Weak universality of the parabolic Anderson model

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July 2017
Stochastic Analysis Symposium
Durham

Joint work with Jörg Martin

Motivation: branching random walks

Random i.i.d. potential $(\eta(x))_{x\in\mathbb{Z}^d}$.

- Independent particles on $\mathbb{Z}^d$ follow random walks (cont. time);
- at site x particle has branching rate $\eta(x)^+$; killing rate $\eta(x)^-$;
- branching: new independent copy, follows same dynamics;
- killing: particle disappears.

Complicated $\Rightarrow$ consider simple statistics:

$$u(t,x) = \mathbb{E}[\#\text{ particles in }(t,x)|\eta].$$

Get ∞-dim ODE "parabolic Anderson model" (PAM)

$$\partial_t u = \Delta_{\mathbb{Z}^d} u + u\eta.$$

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Comparison with super-Brownian motion

- Indep. branching particles on $\mathbb{Z}^d$ with det. branching/killing rate 1.
- $u^N(0,x) = Nu_0(x)$, $x \in \mathbb{Z}^d$.
- Send only no. of particles $\to \infty$:

$$u(t,x) = \lim_{N\to\infty} \frac{u^N(t,x)}{N} = \mathbb{E}[\# \text{ particles in } (t,x)],$$

then limit is discrete heat equation

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- Also zoom out as no. of particles increases:

$$v(t,x) = \lim_{N\to\infty} \frac{u^N(N^2 t, Nx)}{N},$$

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Large scale behavior of PAM

$$\partial_t u = \Delta_{\mathbb{Z}^d} u + u\eta.$$

- Intensely studied in past decades (Carmona, Molchanov, Gärtner, König, ... MANY more);
- if η is "truely random": u is intermittent, mass concentrated in few, small, isolated islands; survey König '16;
- $\Rightarrow$ only possible scaling limit is (finite sum of) Dirac deltas.

Competition between disorder:

$$\partial_t u = u\eta \qquad \Rightarrow \qquad u(t,x) = e^{t\eta(x)} u_0(x)$$

and smoothing:

$$\partial_t u = \Delta_{\mathbb{Z}^d} u \qquad \Rightarrow \qquad u(t,x) = P_t^{\mathbb{Z}^d} * u_0(x).$$

Intermittency: disorder always wins; to see nontrivial limit: weaken disorder; expect phase transition(s) between intermittence and smoothness.

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Weak disorder

$$\partial_t u = \Delta_{\mathbb{Z}^d} u + \varepsilon^{\alpha} u \eta.$$

- To preserve scaling of $\Delta_{\mathbb{Z}^d}$:

$$u^{\varepsilon}(t,x) = \varepsilon^{\beta} u(t/\varepsilon^2, x/\varepsilon).$$

- Then

$$\partial_t u^{\varepsilon} = \Delta_{\varepsilon\mathbb{Z}^d} u^{\varepsilon} + u^{\varepsilon} \varepsilon^{-2+\alpha} \eta(\cdot/\varepsilon),$$

  and $\varepsilon^{-d/2}\eta(\cdot/\varepsilon) \Rightarrow \xi$ (white noise), so

$$-2+\alpha = -d/2 \qquad \Leftrightarrow \qquad \alpha = 2 - d/2.$$

- Something goes wrong for $d \geq 4$.
- Conjecture for $d < 4$ and centered η: $u^{\varepsilon} \Rightarrow v$,

$$\partial_t v = \Delta_{\mathbb{R}^d} v + v\xi \qquad \text{(continuous PAM)}$$

  where ξ = space white noise.
- Continuous PAM only makes sense for $d < 4$, critical in $d = 4$ and supercritical for $d > 4$!

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Phase transition

$$\partial_t u = \Delta_{\mathbb{Z}^d} u + \lambda \varepsilon^{2-d/2} u\eta, \qquad u^\varepsilon(t,x) = \varepsilon^\beta u(t/\varepsilon^2, x/\varepsilon).$$

- Assume we showed $u^\varepsilon \to v$ solving continuous PAM

$$\partial_t v = \Delta_{\mathbb{R}^d} v + \lambda v\xi.$$

- Conjecture: v is also intermittent ($d = 1$: Chen '16, Dumaz-Labbé in progress; $d = 2$: first results in progress by Chouk, van Zuijlen).
- Scaling invariance of ξ: large scales for $v \Leftrightarrow$ large λ.
- So $\lambda \to \infty$: intermittency, $\lambda \to 0$: smoothness;
  $\Rightarrow$ discrete PAM has phase transition at noise strength $O(\varepsilon^{2-d/2})$.

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Generalization: branching interaction

Same dynamics as before, but include interaction:

- at site x particle has branching rate

$$f(\#\text{ particles in }(x))\eta(x)^{+};$$

  killing rate

$$f(\#\text{ particles in }(x))\eta(x)^{-};$$

- example that models limited resources: $f(u) = 1 - u/C$;
- could include interaction through jump rate, but did not work this out.

Very complicated $\Rightarrow$ consider simple statistics:

$$u(t,x) = \mathbb{E}[\#\text{ particles in }(t,x)|\eta].$$

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Scaling of generalized PAM

Consider $d = 2$ from now on. Will work in $d = 1$ (easier) and $d = 3$ (much harder).

$$\partial_t u = \Delta_{\mathbb{Z}^2} u + \varepsilon F(u)\eta, \qquad u(0,x) = \mathbb{1}_{x=0}.$$

- Natural conjecture: If $\mathbb{E}[\eta(0)] = 0$, $\mathrm{var}(\eta(0)) = 1$, then

  $$\lim_{\varepsilon \to 0} \varepsilon^{-2} u(t/\varepsilon^2, x/\varepsilon) = v,$$

  where v solves generalized continuous PAM

  $$\partial_t v = \Delta_{\mathbb{R}^2} v + F(v)\xi, \qquad v(0,x) = \delta(x - 0).$$

- Problem 1: (generalized) continuous PAM needs renormalization!
  $\Rightarrow$ assume instead $\mathbb{E}[\eta(0)] = -\varepsilon F'(0) c^\varepsilon$ with $c^\varepsilon \simeq |\log \varepsilon|$.
- Problem 2: continuous generalized PAM cannot be started in δ unless $F(u) = \lambda u$.

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Weak universality of PAM

$$\partial_t u = \Delta_{\mathrm{rw}} u + \varepsilon F(u)\eta, \qquad u(0,x) = \mathbb{1}_{x=0}.$$

Assume:

- Δ_{rw} generator of random walk with sub-exponential moments;
- $F'' \in L^\infty$, $F(0) = 0$;
- $(\eta(x))_{x\in\mathbb{Z}^2}$ independent, $\mathbb{E}[\eta(x)] = -\varepsilon F'(0)c^\varepsilon$, $\mathrm{var}(\eta(x)) = 1$, $\sup_x \mathbb{E}[|\eta(x)|^p] < \infty$ for some $p > 14$ (might treat $p > 4$ by truncation).
- We may also generalize $\mathbb{Z}^2$ to any two-dimensional "crystal lattice".

Theorem (Martin-P. '17)

Under these assumptions $\lim_{\varepsilon\to 0} \varepsilon^{-2} u(t/\varepsilon^2, x/\varepsilon) = v$, *where* v *solves linear continuous PAM,*

$$\partial_t v = \Delta_{\mathbb{R}^2} v + F'(0) v\xi, \qquad v(0,x) = \delta(x-0).$$

Weak universality of PAM

Call this weak universality since model changes with scaling:

$$\partial_t u = \Delta_{\mathrm{rw}} u + \varepsilon F(u)\eta, \qquad u(0,x) = \mathbb{1}_{x=0}.$$

- Continuous PAM treated pathwise (rough paths, regularity structures, paracontrolled distributions);
- pathwise approaches need subcriticality: nonlinearity unimportant on small scales
  $\Rightarrow$ solutions not scale-invariant
  $\Rightarrow$ fixed model cannot rescale to continuous PAM.

Comparison: weak universality of the KPZ equation

Conjecture ("Weak KPZ universality conjecture")

All (appropriate) $1+1$-dimensional weakly asymmetric interface growth models scale to the KPZ equation

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi.$$

Example: WASEP with open boundaries, Gonçalves-P.-Simon '17.

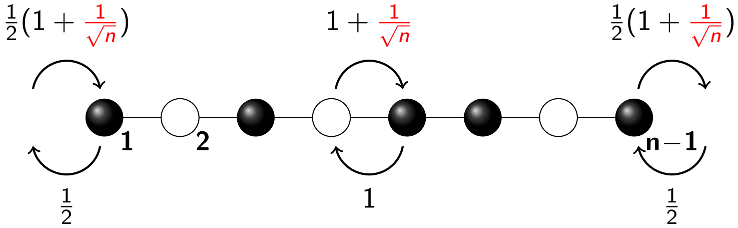


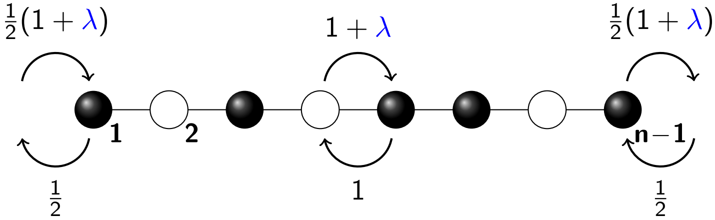
Figure: Jump rates. Leftmost and rightmost rates are entrance/exit rates. Compare also Corwin-Shen '16.

Strong KPZ universality

Conjecture ("Strong KPZ universality conjecture")

All (appropriate) $1+1$-dimensional asymmetric interface growth models show the same large scale behavior as the KPZ equation.

- Much harder than weak KPZ universality.
- Example: ASEP with open boundaries



Solution of the continuous PAM

$$\partial_t v = \Delta_{\mathbb{R}^2} v + v\xi.$$

- Difficulty: ξ only noise in space, no martingales around;
- Analysis: $\xi \in C^{-1-}_{\mathrm{loc}} \Rightarrow$ expect $v \in C^{1-}_{\mathrm{loc}}$;
  $\Rightarrow$ sum of regularities < 0, so $v\xi$ ill-defined.
- Subcriticality: on small scales v should look like Ξ,

$$\partial_t \Xi = \Delta_{\mathbb{R}^2} \Xi + \xi.$$

- Direct computation ((Ξ, ξ) is Gaussian):

$$\Xi\xi = \lim_{\varepsilon \to 0} [(\Xi * \delta_\varepsilon)(\xi * \delta_\varepsilon) - c^\varepsilon], \qquad c^\varepsilon \simeq |\log \varepsilon|,$$

  is well defined and in C^{0-}_{loc}.
- Philosphy of rough paths: also $v\xi$ is well defined.
- Implement this with paracontrolled distributions Gubinelli-Imkeller-P. '15.

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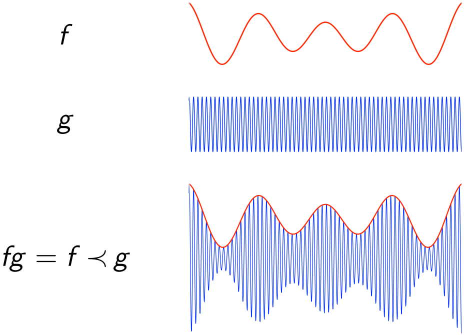
Crash course on paracontrolled distributions I

- Littlewood-Paley blocks: Δ_m are contributions of f on scale 2^{-m}

$$f = \sum_m \mathcal{F}^{-1}(\mathbb{1}_{[2^m, 2^{m+1})}(|\cdot|)\mathcal{F}f) = \sum_m \Delta_m f.$$

- Formally: $fg = \sum_{m,n} \Delta_m f \Delta_n g$.
- Bony '81: paraproduct $f \prec g = \sum_{m \le n-2} \Delta_m f \Delta_n g$ always well defined, inherits regularity of g.
- We interpret $f \prec g$ as frequency modulation of g:



Crash course on paracontrolled distributions II

- Intuition: $f \prec g$ "looks like" g (call it paracontrolled).
- Gubinelli-Imkeller-P. '15: if gh is given, $(f \prec g)h$ is well defined and paracontrolled by h.
- Solutions to SPDEs often paracontrolled (paraproduct + smooth rest)

Example PAM:

$$\partial_t v = \Delta_{\mathbb{R}^2} v + v\xi.$$

- $v = v \prec \Xi + v^\sharp$ with $v^\sharp \in C^{2-}_{\mathrm{loc}}$.
- $\Rightarrow v\xi$ ok if $\Xi\xi$ ok, this we can control with Gaussian analysis.
- Gubinelli-Imkeller-P. '15, Hairer '14: for periodic white noise ξ; non-periodic: Hairer-Labbé '15.
- v depends continuously on $(\xi, \Xi\xi)$; good for proving convergence!

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Back to our model

$$\partial_t u = \Delta_{\mathrm{rw}} u + \varepsilon F(u)\eta, \qquad u(0,x) = \mathbb{1}_{x=0}.$$

Problem: u lives on lattice, not $\mathbb{R}^2$. Possible solutions:

- Interpolation: find $\tilde{u}$ on $\mathbb{R}^2$ with $\tilde{u}|_{\mathbb{Z}^2} = u$ and such that $\tilde{u}$ solves "similar" equation, e.g. Mourrat-Weber '16, Gubinelli-P. '17, Zhu-Zhu '15, Chouk-Gairing-P. 17, Shen-Weber '16.
  Needs random operators, highly technical.
- Discretization of regularity structures: Hairer-Matetski '16, Cannizzaro-Matetski '16, Erhard-Hairer '17.
- Paracontrolled distributions via semigroups: Replace Fourier transform by heat semigroup, works on manifolds and on discrete spaces Bailleul-Bernicot '16.

We want to be similar to continuous paracontrolled distributions.

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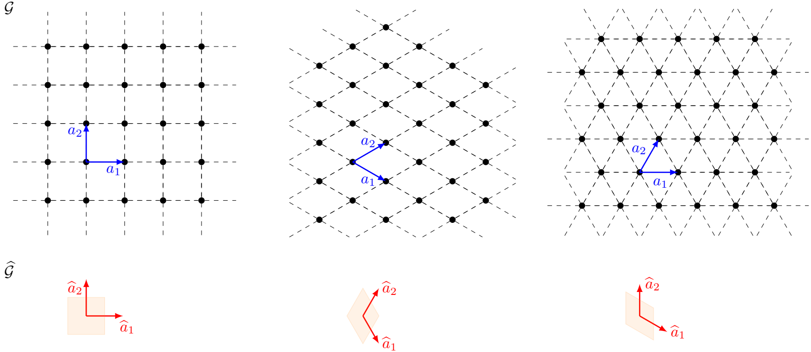
Crystal lattices

Consider lattices $\mathcal{G}$ that allow for Fourier transform:



Fourier transform lives on "reciprocal Fourier cell" $\widehat{\mathcal{G}}$:

$$\mathcal{F}\varphi(x) := \widehat{\varphi}(x) := |\mathcal{G}| \sum_{k \in \mathcal{G}} \varphi(k) e^{-2\pi\imath k \cdot x}, \qquad x \in \widehat{\mathcal{G}}.$$

Example: $\mathcal{G} = \varepsilon \mathbb{Z}^d$ then $\widehat{\mathcal{G}} = \varepsilon^{-1}(\mathbb{R}/\mathbb{Z})^d$.

Paracontrolled distributions on crystal lattices

- Given Fourier transform we define Littlewood-Paley blocks as on $\mathbb{R}^d$:

$$\Delta_m f = \mathcal{F}^{-1}(\mathbb{1}_{[2^m, 2^{m+1})}(|\cdot|)\mathcal{F}f).$$

- Should not interpret $\mathcal{F}f$ as periodic function but embed $\widehat{\mathcal{G}} = \varepsilon^{-1}(\mathbb{R}/\mathbb{Z})^d$ in $\mathbb{R}^d$.
- $\varepsilon \gg 0$: maybe $\Delta_m f = 0$ for all $m \geq 0$, but nontrivial decomposition for $\varepsilon \to 0$.
- From here paracontrolled analysis exactly as in continuous space.

Weighted paracontrolled distributions

Next difficulty: equation lives on unbounded domain

$$\partial_t u = \Delta_{\mathrm{rw}} u + \varepsilon F(u)\eta, \qquad u(0,x) = \mathbb{1}_{x=0}.$$

$\Rightarrow$ cannot control η in C^{-1-}, but only in weighted Hölder space.

- Develop paracontrolled distributions in weighted spaces.
- Trick from Hairer-Labbé '15: convenient to allow (sub-)exponential growth of u, but then u is no tempered distribution, i.e. we have no Fourier transform!
- $\Rightarrow$ consider ultra-distributions (can grow faster than polynomially, still have Fourier transforms.) Similar to Mourrat-Weber '15, but their approach does not work on L^∞ spaces.

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Analytic convergence proof

Rescale:

$$\partial_t u = \Delta_{\mathrm{rw}} u + \varepsilon F(u)\eta, \qquad u(0,x) = \mathbb{1}_{x=0},$$

$\Rightarrow$ for $u^\varepsilon(t,x) = \varepsilon^{-2} u(t/\varepsilon^2, x/\varepsilon)$

$$\partial_t u^\varepsilon = \Delta_{\varepsilon\mathbb{Z}^2} u^\varepsilon + \varepsilon^{-2} F(\varepsilon^2 u^\varepsilon)\xi^\varepsilon, \qquad u^\varepsilon(0) = \delta^\varepsilon.$$

- Taylor expansion:

$$\varepsilon^{-2} F(\varepsilon^2 u^\varepsilon)\xi^\varepsilon = F'(0) u^\varepsilon \xi^\varepsilon + o(1)$$

- Paracontrolled analysis of rescaled, Taylor expanded equation.
- Key ingredient: Schauder estimates for semigroup generated by Δ_{rw}.
- Final result: If $\xi^\varepsilon \to \xi$ and $\Xi^\varepsilon \xi^\varepsilon \to \Xi\xi$ in appropriate spaces, then $u^\varepsilon \to v$,

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Convergence of the stochastic data

Remains to study convergence of $(\xi^\varepsilon, \Xi^\varepsilon \xi^\varepsilon)$.

- Central limit theorem: $\xi^\varepsilon \to \xi$.
- Convergence of $\Xi^\varepsilon \xi^\varepsilon$ often via diagonal sequence argument (Mourrat-Weber '16, Hairer-Shen '16, Chouk-Gairing-P. '16, ...).
  Here: Use Wick product to write $\Xi^\varepsilon \xi^\varepsilon$ as discrete multiple stochastic integral; apply results of Caravenna-Sun-Zygouras '17 to identify limit.
- Regularity from Kolmogorov's criterion $\Rightarrow$ need high moments; obtain bounds via martingale arguments and Wick products.

This concludes the convergence proof.

Conclusion

- Consider interacting branching population in a random potential.
- Model too complicated $\Rightarrow$ average over particle dynamics, formally get generalized discrete PAM.
- Generalized discrete PAM with small potential on large scales universally described by linear continuous PAM.
- To prove this we develop paracontrolled distributions on lattices,
- and we provide a systematic approach based on Caravenna-Sun-Zygouras '17 and Wick products to control multilinear functionals of i.i.d. variables.

Thank you