Existence and uniqueness of absolutely continuous solutions to continuity equations on Hilbert spaces

Michael Röckner (Bielefeld University)

joint work with: Giuseppe Da Prato

Work supported by Deutsche Forschungsgemeinschaft (DFG) through "Collaborative Research Centre (CRC) 1283", GNAMPA and INDAM.

Contents

1. Introduction
2. Main Existence Result
3. A Deterministic Feynman–Kac formula
4. Sketch of Proof of Theorem
5. References

1. Introduction

We are given a separable Hilbert space H (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$), a Borel vector field $F:[0,T]\times H\to H$ and a Borel probability measure ζ on H. Consider the following continuity equation,

$$\begin{aligned}\int_0^T\int_H\big[D_t u(t,x)+\langle D_x u(t,x),F(t,x)\rangle\big]\,\nu_t(dx)\,dt\\ =-\int_H u(0,x)\,\zeta(dx),\quad \forall\, u\in\mathcal{F}C^1_{b,T},\end{aligned}\tag{CE}$$

where the unknown $\nu=(\nu_t)_{t\in[0,T]}$ is a probability kernel such that $\nu_0=\zeta$. Moreover, D_x denotes the gradient operator and $\mathcal{F}C^1_{b,T}$ is defined as follows: for $k\in\mathbb{N}\cup\{\infty\}$ let $\mathcal{F}C^k_b$ and $\mathcal{F}C^k_0$ denote the $\mathbb{R}$-linear span of all functions $f:H\to\mathbb{R}$ of the form

$$f(x)=\widetilde{f}(\langle h_1,x\rangle,\cdots,\langle h_N,x\rangle),\quad x\in H,$$

where $N\in\mathbb{N}$, $\widetilde{f}\in C^k_b(\mathbb{R}^N)$, $C^k_0(\mathbb{R}^N)$ respectively, and $h_1,\cdots,h_N\in Y$, where Y is a dense linear subspace of H to be specified later.

1. Introduction

Then $\mathcal{F}C^k_{b,T}$ and $\mathcal{F}C^k_{0,T}$ are defined to be the $\mathbb{R}$–linear span of all functions $u : [0,T] \times H \to \mathbb{R}$ of the form

$$u(t,x) = g(t)f(x), \quad (t,x) \in [0,T] \times H,$$

where $g \in C^1([0,T])$ with $g(T) = 0$ and $f \in \mathcal{F}C^k_b$, $\mathcal{F}C^k_0$ respectively.
Correspondingly, let $\mathcal{V}\mathcal{F}C^k_{0,T}$ be the $\mathbb{R}$-linear span of all maps $G : [0,T] \times H \to H$ of the form

$$G(t,x) = \sum_{i=1}^{N} u_i(t,x)h_i, \quad (t,x) \in [0,T] \times H, \tag{1}$$

where $N \in \mathbb{N}$, $u_1, \cdots, u_N \in \mathcal{F}C^k_{0,T}$ and $h_1, \cdots, h_N \in Y$. Clearly, $\mathcal{F}C^\infty_{0,T}$ is dense in $L^p([0,T] \times H, \nu)$ for all finite Borel measures ν on $[0,T] \times H$ and all $p \in [1,\infty)$. $\mathcal{V}\mathcal{F}C^k_b$ denotes the set of all G as in (1) with $u_i \in \mathcal{F}C^k_{0,T}$ replaced by $u_i \in \mathcal{F}C^k_b$.

It is well known that problem (CE) in general admits several solutions even when H is finite dimensional. So, it is natural to look for well posedness of (CE) within the special class of measures $(\nu_t)_{t\in[0,T]}$ which are absolutely continuous with respect to a given *reference measure* γ.

1. Introduction

In this case, denoting by $\rho(t,\cdot)$ the density of ν_t with respect to γ,

$$\nu_t(dx) = \rho(t,x)\gamma(dx), \quad t \in [0,T],$$

equation (CE) becomes

$$\begin{aligned}&\int_0^T \int_H [D_t u(t,x) + \langle D_x u(t,x), F(t,x)\rangle]\, \rho(t,x)\,\gamma(dx)\,dt \\ &= -\int_H u(0,x)\,\rho_0(x)\gamma(dx), \quad \forall\, u \in \mathcal{F}C^1_{b,T}.\end{aligned} \qquad (\mathrm{CE}_\rho)$$

Here $\rho_0 := \rho(0,\cdot)$ is given and $\rho(t,\cdot),\ t \in [0,T]$, is the unknown.

1. Introduction

Our basic assumption on γ is the following

Hypothesis 1

γ is a nonnegative measure on $(H, \mathcal{B}(H))$ with $\gamma(H) < \infty$ such that there exists a dense linear subspace $Y \subset H$ having the following properties:
For all $h \in Y$ there exists $\beta_h : H \to \mathbb{R}$ Borel measurable such that for some $c_h > 0$

$$\int_H e^{c_h|\beta_h|}\, d\gamma < \infty$$

and

$$\int_H \partial_h u\, d\gamma = -\int_H u\beta_h\, d\gamma,$$

where $\partial_h u$ denotes the partial derivative of u in the direction h.

1. Introduction

Assume from now on that γ satisfies Hypothesis 1.

Remark

It is well known that the operator D_x = Fréchet–derivative with domain $\mathcal{F}C_b^1$ is closable in $L^p(H,\gamma)$ for all $p \in [1,\infty)$, see e.g. [AlRo90]. Its closure will again be denoted by D_x and its domain will be denoted by $W^{1,p}(H,\gamma)$.

Let $D_x^* : dom(D_x^*) \subset L^2(H,\gamma;H) \to L^2(H,\gamma)$ denote the adjoint of D_x.

Lemma 1

$\mathcal{V}\mathcal{F}C_b^1 \subset dom(D_x^*)$ and for $G \in \mathcal{V}\mathcal{F}C_b^1$, $G = \sum_{i=1}^N u_i h_i$ we have

$$D_x^* G = -\sum_{i=1}^N (\partial_{h_i} u_i + \beta_{h_i} u_i).$$

1. Introduction

Proof

For $v \in \mathcal{F}C_b^1$ we have

$$\int_H \langle D_x v, G\rangle_H \, d\gamma = \sum_{i=1}^N \int_H \partial_{h_i} v \, u_i \, d\gamma$$

$$= \sum_{i=1}^N \int_H \partial_{h_i}(v \, u_i) \, d\gamma - \sum_{i=1}^N \int_H v \, \partial_{h_i} u_i \, d\gamma$$

$$= - \int_H v \sum_{i=1}^N (\partial_{h_i} u_i + \beta_{h_i} u_i) \, d\gamma.$$

□

We stress that if H is infinite dimensional, β_h is typically not bounded and not continuous. Here are some examples.

1. Introduction

Examples

(i) (Gaussian case) Let Q be a symmetric positive defined operator of trace class on H and $\gamma := N(0, Q)$, i.e. the centered Gaussian measure on H with covariance operator Q. Assume that $\ker Q = \{0\}$ and let Y be the linear span of all eigenvectors of Q. Then Hypothesis 1 is fulfilled with this Y and for $h \in Y$, $h = c_i h_1 + \cdots + c_N h_N$ with $Qh_i = \lambda_i^{-1} h_i$, we have

$$\beta_h(x) = -\sum_{i=1}^{N} c_i \lambda_i \langle h_i, x \rangle_H, \quad x \in H.$$

1. Introduction

(ii) (Case of symmetric reaction diffusions) Let $H := L^2((0,1), d\xi)$ and $A := -\Delta$ with zero boundary conditions. Define

$$\gamma(dx) := \frac{1}{Z}\, e^{-\frac{1}{4}\int_0^1 |x(\xi)|^4 d\xi}\, N(0, -\tfrac{1}{2}\, A^{-1})(dx),$$

where

$$Z := \int_H e^{-\frac{1}{4}\int_0^1 |x(\xi)|^4 d\xi}\, N(0, -\tfrac{1}{2}\, A^{-1})(dx).$$

Then with Y as in (i) for $Q = -\frac{1}{2}\, A^{-1}$ we find for $h = c_i h_1 + \cdots + c_N h_N$ as in (i)

$$\beta_h(x) = -\sum_{i=1}^{N} c_i \lambda_i \langle h_i, x\rangle_H - \int_0^1 h_i(\xi)\, x(\xi)^3\, d\xi, \quad \text{for } N(0, -\tfrac{1}{2}\, A^{-1})\text{–a.e. } x \in H$$

and obviously the exponential integrability condition holds in Hypothesis 1.

(iii) Non-symmetric diffusion also ok!

1. Introduction

Concerning F in (CE) we assume:

Hypothesis 2

(i) $F : [0, T] \times H \to H$ is Borel measurable and bounded.

(ii) There exist $F_j \in \mathcal{VFC}^2_{0,T}$, $j \in \mathbb{N}$, uniformly bounded, such that

$$\begin{cases} \lim\limits_{j \to \infty} F_j = F \quad dt \otimes \gamma\text{-a.e.} \\ \sup\limits_{j \in \mathbb{N}} C_{F_j} < \infty, \end{cases}$$

where C_{F_j} is defined below.

Lemma 2

Assume, besides Hypothesis 1, that $F \in$ dom (D_x^*) and $\varphi \in C_b^1(H)$. Then $\varphi F \in$ dom (D_x^*) and we have

$$D_x^*(\varphi F) = \varphi\, D_x^*(F) - \langle D_x \varphi, F \rangle.$$

2. Main Existence Result

First, we note that if $F \in \mathrm{dom}\,(D_x^*)$ then a regular solution ρ to (CE$_\rho$) solves the equation

$$\begin{cases} D_t\rho + \langle F, D_x\rho\rangle - D_x^* F\,\rho = 0, \\ \rho(0,\cdot) = \rho_0, \end{cases} \tag{CE$_\varrho$ diff}$$

and vice versa. In fact, since for all $u \in \mathcal{VFC}^1_{b,T}$

$$\int_0^T D_t u(t,x)\,\rho(t,x)\,dt = -\int_0^T u(t,x)\,D_t\rho(t,x)\,dt - u(0,x)\rho_0(x), \quad x \in H$$

and (by Lemma 2)

$$\int_H \langle D_x u(t,x), F(t,x)\rangle\,\rho(t,x)\,\gamma(dx) = \int_H \langle D_x u(t,x), \rho(t,x)F(t,x)\rangle\,\gamma(dx)$$

$$= \int_H u(t,x)\,D_x^*(\rho F)(t,x)\,\gamma(dx) = \int_H u(t,x)\,\rho(t,x)\,D_x^* F(t,x)\,\gamma(dx)$$

$$-\int_H u(t,x)\,\langle D_x\rho(t,x), F(t,x)\rangle\,\gamma(dx).$$

2. Main Existence Result

This implies that (CE_ρ) is equivalent to $(\mathrm{CE}_\varrho \text{ diff})$ by the density of $\mathcal{F}C^1_{b,T}$ in $L^2([0,T] \times H, dt \otimes d\gamma)$.

Theorem

Assume that Hypotheses 1 and 2 hold. Let $\zeta := \rho_0 \cdot \gamma$ *be a probability measure on* $(H, \mathcal{B}(H))$ *such that*

$$\int_H \rho_0 \ln \rho_0 \, d\gamma < \infty.$$

Then there exists $\rho : [0,T] \times H \to \mathbb{R}_+$, $\mathcal{B}([0,T] \times H)$*–measurable such that* $\nu_t(dx) = \rho(t,x)\gamma(dx)$, $t \in [0,T]$, *are probability measures on* $(H, \mathcal{B}(H))$ *satisfying* (CE). *In addition*

$$\int_0^T \int_H \rho(t,x) \ln \rho(t,x) \, \gamma(dx) \, dt < \infty. \tag{2}$$

Sketch of proof in Section 4.

3. A Deterministic Feynman–Kac formula

Consider the equation

$$\begin{cases} \dfrac{d}{dt}\xi(t) = \widetilde{F}(t,\xi(t)), \\ \xi(s) = x, \quad x \in \mathbb{R}^d, \end{cases} \tag{FE}$$

with $\widetilde{F}$ regular. Let $V\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be also regular. We want to solve

$$\begin{cases} v_s(s,x) \quad + \quad \langle D_x v(s,x), \widetilde{F}(s,x)\rangle - V(s,x)v(s,x) = 0, \quad 0 \leq s < T, \\ v(T,x) \quad = \quad \varphi(x), \quad x \in H. \end{cases} \tag{$*$}$$

3. A Deterministic Feynman–Kac formula

Proposition

Assume $\widetilde{F} \in C_b([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ *such that* $\widetilde{F}(t,\cdot) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ *for all* $t \in [0,T]$ *and let* $V \in C([0,T] \times \mathbb{R}^d)$ *such that* $V(t,\cdot) \in C^1(\mathbb{R}^d)$ *for all* $t \in [0,T]$ *such that* $D_x V : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *is continuous. Let* $\varphi \in C^1(\mathbb{R}^d)$*. Then the solution to* $(*)$ *is given by*

$$v(s,x) = \varphi(\xi(T,s,x)) e^{\int_s^T V(u,\xi(u,s,x))du}, \qquad (s,x) \in [0,T] \times \mathbb{R}^d, \tag{RF}$$

where for $s \leq t$, $\xi(t,s,x)$ *denotes the solution to* (FE) *at time* t *when started at time* s *at* $x \in \mathbb{R}^d$*. In particular,* $v(\cdot,x) \in C^1([0,T])$ *for every* $x \in \mathbb{R}^d$ *and* $D_t v \in C([0,T] \times \mathbb{R}^d)$*.*

3. A Deterministic Feynman–Kac formula

Proof

We only present the main steps. For any partition $\{s = s_0 < s_1 < \cdots < s_n = T\}$ of $[s, T]$ we write

$$v(s,x) - \varphi(x) = -\sum_{k=1}^{n} [v(s_k, x) - v(s_{k-1}, x)],$$

which is equivalent to,

$$v(s,x) - \varphi(x) = -\sum_{k=1}^{n} [v(s_k, x) - v(s_k, \xi(s_k, s_{k-1}, x))]$$

$$-\sum_{k=1}^{n} [v(s_k, \xi(s_k, s_{k-1}, x)) - v(s_{k-1}, x)] =: J_1 - J_2.$$

Concerning J_1 we write thanks to Taylor's formula

$$J_1 \sim \sum_{k=1}^{n} \langle D_x v(s_k, x), \xi(s_k, s_{k-1}, x) - x \rangle \sim \sum_{k=1}^{n} \langle D_x v(s_k, x), \widetilde{F}(s_k, x) \rangle (s_k - s_{k-1})$$

$$\to \int_s^T \langle D_x v(r, x), \widetilde{F}(r, x) \rangle dr.$$

3. A Deterministic Feynman–Kac formula

Concerning J_2 we write

$$
\begin{aligned}
J_2 &= \sum_{k=1}^{n} v(s_k, \xi(s_k, s_{k-1}, x)) - v(s_{k-1}, x)) \\
&= \sum_{k=1}^{n} \varphi(\xi(T, s_k, \xi(s_k, s_{k-1}, x))) e^{\int_{s_k}^{T} V(u,\xi(u,s_k,\xi(s_k,s_{k-1},x)))du} \\
&\quad - \sum_{k=1}^{n} \varphi(\xi(T, s_{k-1}, x)) e^{\int_{s_{k-1}}^{T} V(u,\xi(u,s_{k-1},x))du} \\
&= \sum_{k=1}^{n} \varphi(\xi(T, s_{k-1}, x)) \left[e^{\int_{s_k}^{T} V(u,\xi(u,s_{k-1},x))du} - e^{\int_{s_{k-1}}^{T} V(u,\xi(u,s_{k-1},x))du} \right] \\
&= \sum_{k=1}^{n} v(s_{k-1}, x)) \left(e^{-\int_{s_{k-1}}^{s_k} V(u,\xi(u,s_{k-1},x))du} - 1 \right) \\
&\sim - \sum_{k=1}^{n} v(s_{k-1}, x) V(s_{k-1}, x)(s_k - s_{k-1}) \to - \int_s^T v(r, x) V(r, x) dr.
\end{aligned}
$$

3. A Deterministic Feynman–Kac formula

Replacing J_1 and J_2 yields

$$v(s,x) = \varphi(x) + \int_s^T \langle D_x v(r,x), \widetilde{F}(r,x)\rangle dr + \int_s^T v(r,x)V(r,x)dr$$

and the claim is proved. $\square$

3. A Deterministic Feynman–Kac formula

As a trivial consequence we obtain

Corollary

Suppose $H = \mathbb{R}^d$ *and* γ *satisfies Hypothesis 1. Let* $F \in C_b([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ *such that* $F(t,\cdot) \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ *and* $D_x^* F(t,\cdot) \in C^1(\mathbb{R}^d)$ *for all* $t \in [0,T]$, *and* $D_x^* F \in C([0,T] \times \mathbb{R}^d)$, $D_x D_x^* F \in C([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$. *Then for every* $\rho_0 \in C^1(\mathbb{R}^d)$, $\rho_0 \geq 0$,

$$\rho(t,x) := \rho_0(\xi(T, T-t, x)) e^{\int_0^t D_x^* F(T-u, \xi(T-u, T-t, x)) du}$$

is a solution of $(\mathrm{CE}_\varrho \text{ diff})$, *where* $\xi(\cdot, s, x)$ *is the solution to* (FE) *started at time* s *at* $x \in \mathbb{R}^d$, *with* $\widetilde{F}(t,x) := -F(T-t,x)$, $(t,x) \in [0,T] \times \mathbb{R}^d$. *Furthermore,* $\rho(\cdot,x) \in C^1([0,T])$ *for every* $x \in \mathbb{R}^d$ *and* $D_t \rho \in C([0,T] \times \mathbb{R}^d)$.

Proof

Apply Proposition with $\widetilde{F}$ as in the assertion above,

$$V(t,x) = D_x^* F(T-t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^d$$

and $\varphi := \rho_0$. □

4. Sketch of Proof of Theorem

By disintegration we shall reduce the proof to the case $H = \mathbb{R}^N$ and by regularization to the Corollary in Section 3.

Case 1 Suppose $F \in \mathcal{V}\mathcal{F}C^2_{0,T}$, $\rho_0 \in \mathcal{F}C^1_b$, $\rho_0 \geq 0$.

In this case we can find an orthonormal basis $\{e_i : i \in \mathbb{N}\}$ of H which consists of elements in Y such that for some $N \in \mathbb{N}$ (which we fix below)

$$F(t,x) = \sum_{i=1}^{N} g_i(t)\, f_i(x)\, e_i, \quad (t,x) \in [0,T] \times H,$$

where for $1 \leq i \leq N$, $g_i \in C^1([0,T])$ with $g_i(T) = 0$ and $f_i \in \mathcal{F}C^2_0$ such that for $x \in H$

$$f_i(x) = \widetilde{f}_i(\langle e_1, x\rangle, ..., \langle e_N, x\rangle)$$

and

$$\rho_0(x) = \widetilde{\rho}_0(\langle e_1, x\rangle, ..., \langle e_N, x\rangle)$$

with $\widetilde{f}_i \in C^2_0(\mathbb{R}^N)$, $\widetilde{\rho}_0 \in C^1_b(\mathbb{R}^N)$.

4. Sketch of Proof of Theorem

Define

$$H_N := \text{lin span}\{e_1, ..., e_N\}$$

and let $\Pi_N : H \to H_N$ be the orthogonal projection. Let $E := H_N^\perp$ be the orthogonal complement of H_N, i.e.

$$H = H^N \oplus E \equiv \mathbb{R}^N \times E,$$

hence, for $z \in H$, $z = (x, y)$ with unique $x \in \mathbb{R}^N$, $y \in E$.
Letting $\nu := \gamma \circ \Pi_N^{-1}$ be the image measure on $(E, \mathcal{B}(E))$ of γ under Π_N^{-1}. Then we have the well known disintegration result for γ

4. Sketch of Proof of Theorem

Lemma 3

There exists $\Psi : \mathbb{R}^N \times E \to [0, \infty)$, $\mathcal{B}(\mathbb{R}^N \times E)$*–measurable such that*

$$\gamma(dz) = \gamma(dx\, dy) = \Psi^2(x, y) dx\, \nu(dy),$$

where dx *denotes Lebesgue measure on* $\mathbb{R}^N$.
Furthermore, for every $y \in E$

$$\Psi(\cdot, y) \in H^{1,2}(\mathbb{R}^N, dx),$$

i.e. the Sobolev space of order 1 *in* $L^2(\mathbb{R}^N, dx)$.

Proof

See [AlRoZh93, Proposition 4.1]. □

4. Sketch of Proof of Theorem

We have by Hypothesis 1 that for all $1 \leq i \leq N$ there exists $c_i \in (0, \infty)$ such that

$$\begin{aligned} \infty \quad > \quad & \int_H e^{c_i \beta_{e_i}} \, d\gamma = \int_E \int_{\mathbb{R}^N} e^{c_i \beta_{e_i}(x,y)} \, \Psi^2(x,y) \, dx \, \nu(dy) \\ = \quad & \int_E \int_{\mathbb{R}^N} \exp\left\{ c_i \, \frac{\partial}{\partial x_i} \, \Psi^2(x,y)/\Psi^2(x,y) \right\} \Psi^2(x,y) dx \, \nu(dy), \end{aligned}$$

where we used that

$$\beta_{e_i}(x,y) = \frac{\partial}{\partial x_i} \, \Psi^2(x,y)/\Psi^2(x,y), \quad (x,y) \in \mathbb{R}^N \times E = H,$$

and the right hand side is defined to be zero on $\{\Psi = 0\}$. Hence we can find $E_0 \in \mathcal{B}(E)$ such that $\nu(E_0) = 1$ and

$$\int_{\mathbb{R}^N} \exp\left\{ c_i \, \frac{\partial}{\partial x_i} \, \Psi^2(x,y)/\Psi^2(x,y) \right\} \Psi^2(x,y) dx < \infty$$

for $y \in E_0$. Below we fix $y \in E_0$.

4. Sketch of Proof of Theorem

Define for $M, l \in \mathbb{N}$ and $(x,y) \in \mathbb{R}^N \times E(\equiv H)$

$$\Psi_M(x,y) := \Big(\Psi^2(x,y) \wedge M\Big)^{1/2},$$

$$\Psi_{M,l}(x,y) := \Big(\Psi_M^2(\cdot,y) * \delta_l\Big)^{1/2}(x),$$

where $\delta_l(x) = l^N \eta(lx)$, $x \in \mathbb{R}^N$, $\eta \in \mathcal{S}(\mathbb{R}^N)$ (:= set of Schwartz test functions) $\eta > 0$, $\eta(x) = \eta(-x)$, $x \in \mathbb{R}^N$ and $\int_{\mathbb{R}^N} \eta\, dx = 1$.) Then by the Corollary in Section 3 applied with the measure $\gamma_{M,l,y}(\mathrm{d}x) = \Psi_{M,l}^2(x,y)\mathrm{d}x$ replacing $\gamma(\mathrm{d}x)$, we know that

$$\rho_{M,l}(t,(x,y)) := \rho_0(\xi(T, T-t, x))\, e^{\int_0^t D_{M,l}^* F(T-u,(\xi(T-u,T-t,x),y))\, du}, \ (t,x) \in [0,T] \times \mathbb{R}^N,$$

where

$$D_{M,l}^* F(r,(x,y)) := -\sum_{i=1}^N g_i(r) \left(\partial_{e_i} f_i(x) + f_i(x) \frac{\partial}{\partial x_i} \Psi_{M,l}^2(x,y) / \Psi_{M,l}^2(x,y) \right),$$

$r \in [0,T]$, $x \in \mathbb{R}^N$, solves

$$\begin{cases} D_t \rho_{M,l}(t,(x,y)) + \langle F(t,x), D_x \rho_{M,l}(t,(x,y)) \rangle - D_{M,l}^*(t,(x,y)) \rho_{M,l}(t,(x,y)) = 0, \\ \rho_{M,l}(0,(x,y)) = \rho_0(x). \end{cases}$$

4. Sketch of Proof of Theorem

Lemma 4 (crucial!)

Let $\epsilon > 0$*. Then for all* $1 \leq N, l, M \in \mathbb{N}$

$$\int_{\mathbb{R}^N} \exp\left[\epsilon \left| \frac{\partial \Psi^2_{M,l}}{\partial x_i}(x,y) / \Psi^2_{M,l}(x,y) \right|\right] \Psi^2_{M,l}(x,y)\, dx$$

$$\leq \int_{\mathbb{R}^N} \exp\left[\epsilon \left| \frac{\partial \Psi^2_{M}}{\partial x_i}(x,y) / \Psi^2_{M}(x,y) \right|\right] \Psi^2_{M}(x,y)\, dx$$

$$\leq \int_{\mathbb{R}^N} \exp\left[\epsilon \left|\beta_{e_i}(x,y)\right|\right] \Psi^2(x,y)\, dx.$$

4. Sketch of Proof of Theorem

Proof

Obviously, the left hand side is dominated by

$$\int_{\mathbb{R}^N} \exp\left[\epsilon \int_{\mathbb{R}^N} \left(\left|\frac{\partial \Psi_M^2}{\partial x_i}\right| / \Psi_M^2\right)(\tilde{x}, y)\, \Psi_M^2(\tilde{x}, y)\, \delta_l(x - \tilde{x})\, d\tilde{x} (\Psi_{M,l}^2(x, y))^{-1}\right] \Psi_{M,l}^2(x, y))dx, \quad (3)$$

where we used that $\frac{\partial}{\partial_{x_i}} \Psi_M^2 = 0$ dx–a.e. on $\{\Psi_M^2 = 0\}$.
Applying Jensen's inequality for fixed $x \in \mathbb{R}^N$ to the probability measure

$$(\Psi_{M,l}^2(x, y))^{-1}\, \Psi_M^2(\tilde{x}, y)\, \delta_l(x - \tilde{x})\, d\tilde{x}$$

and the convex function $r \to e^{\epsilon r}$, we obtain that (3) is dominated by

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \exp\left[\epsilon \left(\left|\frac{\partial \Psi_M^2}{\partial x_i}\right| / \Psi_M^2\right)(\tilde{x}, y)\right] \Psi_M^2(\tilde{x}, y)\, \delta_l(x - \tilde{x})\, d\tilde{x}\, dx.$$

4. Sketch of Proof of Theorem

By Young's Inequality and since $\|\delta_l\|_{L^1(\mathbb{R}^N)} = 1$, the latter is dominated by

$$\int_{\mathbb{R}^N} \exp\left[\epsilon\left(\left|\frac{\partial \Psi_M^2}{\partial x_i}\right| / \Psi_M^2\right)(x,y)\right] \Psi_M^2(x,y)\,dx. \tag{4}$$

Hence the fist inequality of the assertion is proved. To show the second we note that

$$\frac{\partial \Psi_M^2}{\partial x_i} = 1_{\Psi^2 < M} \frac{\partial \Psi^2}{\partial x_i}$$

Hence (4) is dominated by

$$\int_{\mathbb{R}^N} \exp\left[\epsilon 1_{\Psi^2 < M}\left(\left|\frac{\partial \Psi^2}{\partial x_i}\right| / \Psi^2\right)(x,y)\right] \Psi^2(x,y)\,dx,$$

which implies the second inequality of the assertion. □

4. Sketch of Proof of Theorem

Let

$$\delta := \inf_{1 \leq i \leq N} \frac{c_i}{N\left(\|g_i\|_\infty \|f_i\|_\infty\right) + 1}.$$

Then by Lemma 4

$$C_F := \int_0^T \int_{\mathbb{R}^N} \exp\left[-\delta \sum_{i=1}^N g_i(t)\partial_{e_i} f_i(x)\right]^+ \exp\left[\delta \sum_{i=1}^N \|g_i\|_\infty \|f_i\|_\infty \left(\left|\frac{\partial_i \Psi_{M,l}^2}{\partial x_i}\right| / \Psi_{M,l}^2\right)(x,y)\right] \Psi_{M,l}^2(x,y)\, \mathrm{d}x\mathrm{d}t < \infty. \quad (!)$$

4. Sketch of Proof of Theorem

Lemma 5

(i) *For* $\mathrm{d}x$*-a.e.* $x \in \{\Psi(\cdot, y) > 0\}$ *and* $\forall t \in [0, T]$

$$\lim_{M\to\infty} \lim_{k\to\infty} \rho_{M,l_k}(t,(x,y)) = \rho(t,(x,y)) \qquad \textit{(from Corollary)}$$

(ii) *(uniform entropy estimate)*

$$\int_{\mathbb{R}^N} \rho_{M,l}(t,(x,y))(\ln \rho_{M,l}(t,(x,y)) - 1)\Psi^2_{M,l}(x,y)\,\mathrm{d}x$$

$$\leq e^{\frac{t}{\delta}} \Bigg[\int_{\mathbb{R}^N} \rho_0(x)|\ln \rho_0(x) - 1|\Psi^2_{M,l}(x,y)\,\mathrm{d}x + C_F$$

$$+ \frac{t}{\delta}|\ln \frac{1}{\delta}| \int_{\mathbb{R}^N} \rho_0(x)\Psi^2_{M,l}(x,y)\,\mathrm{d}x + \int \Psi^2_{M,l}(x,y)\,\mathrm{d}x \Bigg] \;\; \forall t \in [0,T]$$

! Can pass to the limit to get the same entropy estimate for ρ.
Hence can pass to the limit in (CE) and complete the proof of Step 1.

4. Sketch of Proof of Theorem

Before we proceed to the general case and go from F_j and their corresponding ρ_j to F and corresponding ρ, let us note that we have made the following underlying (standard) heuristics rigorous: Multiplying (CE$_\rho$ diff) by $\ln \rho_j$ and integrating with γ, we find

$$\int_H D_t\rho_j \ \ln \rho_j \, d\gamma$$

$$= -\int_H \langle F_j, D_x\rho_j\rangle_H \ \ln \rho_j \, d\gamma + \int_H D^*(F_j)\rho_j \ \ln \rho_j \, d\gamma$$

$$= -\int_H \langle F_j, D_x(\rho_j \ln \rho_j - \rho_j)\rangle_H \, d\gamma + \int_H D^*(F_j)(\rho_j \ln \rho_j - \rho_j) \, d\gamma + \int_H D^*(F_j)\rho_j \, d\gamma$$

$$\leq \int_H e^{\delta(D^*(F_j))_-} \, d\gamma + \int_H \left(\tfrac{1}{\delta}\rho_j \ln(\tfrac{1}{\delta}\rho_j) - \tfrac{1}{\delta}\rho_j\right) \, d\gamma,$$

where the last step follows by Young's Inequality. Since $\int_H \rho_j \, d\gamma = 1$, this implies that

$$\int_0^T \int_H \rho_j \ \ln \rho_j \, d\gamma \, dt \leq \left(M + \tfrac{1}{\delta} \ln \tfrac{1}{\delta} - \tfrac{1}{\delta}\right) e^{\frac{1}{\delta}\, T} \, T. \qquad (**)$$

4. Sketch of Proof of Theorem

We get $(**)$ rigorously by passing to the limit in Lemma 5 (ii).
Hence (selecting a subsequence if necessary)

$$\rho_j \to \rho \quad \text{weakly in } L^1([0,T] \times H, dt \otimes d\gamma).$$

Now let us show that ρ solves (CE):
We have for all $u \in \mathcal{F}C^1_{b,T}$

$$\int_0^T \int_H \left[\frac{d}{dt} u(t,x) + \langle D_x u(t,x), F_j(t,x) \rangle_H \right] \rho_j(t,x) \, \gamma(dx) \, dt$$

$$= - \int_H u(0,x) \, \rho_j(0,x) \gamma(dx).$$

So, if $\rho_j(0,\cdot) \to \rho_j(0,\cdot)$ in $L^1(H,\gamma)$, we only have to consider the convergence of the left hand side, more precisely only the part of it involving F_j.

4. Sketch of Proof of Theorem

But

$$\left|\int_0^T \int_H (\langle D_x u, F_j\rangle_H \, \rho_j - \langle D_x u, F\rangle_H \, \rho) \, d\gamma \, dt\right|$$

$$\leq \|Du\|_\infty \int_0^T \int_H |F_j - F|_H \, \rho_j \, d\gamma \, dt + \left|\int_0^T \int_H \langle F, Du\rangle \, (\rho_j - \rho) \, d\gamma \, dt\right|$$

Because of the boundedness of $\langle F, Du\rangle$ the second term on the right hand side converges to 0 if $j \to \infty$. Let $\epsilon > 0$. Then, by Young's Inequality, the first term on the right hand side is up to a constant dominated by

$$\int_0^T \int_H e^{\frac{1}{\epsilon}|F_j - F|_H} \, d\gamma \, dt + \epsilon \int_0^T \int_H \rho_j \ln(\epsilon \rho_j) \, d\gamma \, dt,$$

of which the first summand converges to zero as $j \to \infty$, since F_j, F are uniformly bounded, while the second summand is dominated by

$$\epsilon \int_0^T \int_H \rho_j \ln \rho_j \, d\gamma \, dt + \epsilon \ln \epsilon,$$

which can be made arbitrarily small uniformly in j because of $(**)$. The entropy condition for ρ in the Theorem then follows by Komlos' Lemma.

5. References I

S. Albeverio, M. Röckner, *Classical Dirichlet forms on topological vector spaces–closability and a Cameron-Martin formula.* J. Funct. Anal. **88**, no. 2, 395–436, 1990.

S. Albeverio, M. Röckner, T. Zhang, *Markov uniqueness for a class of infinite-dimensional Dirichlet operators.* Stochastic processes and optimal control (Friedrichroda, 1992), 1–26, Stochastics Monogr., 7, Gordon and Breach, Montreux, 1993.

L. Ambrosio, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math., **158** , no. 2, 227–260, 2004.

L. Ambrosio, A. Figalli, *On flows associated to Sobolev vector fields in Wiener spaces: an approach à la Di Perna - Lions*, J. Funct. Anal. **256**, no. 1, 179–214, 2009.

L. Ambrosio, G. Savaré and L. Zambotti, *Existence and stability for Fokker-Planck equations with log-concave reference measure.* Probab. Theory Related Fields **145**, no. 3-4, 517–564, 2009.

L. Ambrosio and D. Trevisan, *Well posedness of Lagrangian flows and continuity equations in metric measure spaces*, Anal. PDE 7, no. 5, 1179–1234, 2014.

5. References II

- V. I. Bogachev, E. Mayer-Wolf, Absolutely continuous flows generated by Sobolev class vector fields in finite in infinite dimensions, JFA **167**, 1–68, 1999.
- G. Da Prato, *Kolmogorov equations for stochastic PDEs*, Birkäuser 2004.
- G. Da Prato and A. Debussche, *Existence of the Fomin derivative of the invariant measure of a stochastic reaction–diffusion equation*, Research Institute for Mathematical Sciences), Kyoto University, Kyoto, Japan, 121–134, 2014, arXiv:1193405.
- G. Da Prato and A. Debussche, *Estimate for $P_t D$ for the stochastic Burgers equation,* Ann. Inst. Henri Poincaré Probab. Stat. **52**, no. 3, 1248–1258, 2016.
- G. Da Prato and A. Debussche, An integral inequality for the invariant measure of a stochastic reaction–diffusion equation, J. evol. equ. (to appear) arXiv:1511.07133.
- G. Da Prato, F. Flandoli and M. Röckner, *Uniqueness for continuity equations in Hilbert spaces with weakly differentiable drift,* Stoch. PDE: Anal. Comp., **2**, 121–145, 2014.

5. References III

G. Da Prato and M. Röckner, *Singular dissipative stochastic equations in Hilbert spaces,* Probab. Theory Relat. Fields, **124**, no. 2, 261–303, 2002.

G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, second edition, Cambridge, 2014.

R. J. Di Perna, P. L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces,* Invent. Math., **98**, 511-547, 1989.

K. D. Elworthy and X.-M. Li, *Formulae for the derivatives of heat semigroups,* J. Funct. Anal., **125**, no.1, 252–286, 1994.

S. Fang, D. Luo, *Transport equations and quasi-invariant flows on the Wiener space*, Bull. Sci. Math., **134**, 295–328, 2010.

A.V. Kolesnikov and M. Röckner, *On continuity equations in infinite dimensions with non-Gaussian reference measure.* J. Funct. Anal. **266**, no. 7, 4490–4537, 2014