 \mathbf{1}_{\{\theta_R > T\}}\right] + \frac{2\delta}{p}\mathbb{E}\left[\sup_{0\leq t\leq T} |e(t)|^p\right] \\ & + \frac{1-\frac{2}{p}}{\delta^{2/(p-2)}}\mathbb{P}\big(\tau_R \leq T \text{ or } \rho_R \leq T\big). \end{aligned} \tag{2.6}$$

By (2.4)

$$\mathbb{P}\big(\tau_R \le T \text{ or } \rho_R \le T\big) \le \mathbb{P}\big(\tau_R \le T\big) + \mathbb{P}\big(\rho_R \le T\big) \le \frac{2A}{R^p}.$$

Using this bound along with

$$\mathbb{E}\left[\sup_{0\le t\le T} |e(t)|^p\right] \le 2^{p-1}\mathbb{E}\left[\sup_{0\le t\le T} \big(|\bar{X}_\Delta(t)|^p + |x(t)|^p\big)\right] \le 2^p A$$

in (2.6) gives

$$\mathbb{E}\left[\sup_{0\le t\le T} |e(t)|^2\right] \le \mathbb{E}\left[\sup_{0\le t\le T} |\bar{X}_\Delta(t\wedge\theta_R) - x(t\wedge\theta_R)|^2\right] + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)2A}{p\delta^{2/(p-2)}R^p}. \tag{2.7}$$

Show

$$\mathbb{E}\left[\sup_{0\le t\le T}\left(\bar{X}_\Delta(t\wedge\theta_R)-x(t\wedge\theta_R)\right)^2\right]\le C\Delta(C_R^2+1)e^{4C_R(T+4)},$$

where C is a universal constant independent of Δ, R and δ. Inserting this into (2.7) gives

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right]\le C\Delta(C_R^2+1)e^{4C_R(T+4)}+\frac{2^{p+1}\delta A}{p}+\frac{(1-\frac{2}{p})2A}{\delta^{2/(p-2)}R^p}. \quad (2.8)$$

Given any $\epsilon > 0$, we can choose δ so that $(2^{p+1}\delta A)/p < \epsilon/3$, then choose R so that

$$\frac{(1 - \frac{2}{p})2A}{\delta^{2/(p-2)} R^p} < \frac{\epsilon}{3}$$

and then choose Δ sufficiently small for

$$C\Delta(C_R^2 + 1)e^{4C_R(T+4)} < \frac{\epsilon}{3},$$

so that, in (2.8), $\mathbb{E}[\sup_{0 \le t \le T} |e(t)|^2] < \epsilon$, as required.

Outline

Assumption 3

The functions f and g in (1.1) are C^1 and there exist constants $\mu, c > 0$ such that

$$\begin{aligned}\langle a-b, f(a)-f(b)\rangle &\le \mu|a-b|^2 \quad \forall a,b\in\mathbb{R}^d, && (2.9)\\ |g(a)-g(b)|^2 &\le c|a-b|^2 \quad \forall a,b\in\mathbb{R}^d. && (2.10)\end{aligned}$$

Assumption 4

There exists a pair of positive constants D and q such that for all $a, b \in \mathbb{R}^d$,

$$|f(a) - f(b)|^2 \le D\,(1 + |a|^q + |b|^q)\,|a - b|^2. \tag{2.11}$$

Assumption 5

The SDE and EM solutions satisfy

$$\mathbb{E} \sup_{0 \le t \le T} |x(t)|^p < \infty, \quad \mathbb{E} \sup_{0 \le t \le T} |\bar{X}_\Delta(t)|^p < \infty, \quad \forall p \ge 1.$$

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Theorem 6

Under Assumptions 3, 4 and 5 the Euler–Maruyama solution (1.3) with continuous-time extension (2.1) satisfies

$$\mathbb{E}\left[\sup_{0\le t\le T} |\bar{X}_\Delta(t) - x(t)|^2\right] = O(\Delta).$$

Higham, Mao and Stuart 2002 proposed the bounded condition on the *pth* moments of both exact solution and numerical solution to the underlying SDE and proved the strong convergence theory. Their theory turns the problem of the strong convergence into the verification of the boundedness of the *p*th moments of the exact and numerical solutions under the local Lipschitz condition.

They showed that under the linear growth condition, both exact and numerical solutions by either the Euler-Maruyama (EM) or the stochastic theta method satisfy the moment bounded condition, and hence they proved that the numerical solutions converge to the exact solution in the strong sense under the Local Lipschitz condition and the linear growth condition.

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However, the linear growth condition is still too restrictive. Higham, Mao and Stuart (2002) pointed out that in general, it is not clear when such moment bounds can be expected to hold for the EM method even when both drift coefficient and the diffusion coefficient are C^1 (unbounded derivatives of course).
Hutzenthaler, Jentzen and Kloeden (2011) answered the question negatively by proving that the moment of the explicit EM method will diverge in finite time for those SDEs with either the drift coefficient or the diffusion coefficient being superlinear.

Implicit methods have therefore naturally been used to study the numerical solutions to SDEs without the linear growth condition recently, for example, in Szpruch, Mao, Higham and Pan (2011), Mao and Szpruch (2013), Tretyakov and Zhang (2013).

Methods with variable stepsize also attract a lot of attention. See, for example, Werner and Renate (2006), Valinejad and Hosseini (2010).

Since the classical explicit EM method has its simple algebraic structure, cheap computational cost and acceptable convergence rate under the global Lipschitz condition, it has been attracting lots of attention. Although Hutzenthaler, Jentzen and Kloeden (2011) showed the strong and weak divergence in finite time of the EM method for SDEs with non-globally Lipschitz continuous coefficients, some modified EM methods have recently been developed for the nonlinear SDEs without the linear growth condition. For example:

- The tamed EM method was developed by Hutzenthaler, Jentzen and Kloeden (2012) to approximate SDEs with one-sided Lipschitz drift coefficient and the linear growth diffusion coefficient.
- This method was further developed by Sabanis (2013) while the tamed Milstein method was developed by Wang and Gan (2013).
- The stopped EM method was developed by Liu and Mao (2013) for highly nonlinear SDEs as well.
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Assumption 7

Assume that the coefficients f and g satisfy the local Lipschitz condition: For any $R > 0$, there is a $K_R > 0$ such that

$$|f(x) - f(y)| \vee |g(x) - g(y)| \leq K_R |x - y| \tag{3.1}$$

for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$.

Assumption 8

We also assume that the coefficients satisfy the Khasminskii-type condition: There is a pair of constants $p > 2$ and $K > 0$ such that

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \leq K(1 + |x|^2) \tag{3.2}$$

for all $x \in \mathbb{R}^d$.

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for all $x \in \mathbb{R}^d$.

Lemma 9

Under Assumptions 7 and 8, the SDE (1.1) has a unique global solution $x(t)$ and, moreover,

$$\sup_{0\le t\le T} \mathbb{E}|x(t)|^p < \infty, \quad \forall T > 0. \tag{3.3}$$

Assumptions 7 and 8 cover many nonlinear SDEs, for example, the scalar SDE in financial mathematics

$$dx(t) = (\mu - \alpha x^{\beta}(t))dt + \sigma x^{\theta}(t)dB(t), \quad \beta, \theta > 1,\ \mu, \alpha, \sigma > 0, \quad (3.4)$$

and the stochastic population system

$$dx(t) = \mathrm{diag}(x_1(t), x_2(t), ..., x_d(t))[(b + Ax^2(t))dt + Cx(t)dB(t)], \quad (3.5)$$

where $B(t)$ is a scalar Brownian motion, $b = (b_1, \cdots, b_d)^T$, $x^2 = (x_1^2, \cdots, x_d^2)^T$, $C = (C_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$ and $A = (A_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$ is such that $\lambda_{\max}(A + A^T) < 0$.

To define the truncated EM numerical solutions, we first choose a strictly increasing continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(r) \to \infty$ as $r \to \infty$ and

$$\sup_{|x| \le r} \big(|f(x)| \vee |g(x)| \big) \le \mu(r), \quad \forall r \ge 1. \tag{3.6}$$

Denote by μ^{-1} the inverse function of μ and we see that μ^{-1} is a strictly increasing continuous function from $[\mu(0), \infty)$ to $\mathbb{R}_+$. We also choose a strictly decreasing function $h : (0, 1] \to (0, \infty)$ such that

$$\lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \le 1, \quad \forall \Delta \in (0, 1]. \tag{3.7}$$

For a given stepsize $\Delta \in (0, 1)$, let us define the truncated functions

$$f_\Delta(x) = f\Big((|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}\Big) \quad \text{and} \quad g_\Delta(x) = g\Big((|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}\Big) \tag{3.8}$$

for $x \in \mathbb{R}^d$, where we set $x/|x| = 0$ when $x = 0$. It is easy to see that

$$|f_\Delta(x)| \vee |g_\Delta(x)| \le \mu(\mu^{-1}(h(\Delta))) = h(\Delta) \quad \forall x \in \mathbb{R}^d. \tag{3.9}$$

That is, both truncated functions f_Δ and g_Δ are bounded although both f and g may not. Moreover, these truncated functions preserve the Khasminskii-type condition for all $\Delta \in (0, 1]$ as described in the following lemma.

Lemma 10

Let Assumption 8 hold. Then, for all $\Delta \in (0,1]$, we have

$$x^T f_\Delta(x) + \frac{p-1}{2}|g_\Delta(x)|^2 \le \hat{K}(1+|x|^2), \quad \forall x \in \mathbb{R}^d, \tag{3.10}$$

where $\hat{K} = 2K\big(1 \vee [1/\mu^{-1}(h(1))]\big)$.

We can now form the discrete-time truncated EM numerical solutions $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ by setting $X_\Delta(0) = x_0$ and computing

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + f_\Delta(X_\Delta(t_k))\Delta + g_\Delta(X_\Delta(t_k))\Delta B_k, \qquad (3.11)$$

for $k = 0, 1, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$.

Let us now form two versions of the continuous-time truncated EM solutions. The first one is defined by

$$\bar{x}_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k) I_{[t_k, t_{k+1})}(t), \quad t \geq 0. \tag{3.12}$$

This is a simple step process so its sample paths are not continuous. We will refer this as the continuous-time step-process truncated EM solution. The other one is defined by

$$x_\Delta(t) = x_0 + \int_0^t f_\Delta(\bar{x}_\Delta(s))ds + \int_0^t g_\Delta(\bar{x}_\Delta(s))dB(s) \tag{3.13}$$

for $t \geq 0$. We will refer this as the continuous-time continuous-sample truncated EM solution.

We observe that $x_\Delta(t_k) = \bar{x}_\Delta(t_k) = X_\Delta(t_k)$ for all $k \geq 0$. Moreover, $x_\Delta(t)$ is an Itô process with its Itô differential

$$dx_\Delta(t) = f_\Delta(\bar{x}_\Delta(t))dt + g_\Delta(\bar{x}_\Delta(t))dB(t). \tag{3.14}$$

Lemma 11

Let Assumptions 7 and 8 hold. Then

$$\sup_{0<\Delta\le\Delta^*}\ \sup_{0\le t\le T} \mathbb{E}|x_\Delta(t)|^p \le C, \quad \forall T>0, \tag{3.15}$$

where C is a positive constant dependent on T, p, K, x_0 etc but independent of Δ.

Theorem 12

Let Assumptions 7 and 8 hold. Then, for any $q \in [2, p)$,

$$\lim_{\Delta \to 0} \mathbb{E}|x_\Delta(T) - x(T)|^q = 0 \quad \textit{and} \quad \lim_{\Delta \to 0} \mathbb{E}|\bar{x}_\Delta(T) - x(T)|^q = 0. \quad (3.16)$$

Assumption 13

Assume that there is a pair of constants $q > 2$ and $H_1 > 0$ such that

$$(x-y)^T(f(x)-f(y)) + \frac{q-1}{2}|g(x)-g(y)|^2 \le H_1|x-y|^2 \qquad (3.17)$$

for all $x, y \in \mathbb{R}^d$.

Assumption 14

Assume that there is a pair of positive constants ρ and H_2 such that

$$|f(x)-f(y)|^2 \vee |g(x)-g(y)|^2 \le H_2(1+|x|^\rho+|y|^\rho)|x-y|^2 \qquad (3.18)$$

for all $x, y \in \mathbb{R}^d$.

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for all $x, y \in \mathbb{R}^d$.

Note that Assumption 14 implies

$$|f(x)| \vee |g(x)| \leq H_3|x|^{(2+\rho)/2}, \quad \forall |x| \geq 1, \tag{3.19}$$

where $H_3 = \sqrt{2H_2} + |f(0)| + |g(0)|$.

Theorem 15

Let Assumptions 7, 13 and 14 hold and let Assumption 8 hold for any $p > 2$. Define

$$\mu(u) = H_3 u^{(2+\rho)/2}, \quad u \geq 0, \tag{3.20}$$

and let

$$h(\Delta) = \Delta^{-\varepsilon} \quad \text{for some } \varepsilon \in (0, 1/4]. \tag{3.21}$$

Then, for any $\bar{q} \in [2, q)$,

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar{q}} \leq O\big(\Delta^{\bar{q}(1-2\varepsilon)/2}\big) \tag{3.22}$$

and

$$\mathbb{E}|x(T) - \bar{x}_\Delta(T)|^{\bar{q}} \leq O\big(\Delta^{\bar{q}(1-2\varepsilon)/2}\big). \tag{3.23}$$

Example

Consider the scalar stochastic Ginzburgh–Landau equation

$$dx(t) = (ax(t) - bx^3(t))dt + cx(t)dB(t), \tag{3.24}$$

where $B(t)$ is a scalar Brownian motion and a, b, c are three positive numbers. Clearly, its coefficients $f(x) = ax - bx^3$ and $g(x) = cx$ are locally Lipschitz continuous for $x \in \mathbb{R}$, namely, satisfy Assumption 7. Also, for any $p > 2$, we have

$$xf(x) + \frac{p-1}{2}|g(x)|^2 = ax^2 - bx^4 + \frac{(p-1)c^2}{2}x^2 \le \frac{1}{16}b(2a+(p-1)c^2)^2.$$

That is, Assumption 8 is satisfied for any $p > 2$.

Moreover, for any $q > 2$,

$$\begin{aligned}&(x-y)(f(x)-f(y)) + \frac{q-1}{2}|g(x)-g(y)|^2 \\ \le\ & \big(a + 0.5c^2(q-1)\big)(x-y)^2, \quad x, y \in \mathbb{R}.\end{aligned}$$

This means that Assumption 13 is satisfied for any $q > 2$ with $H_1 = a + 0.5c^2(q-1)$.

Furthermore, we can show

$$|f(x)-f(y)|^2 \vee |g(x)-g(y)|^2 \le H_2(1+|x|^4+|y|^4)|x-y|^2,$$

where $H_2 = a^2 + 9b^2 + c^2$. So, Assumption 14 is also satisfied with $\rho = 4$.

Finally, we need to design functions $\mu(\cdot)$ and $h(\cdot)$. Noting that

$$\sup_{|x|\le u}\big(|f(x)|\vee|g(x)|\big)\le \alpha u^3,\quad \forall u\ge 1,$$

where $\alpha=a+b+c$, we can have $\mu(u)=\alpha u^3$ and its inverse function $\mu^{-1}(u)=(u/\alpha)^{1/3}$ for $u\ge 0$. For $\varepsilon\in(0,1/4]$, we define $h(\Delta)=\Delta^{-\varepsilon}$ for $\Delta>0$. We can therefore conclude by Theorem 15 that the truncated EM solutions of the SDE (3.24) satisfy

$$\mathbb{E}|x(T)-x_\Delta(T)|^{\bar q}=O(\Delta^{\bar q(1-2\varepsilon)/2})$$

and

$$\mathbb{E}|x(T)-\bar x_\Delta(T)|^{\bar q}=O(\Delta^{\bar q(1-2\varepsilon)/2}).$$

That is, the order of $L^{\bar q}$-convergence can be arbitrarily close to $\bar q/2$.

- This talk was concerned with L^q convergence theory for numerical SDE simulations beyond the realm of globally Lipschitz problems.
- We reviewed the contributions of Higham, Mao and Stuart 2002.
- We reviewed the recent developments on the modified EM methods with some emphasize on the truncated EM.

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