BSDEs, martingale problems, pseudo-PDEs and applications.

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Outline

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7. Special case of the Föllmer-Schweizer decomposition.
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Basic Reference

Ismail Laachir and Francesco Russo.

BSDEs, càdlàg martingale problems and orthogonalization under basis risk.

SIAM Journal on Financial Mathematics, vol. 7, pp. 308-356 (2016)

Related references.

- A. Barrasso and F. Russo.

  Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations.

  https://hal.inria.fr/hal-01431559

- A. Barrasso and F. Russo.

  Decoupled Mild solutions for Pseudo Partial Differential Equations versus Martingale driven forward-backward SDEs.
  https://hal.archives-ouvertes.fr/hal-01505974

Available preprints and publications.

http://uma.ensta.fr/∼russo/

1 General mathematical context

- Interface between “stochastic processes” and “deterministic world”.
- Benchmark situation: bridge between semilinear PDEs and BSDEs.

PDE:

$$\begin{cases} \partial_s u(s,x) + L_s u(s,x) + f(s,x,u(s,x),\sigma\partial_x u(s,x)) = 0 \\ u(T,x) = g(x),\ s \in [0,T], x \in E = \mathbb{R}^d, \end{cases} \tag{1}$$

where L_s is the generator of a diffusion of the type

$$dX_s = \sigma(s,X_s)dW_s + b(t,X_s)ds, X_t = x. \tag{2}$$

BSDE: (2) is coupled with

$$Y_s = g(X_T) + \int_s^T f(s, X_r, Y_r, Z_r)dr - \int_s^T Z_r dW_r. \qquad (3)$$

The link is the following.

1. If u is a classical solution of (1) then

$$Y_s = u(s, X_s), Z_s = \sigma(s, X_s)\nabla u(s, X_s)$$

   provide a solution to (3).

2. Viceversa if, given $(t, x) \in [0, T] \times E$ and $X^{t,x}$ is given by (2), $(X^{t,x}, Y^{t,x}, Z^{t,x})$ is a solution to (3), then $u(t, x) := Y_t^{t,x}$ is a *viscosity solution* to (1).

What about $v(t,x) := Z_t^{t,x}$?

- If u is of class $C^{0,1}$ then $v(t,x) = \sigma(t,x)\nabla u(t,x)$.
- What happens in general? Only partial answers even in the Brownian case.
- This talk and the mentioned references discuss some issues related to this problem when W is replaced by a cadlag martingale.

2 Financial Motivations

2.1 Hedging in a complete market

Let $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $\mathcal{F}_0$ being the trivial σ-algebra.

- S price of a risky asset.
- B price of a riskless asset.

Complete market.

For any random variable h, there exists a self-financing strategy $(\nu_t)_{t\in[0,T]}$ perfectly replicating h, i.e. a trading strategy that starts from an initial wealth V_0 and re-invests the gain/loss from S on the riskless asset B.

If we suppose that the riskless asset price is constant, this reduces to

$$V_0 + \int_0^T \nu_u dS_u = h.$$

2.2 Hedging in the presence of basis risk

Basis risk.

Risk arising when a derivative product h is *based* on a non-traded or illiquid underlying, but observable, and the replicating (hedging) portfolio is constituted of traded and liquid additional assets which are correlated with the original one.

Example:

- Basket option hedged with a subset of the composing assets.
- Airline companies hedging kerosene exposure with correlated contacts, as crude oil or heating oil.

Consider a pair of processes (X, S) and a contingent claim of the type $h := g(X_T, S_T)$.

- X is a non traded or illiquid, but observable asset.
- S is a traded asset, correlated to X.
- B is riskless asset. We suppose B to be constant.

Hedging problem: construct a trading strategy on the assets (B, S) in order to replicate the random variable h.

In this case, the market is incomplete: perfect replication with a self-financing strategy is not possible. One should define a risk aversion criterion, for example the following.

- **Utility-based** criterion.
- **Quadratic risk criteria**: *local risk minimization* and *mean-variance minimization*.

2.3 Quadratic hedging: local and global risk minimization.

- Introduced by Föllmer and Sondermann [1985], for S being a (local) martingale. In this case, the unique (local) risk-minimizing strategy is determined by the Kunita-Watanabe (K-W) representation of martingales.
- Extension to the semimartingale case is more delicate, and was handled by Schweizer [1988, 1991]. Its existence is linked to the existence of the so-called Föllmer-Schweizer (F-S) decomposition, a generalization of the (K-W) representation.
- **Global risk minimization.** Again F-S decomposition.

2.4 Föllmer-Schweizer decomposition

Mean-variance hedging is closely related to the so called Föllmer-Schweizer (F-S) decomposition.

Definition 1 *Let* $S = M^S + V^S,\ V_0^S = 0$ *be a special semimartingale. A square integrable random variable* h *admits an F-S decomposition if*

$$h = h_0 + \int_0^T Z_u dS_u + O_T,$$

where $h_0 \in \mathbb{R}, Z \in \Theta$ *and* O *is a square integrable martingale, strongly orthogonal to* M^S*.*

Definition 2 *Let L and N be two $\mathcal{F}_t$-local martingales, with null initial value. L and N are said to be strongly orthogonal if LN is a local martingale.*

Example 3 *If L and N are locally square integrable, then they are strongly orthogonal if and only if $\langle L, N \rangle = 0$.*

2.5 F-S decomposition via a backward SDE

If (h_0, Z, O) is an F-S decomposition, then the process $Y_t := h_0 + \int_0^t Z_u dS_u + O_t$ verifies

$$Y_t := h - \int_t^T Z_u dM_u^S - \int_t^T Z_u dV_u^S - (O_T - O_t),$$

which is a Backward Stochastic Differential Equation, driven by a local martingale, where the final condition $Y_T = h$ is known.
The resolution of the BSDE is a method to determine the F-S decomposition.

3 Backward Stochastic Differential Equations

3.1 BSDEs driven by a Brownian motion

BSDEs were introduced by Pardoux and Peng [1990].
Pioneering work by Bismut [1973].

- Given a pair $(h, \hat{f})$ called *terminal condition* and *driver*.
- One looks for a pair of (adapted) processes (Y, Z), satisfying

$$Y_t = h + \int_t^T \hat{f}(\omega, s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad (4)$$

and

$$\mathbb{E}\int_0^T |Z_t|^2 dt < \infty.$$

3.2 Existence and uniqueness

- Pardoux and Peng [1990] showed existence and uniqueness when $\hat{f}$ is globally Lipschitz with respect to (y, z) and h being square integrable.
- Conditions on the driver $\hat{f}$ were first relaxed to a monotonicity condition on y, later to a quadratic growth condition and other generalizations, see e.g. Hamadene [1996], Lepeltier and San Martín [1998], Kobylanski [2000], Briand and Hu [2006, 2008].
- Applications to finance: El Karoui et al. [1997].
- Extension to *reflected* BSDEs...

3.3 BSDEs and semi-linear parabolic PDEs

Consider the BSDE

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(s, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r, \quad (5)$$

where $\{X_s^{t,x}, t \leq s \leq T\}$ is a solution of the SDE

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r, \ t \leq s \leq T.$$

Link with the semi-linear parabolic PDE.

$$\begin{cases} \partial_t u(t,x) + L_t u(t,x) + f(t,x,u(t,x),\sigma\partial_x u(t,x)) = 0 \\ u(T,x) = g(x),\ t \in [0,T], x \in \mathbb{R}. \end{cases} \tag{6}$$

3.4 From semi-linear parabolic PDEs to BSDEs

Theorem 4 (Pardoux and Peng [1992]) *Let* $u \in C^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$ *be a classical solution of* (6) *such that*

$$|\partial_x u(t,x)| \leq c(1 + |x|^q), \text{ for some } c, q > 0.$$

Then, $\forall (t,x)$, $(u(s, X_s^{t,x}), (\sigma \partial_x u)(s, X_s^{t,x}))_{s \in [t,T]}$ *is solution of the BSDE* (5)*.*
In particular, under the conditions of well-posedness of the BSDE

$$u(t,x) = Y_t^{t,x}.$$

3.5 From BSDEs to semi-linear parabolic PDEs

Theorem 5 (Pardoux and Peng [1992]) *Let* $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$ *be the solution of the BSDE* (5)*, then* $u(t,x) := Y_t^{t,x}$ *is a continuous function and it is a viscosity solution of the PDE* (6)*.*

This representation theorem can be seen as an extension of Feynman-Kac formula.

3.6 Extensions of BSDEs driven by Brownian Motion

- BSDE driven by a Brownian motion and a compensated random measure.
- BSDE driven by a càdlàg martingale.

3.7 BSDEs driven by a càdlàg Martingale

Given a càdlàg (local) martingale M^S and a bounded variation process V^S, one looks for a triplet (Y, Z, O) verifying

$$Y_t = h + \int_t^T \hat{f}(\omega, s, Y_{s-}, Z_s) dV_s^S - \int_t^T Z_s dM_s^S - (O_T - O_t), \quad (7)$$

where O is (local) martingale strongly orthogonal to M^S.

- First contribution by Buckdahn [1993].
- Other contributions, e.g. El Karoui and Huang [1997]. See also Briand et al. [2002], as side-effect of a convergence scheme.
- More recent setting for sufficient conditions for existence and uniqueness for (7) has been given by Carbone et al. [2007].
- BSDEs with partial information driven by càdlàg martingales were investigated by Ceci, Cretarola, Russo in Ceci et al. [2014a,b].

4 Contributions of the work

A forward BSDE, where the forward process solves a *strong martingale problem*. We focus on four tasks.

- Characterize forward-backward SDEs via the solution of a deterministic problem generalizing the classical PDE appearing in the case of Brownian martingales.
- Give applications to the hedging problem in the case of basis risk via the Föllmer-Schweizer decomposition.

- Give explicit expressions when the pair of processes (X, S) is an exponential of additive processes.
- Extensions to the case when the forward process is given in law: strict and generalized solutions of the deterministic problem.

5 Strong Martingale Problem

5.1 Definition

Definition 6 *Let* $\mathcal{O}$ *be an open set of* $\mathbb{R}^2$ *and* (A_t) *be an* $\mathcal{F}_t$*-adapted b.v. continuous process, such that, a.s.* $dA_t \ll d\rho_t$*, for some b.v. function* ρ*, and* $\mathcal{A}$ *a map*

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{C}([0,T] \times \mathcal{O}, \mathbb{C}) \longrightarrow \mathcal{L}.$$

We say that (X, S) is a solution of the **strong martingale problem** related to $(\mathcal{D}(\mathcal{A}), \mathcal{A}, A)$, if for any $g \in \mathcal{D}(\mathcal{A})$, $(g(t, X_t, S_t))_t$ is a semimartingale such that

$$t \longmapsto g(t, X_t, S_t) - \int_0^t \mathcal{A}(g)(u, X_{u-}, S_{u-}) dA_u$$

is an $\mathcal{F}_t$- local martingale.

Notations 7

- $id : (t, x, s) \longmapsto s$, $s^2 : (t, x, s) \longmapsto s^2$.
- *For any* $y \in \mathcal{C}([0, T] \times \mathcal{O})$, $\widetilde{y} := y \times id$.
- *Suppose that* $id \in \mathcal{D}(\mathcal{A})$. *For* $y \in \mathcal{D}(\mathcal{A})$ *such that* $\widetilde{y} \in \mathcal{D}(\mathcal{A})$, *we set* $\widetilde{\mathcal{A}}(y) := \mathcal{A}(\widetilde{y}) - y\mathcal{A}(id) - id\mathcal{A}(y).$

Proposition 8 *Suppose that $id, s^2 \in \mathcal{D}(\mathcal{A})$. Then S is a special semimartingale with decomposition $M^S + V^S$ given below.*

1. $V_t^S = \int_0^t \mathcal{A}(id)(u, X_{u-}, S_{u-}) dA_u$.
2. $\langle M^S \rangle_t = \int_0^t \widetilde{\mathcal{A}}(id)(u, X_{u-}, S_{u-}) dA_u$.

Proof.

Item 2. follows from the following more general result.

Lemma 9 *If* $Y_t = y(t, X_t, S_t)$, $y, y \times id \in \mathcal{D}(\mathcal{A})$, *then*

$$\langle M^Y, M^S \rangle_t = \int_0^t \tilde{\mathcal{A}}(y)(u, X_{u-}, S_{u-}) dA_u.$$

5.2 Examples

- Diffusion process: the operator $\mathcal{A}$ has the form

$$
\begin{aligned}
\mathcal{A}(f) &= \partial_t f + b_S \partial_s f + b_X \partial_x f \\
&+ \frac{1}{2}\left\{ |\sigma_S|^2 \partial_{ss} f + |\sigma_X|^2 \partial_{xx} f + 2\langle \sigma_S, \sigma_X \rangle \partial_{sx} f \right\},
\end{aligned}
$$

- S is a Markov process, with related Markov semigroup of generator L: the operator $\mathcal{A}$ has the form

$$\mathcal{A}(g)(t,s) = \frac{\partial g}{\partial t}(t,s) + Lg(t,\cdot)(s).$$

5.3 Exponential of additive processes

Definition 10 *(Z^1, Z^2) is said to be an* **additive process** *if $(Z^1, Z^2)_0 = 0$, (Z^1, Z^2) is continuous in probability and it has independent increments. The generating function of (Z^1, Z^2) is defined by*

$$\exp(\kappa_t(z_1, z_2)) = \mathbb{E}e^{z_1 Z_t^1 + z_2 Z_t^2},\ \forall (z_1, z_2) \in D,$$

where $D := \{z = (z_1, z_2) \in \mathbb{C}^2 | \quad \mathbb{E}e^{\mathrm{Re}(z_1) Z_T^1 + \mathrm{Re}(z_2) Z_T^2} < \infty\}$.
We denote also, for $(z_1, z_2), (y_1, y_2) \in D/2$

$$\rho_t(z_1, z_2, y_1, y_2) := \kappa_t(z_1 + y_1, z_2 + y_2) - \kappa_t(z_1, z_2) - \kappa_t(y_1, y_2),$$

$$\rho_t^S := \kappa_t(0, 2) - 2\kappa_t(0, 1),\ \textit{if}\ (0, 1) \in D/2.$$

We always suppose the validity of the following.

Assumption 11 (Basic assumption) $(0,2) \in D$. *This is equivalent to the existence of the second order moment of* $S = e^{Z^2}$*.*

5.4 First decomposition

We consider two processes $X = \exp(Z^1),\ S = \exp(Z^2)$.

Lemma 12 *Let* $\lambda : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$ *such that, for any* $(z_1, z_2) \in D$, $d\lambda(t, z_1, z_2) \ll d\rho^S_t$. *Then for any* $(z_1, z_2) \in D$,

$$\begin{aligned} t &\mapsto M^\lambda_t(z_1, z_2) := X^{z_1}_t S^{z_2}_t \lambda(t, z_1, z_2) \\ &- \int_0^t X^{z_1}_{u-} S^{z_2}_{u-} \Big\{ \frac{d\lambda(u, z_1, z_2)}{d\rho^S_u} + \lambda(u, z_1, z_2) \frac{d\kappa_u(z_1, z_2)}{d\rho^S_u} \Big\} \rho^S_{du}, \end{aligned}$$

is a martingale. Moreover, if $(z_1, z_2) \in D/2$ *then* $M^\lambda(z_1, z_2)$ *is a square integrable martingale.*

5.5 Strong Martingale Problem for exponential of additive processes

Theorem 13 *Under some technical assumptions,* (X, S) *is a solution of the strong martingale problem related to* $(\mathcal{D}(\mathcal{A}), \mathcal{A}, \rho^S)$ *where,* $\mathcal{D}(\mathcal{A})$ *is the set of*

$$f : (t, x, s) \mapsto \int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2} \lambda(t, z_1, z_2),$$

where Π *is a finite Borel measure on* $\mathbb{C}^2$*,*
$\lambda : [0, T] \times \mathbb{C}^2 \to \mathbb{C}$ *Borel verifying a set of conditions,*

$$\mathcal{A}(f)(t,x,s) \quad = \quad \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2} \left\{ \frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} + \lambda(t,z_1,z_2) \frac{d\kappa_t(z_1,z_2)}{d\rho_t^S} \right\}.$$

6 Deterministic problem related to BSDEs driven by a martingale

6.1 Forward-backward SDE

We consider a pair of $\mathcal{F}_t$-adapted processes (X, S) fulfilling the martingale problem related to $(\mathcal{D}(\mathcal{A}), \mathcal{A}, A)$. We are interested in the BSDE

$$
\begin{aligned}
Y_t &= g(X_T, S_T) + \int_t^T f(r, X_{r-}, S_{r-}, Y_{r-}, Z_r) dA_r - \int_t^T Z_r dM_r^S \\
&- (O_T - O_t),
\end{aligned}
$$

1. (Y_t) is $\mathcal{F}_t$-adapted, (Z_t) is $\mathcal{F}_t$-predictable
2. $\int_0^T |Z_s|^2 d\langle M^S \rangle_s < \infty$ a.s.
3. $\int_0^t |f(s, X_{s-}, S_{s-}, Y_{s-}, Z_s)| d\|A\|_s < \infty$ a.s.
4. (O_t) is an $\mathcal{F}_t$-local martingale such that $\langle O, M^S \rangle = 0$ and $O_0 = 0$ a.s.

6.2 Related deterministic analysis

Goal. Look for solutions (Y, Z, O) of the BSDE for which there is a function $y \in \mathcal{D}(\mathcal{A})$ such that $\widetilde{y} = y \times id \in \mathcal{D}(\mathcal{A})$ and a locally bounded Borel function $z : [0, T] \times \mathcal{O} \longrightarrow \mathbb{C}$, such that

$$\begin{aligned} Y_t &= y(t, X_t, S_t), \\ Z_t &= z(t, X_{t-}, S_{t-}), \quad \forall t \in [0, T]. \end{aligned}$$

- When M^S is a Brownian motion, y is a solution of a semilinear PDE.
- General case ?

6.3 Deterministic problem (Pseudo-PDE)

Theorem 14 *Suppose the existence of a function y, such that $y, \widetilde{y} := y \times id$ belong to $\mathcal{D}(\mathcal{A})$, and a Borel locally bounded function z, solving the system*

$$\begin{cases} \mathcal{A}(y)(t,x,s) & = & -f(t,x,s,y(t,x,s),z(t,x,s)) \\ \widetilde{\mathcal{A}}(y)(t,x,s) & = & z(t,x,s)\widetilde{\mathcal{A}}(id)(t,x,s), \end{cases}$$

with the terminal condition $y(T,.,.) = g(.,.)$.
Then the triplet (Y,Z,O) defined by

$$Y_t = y(t,X_t,S_t),\ Z_t = z(t,X_{t-},S_{t-})$$

is a solution to the BSDE (8).

7 Special case of the Föllmer-Schweizer decomposition.

7.1 Weak F-S decomposition

Definition 15 *We say that a square integrable $\mathcal{F}_T$-measurable random variable h admits a* **weak F-S decomposition** (h_0, Z, O) *with respect to S if it can be written as*

$$h = h_0 + \int_0^T Z_s dS_s + O_T, \mathbb{P}\text{−a.s.}, \tag{8}$$

where h_0 is an $\mathcal{F}_0$-measurable r.v., Z is a predictable process such that $\int_0^T |Z_s|^2 d\langle M^S\rangle_s < \infty$ a.s., $\int_0^T |Z_s| d\|V^S\|_s < \infty$ a.s. and O is a local martingale such that $\langle O, M^S\rangle = 0$ with $O_0 = 0$.

7.2 Link to BSDEs

Finding a weak F-S decomposition (h_0, Z, O) for some r.v. h is equivalent to provide a solution (Y, Z, O) of the BSDE

$$Y_t = h - \int_t^T Z_s dS_s - (O_T - O_t).$$

The link is given by $Y_0 = h_0$. Here the driver f is linear in z, of the form

$$f(t, x, s, y, z) = -\mathcal{A}(id)(t, x, s)z.$$

$\Rightarrow$ The weak F-S decomposition can be linked to a deterministic problem (Pseudo-PDE).

7.3 Weak Vs True F-S decomposition

Remark 16 *Setting* $h_0 = y(0, X_0, S_0)$*, the triplet* (h_0, Z, O) *is a candidate for a true F-S decomposition. Sufficient conditions for this are the following.*

1. $h = g(X_T, S_T) \in L^2(\Omega)$.
2. $(z(t, X_{t-}, S_{t-}))_t \in \Theta$ *i.e.*
   - $\mathbb{E} \int_0^T |z(t, X_{t-}, S_{t-})|^2 \widetilde{\mathcal{A}}(id)(t, X_{t-}, S_{t-}) dA_t < \infty$.
   - $\mathbb{E}\left(\int_0^T |z(t, X_{t-}, S_{t-})| \, \|\mathcal{A}(id)(t, X_{t-}, S_{t-}) dA\|_t \right)^2 < \infty$.
3. $\left(y(t, X_t, S_t) - \int_0^t \mathcal{A}(y)(u, X_{u-}, S_{u-}) dA_u \right)_t$ *is an* $\mathcal{F}_t$*-square integrable martingale.*

Corollary 17 (Application of the theorem for general BSDEs) *Let y (resp. z): $[0,T] \times \mathcal{O} \to \mathbb{C}$. We suppose the following.*

1. $y, \widetilde{y} := y \times id$ *belong to* $\mathcal{D}(\mathcal{A})$.
2. $\int_0^T z^2(r, X_{r-}, S_{r-}) \widetilde{\mathcal{A}}(id)(r, X_{r-}, S_{r-}) dA_r < \infty$ *a.s.*
3. (y, z) *solves the problem*

$$\begin{cases} \mathcal{A}(y)(t,x,s) & = & \mathcal{A}(id)(t,x,s)z(t,x,s), \\ \widetilde{\mathcal{A}}(y)(t,x,s) & = & \widetilde{\mathcal{A}}(id)(t,x,s)z(t,x,s), \end{cases} \tag{9}$$

with the terminal condition $y(T,.,.) = g(.,.)$.

Then the triplet (Y_0, Z, O), where

$$Y_t = y(t, X_t, S_t),\ Z_t = z(t, X_{t-}, S_{t-}),\ O_t = Y_t - Y_0 - \int_0^t Z_s dS_s,$$

is a weak F-S decomposition of h.

7.4 Application 1: exponential of additive processes

$(X, S) = (e^{Z^1}, e^{Z^2})$ is an exponential of additive processes.

Example 18 *Goal. Use the **Pseudo-PDE** to give explicit expressions of a weak F-S of an $\mathcal{F}_T$-measurable random variable h of the form $h := g(X_T, S_T)$ for a function g of the form*

$$g(x, s) = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2},$$

where Π is finite Borel complex measure.

Existence and uniqueness.

Proposition 19 *Suppose the validity of the Basic assumption and*

$$\int_0^T \left(\frac{d\kappa_t(0,1)}{d\rho_t^S}\right)^2 d\rho_t^S < \infty.$$

Then any square integrable variable admits a unique true F-S decomposition.

The proof makes use of a general existence and uniqueness theorem by Monat and Stricker [1995].

Idea.

In agreement with the definition of $\mathcal{D}(\mathcal{A})$, we select y of the form

$$y(t,x,s) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2} \lambda(t,z_1,z_2),$$

where Π is the same finite complex measure as in the definition of h and $\lambda : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$.

The deterministic equations in the corollary write as

$$\begin{cases} \displaystyle\int_{\mathbb{C}^2} d\Pi(z_1,z_2)x^{z_1}s^{z_2}\left\{\frac{d\lambda(t,z_1,z_2)}{d\rho_t^S}+\lambda(t,z_1,z_2)\frac{d\kappa_t(z_1,z_2)}{d\rho_t^S}\right\} \\ \qquad\qquad \displaystyle = s\frac{d\kappa_t(0,1)}{d\rho_t^S}z(t,x,s) \\ \displaystyle\int_{\mathbb{C}^2} d\Pi(z_1,z_2)\lambda(t,z_1,z_2)x^{z_1}s^{z_2+1}\frac{d\rho_t(z_1,z_2,0,1)}{d\rho_t^S} = s^2 z(t,x,s) \\ y(T,\cdot,\cdot)=g. \end{cases}$$

Unknown: $\lambda \Rightarrow$ can be determined through the resolution of an ODE in t.

Theorem 20 (Weak F-S decomposition) *Let λ be defined as* $\lambda(t, z_1, z_2) = \exp\left(\int_t^T \eta(z_1, z_2, du)\right), \forall (z_1, z_2) \in D/2$, *where*

$$\eta(z_1, z_2, t) = \kappa_t(z_1, z_2) - \int_0^t \frac{d\rho_u(z_1, z_2, 0, 1)}{d\rho_u^S} \kappa_{du}(0, 1).$$

Then, under some technical assumptions, (Y_0, Z, O) *is a weak F-S decomposition of* h, *where*

$$Y_t = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) X_t^{z_1} S_t^{z_2} \lambda(t, z_1, z_2),$$

$$Z_t = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) X_{t-}^{z_1} S_{t-}^{z_2-1} \lambda(t, z_1, z_2) \gamma_t(z_1, z_2),$$

$$O_t = Y_t - Y_0 - \int_0^t Z_s dS_s \quad \text{and}$$

$$\gamma_t(z_1, z_2) = \frac{d\rho_t(z_1, z_2, 0, 1)}{d\rho_t^S}, \ \forall (z_1, z_2) \in D/2, t \in [0, T],$$

Proposition 21 (True F-S decomposition) *Under slightly stronger assumptions as in Theorem above, the weak F-S decomposition of*

$$h = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) X_T^{z_1} S_T^{z_2}$$

above is a true F-S decomposition.
Moreover, if h is real-valued then the decomposition (h_0, Z, O) is real-valued and it is therefore the unique F-S decomposition.

Example 22 *This statement is a generalization of the results of [Oudjane, Goutte and Russo, 2014] to the case of hedging under basis risk.*

7.5 Application 2: diffusion processes

Let (X, S) be a diffusion process with drift (b_X, b_S) and volatility (σ_X, σ_S).

Assumption 23

- *b_X, b_S, σ_X and σ_S are continuous and globally Lipschitz.*
- *$g : \mathcal{O} \to \mathbb{R}$ is continuous.*

(X, S) solve the strong martingale problem related to $(\mathcal{D}(\mathcal{A}), \mathcal{A}, A)$ where $A_t = t$, $\mathcal{D}(\mathcal{A}) = \mathcal{C}^{1,2}([0,T[\times\mathcal{O}) \cap \mathcal{C}^1([0,T] \times \mathcal{O})$ and

$$\begin{aligned}
\mathcal{A}(y) &= \partial_t y + b_S \partial_s y + b_X \partial_x y \\
&+ \frac{1}{2}\left\{|\sigma_S|^2 \partial_{ss} y + |\sigma_X|^2 \partial_{xx} y + 2\langle \sigma_S, \sigma_X \rangle \partial_{sx} y\right\}, \\
\widetilde{\mathcal{A}}(y) &= |\sigma_S|^2 \partial_s y + \langle \sigma_S, \sigma_X \rangle \partial_x y.
\end{aligned}$$

Example 24 ***Goal.*** *characterize the (weak) F-S decomposition of* $h := g(X_T, S_T)$*.*

Theorem 25 (Weak F-S decomposition) *We suppose the validity of Assumption 23. and that $|\sigma_S|$ is always strictly positive. If (y,z) is a solution of the system*

$$\begin{cases} \partial_t y + B\partial_x y + \frac{1}{2}\left(|\sigma_S|^2\partial_{ss} y + |\sigma_X|^2 \partial_{xx} y + 2\langle \sigma_S, \sigma_X\rangle \partial_{sx} y\right) = 0, \\ y(T,.,.) = g(.,.), \text{ where } B = b_X - b_S \frac{\langle \sigma_S, \sigma_X\rangle}{|\sigma_S|^2}, \\ z = \partial_s y + \frac{\langle \sigma_S, \sigma_X\rangle}{|\sigma_S|^2}\partial_x y, \end{cases} \tag{10}$$

such that $y \in \mathcal{D}(\mathcal{A})$, then (Y_0, Z, O) is a weak F-S decomposition of $g(X_T, S_T)$, where

$$Y_t = y(t, X_t, S_t), \ Z_t = z(t, X_t, S_t), \ O_t = Y_t - Y_0 - \int_0^t Z_s dS_s.$$

Remark 26 *1. Under slightly stronger assumption one can give conditions for the existence of a* **true Föllmer-Schweizer** *decomposition.*

2. Black-Scholes was treated by Hulley and McWalter [2008].

8 Extensions: BSDE vs Pseudo-PDE

- Until now we have essentially shown that a solution to a **blue Pseudo-PDE** provide solutions to BSDEs driven by cadlag martingales.
- **More problematic is the converse implication.**

  Barrasso and Russo [2017a,b].

Let E be a Polish space. Let $\mathbb{P}^{t,x}$ be a *Markov class* family of probability measures under which the canonical process X on $D([0,T];E)$ solves a martingale problem to $\mathcal{D}(\mathcal{A}),\mathcal{A},\rho)$. Let us denote $M^S := M_s^{id,t} := S_s - x - \int_t^s \mathcal{A}(id)(S_r)d\rho(r)$. We consider $BSDE(f,g(S_T),M)$, i.e.

$$\begin{aligned} Y_s &= g(S_T) + \int_s^T f(r,S_{r-},Y_{r-},Z_r)d\rho_r - \int_s^T Z_r dM_r^S \\ &- (O_T - O_s), s\in[t,T], \end{aligned} \tag{11}$$

under $\mathbb{P}^{t,x}$.

Let us suppose the following.

- $id \in \mathcal{D}(\mathcal{A})$.
- $\langle M^S \rangle$ is absolutely continuous with respect to ρ.
- Let us suppose suitable growth condition on g and Lipschitz on f.

“Theorems”

- Then (11) admits a unique solution $(Y^{t,x}, Z^{t,x}, O^{t,x})$ in some suitable spaces.
- There is a “unique” couple (y, z) of Borel functions such that $y(t, x) = Y^{t,x}$, and $Z_s^{t,x} = z(s, X_s)$ a.s. under $\mathbb{P}^{t,x}$.
- The couple (y, z) is a so called *decoupled mild solution* of the system.
- There is a *unique* decoupled mild solution of Pseudo-PDE(f, g).

Thank you for your attention!

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