# Weyl calculus with respect to the Gaussian measure and $L^p$-$L^q$ boundedness of the OU semigroup in complex time

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## Overview of the talk

1. Motivation
2. The Ornstein-Uhlenbeck semigroup
3. Position and momentum
4. The Weyl calculus
5. Work in progress

## 1. Motivation

- $\gamma$ – the standard Gaussian measure in $\mathbb{R}^d$,

$$\gamma(\mathrm{d}x) := (2\pi)^{-d/2} \exp(-|x|^2/2)\,\mathrm{d}x$$

- $L$ – the Ornstein–Uhlenbeck operator

$$L := \nabla^* \nabla$$

with $\nabla$ the *Malliavin derivative*: the realisation of the gradient in $L^2(\mathbb{R}^d, \gamma)$.

Integrating by parts, we obtain

$$L = -\Delta + x \cdot \nabla.$$

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$$D^2 = \begin{bmatrix} \nabla^*\nabla & 0 \\ 0 & \nabla\nabla^* \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix}$$

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<u>Key observation:</u>

> *$L$ does not belong to the functional calculus of $\nabla$,*
> *but $\begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix}$ belongs to the functional calculus of $D$.*

In a more general (infinite-dimensional, non-symmetric) setting this enabled us to prove:

**Theorem.** (Maas, vN '09) *For* $1 < p < \infty$ *TFAE:*

1. *The Riesz transform* $\nabla/\sqrt{L}$ *is bounded on* $L^p(\mathbb{R}^d, \gamma_d)$
2. $\underline{L}$ *has a bounded* $H^\infty$*-calculus on* $L^p(\mathbb{R}^d, \gamma_d)$

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Observation: In terms of annihilation and creation operators:

$$\nabla = (a_1, \dots, a_d), \quad \nabla^* = (a_1^\dagger, \dots, a_d^\dagger)$$

Associated with these are the position and momentum operators

$$q_j = \frac{1}{\sqrt{2}}(a_j + a_j^\dagger), \quad p_j = \frac{1}{i\sqrt{2}}(a_j - a_j^\dagger)$$

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$\Longrightarrow$ Study $L$ in terms of $Q = (q_1, \dots, q_d)$ and $P = (p_1, \dots, p_d)$

## 2. The Ornstein-Uhlenbeck semigroup

- $-L$ generates a $C_0$-semigroup of contractions on $L^p(\mathbb{R}^d, \gamma)$ for all $p \in [1, \infty)$, the so-called *Ornstein-Uhlenbeck semigroup*, given by

$$P(t)f(x) = \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\,\mathrm{d}\gamma(y)$$
$$= \int_{\mathbb{R}^d} M_t(x, y)g(y)\,\mathrm{d}y$$

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with

$$M_t(x, y) = \frac{1}{(2\pi(1 - e^{-2t}))^{d/2}} \exp\Big(-\frac{|e^{-t}x - y|^2}{2(1 - e^{-2t})}\Big)$$

the so-called *Mehler kernel*.

**Probabilistic interpretation:**

$$e^{-\frac{1}{2}tL} = \mathbb{E}f(X_t^x)$$

with $X_t^x$ the solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = -\dfrac{1}{2}X_t\,\mathrm{d}t + \mathrm{d}B_t \\ X_0 = x \end{cases}$$

with $(B_t)_{t\geq 0}$ a standard Brownian motion.

**Analyticity:**

- The OU is an analytic $C_0$-semigroup of contractions on $L^p(\mathbb{R}^d, \gamma)$ for all $p \in (1, \infty)$, of angle $\phi_p$, where

$$\cos \phi_p = \left| \frac{2}{p} - 1 \right|$$

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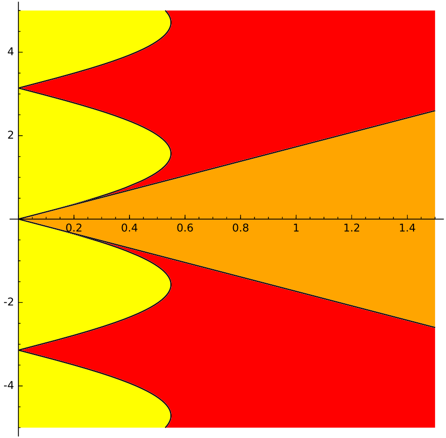
- The angle $\phi_p$ is optimal [Chill-Fašangová-Metafune-Pallara 05]
- The domain of analyticity of $e^{-zL}$ in $L^p(\mathbb{R}^d, \gamma)$ equals the Epperson region

$$E_p = \{x + iy \in \mathbb{C} : |\sin(y)| < \tan(\phi_p) \sinh(x)\}$$

  and $e^{-zL}$ is contractive there.

  [Epperson 89], [García Cuerva-Mauceri-Meda-Sjögren-Torrea 01]




The region $E_p$ for $p = 4/3$ (red)
and the sector of angle $\phi_p$ (orange)

**Hypercontractivity:**

- $e^{-tL}$ is contractive from $L^p(\mathbb{R}^d, \gamma)$ to $L^q(\mathbb{R}^d, \gamma)$ if and only if

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[Nelson 66]

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[Nelson 66]

- $e^{-zL}$ is contractive from $L^p(\mathbb{R}^d,\gamma)$ to $L^q(\mathbb{R}^d,\gamma)$ if and only if for all $w \in \mathbb{C}$

$$(\mathrm{Im}(we^{-z}))^2 + (q-1)(\mathrm{Re}(we^{-z}))^2 \leq (\mathrm{Im}\, w)^2 + (p-1)(\mathrm{Re}\, w)^2$$

[Epperson 89]

## 3. Position and momentum

On $L^2(\mathbb{R}^d)$, consider the position and momentum operators

$$X = (x_1, \ldots, x_d), \quad D = (\frac{1}{i}\partial_1, \ldots, \frac{1}{i}\partial_d).$$

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$$[x_j, x_k] = [D_j, D_k] = 0 \quad [x_j, D_k] = i\delta_{jk}. \tag{1}$$

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Note:

$$\sum_j (D_j^2 + x_j^2) = -\Delta + |x|^2$$

Let $m(\mathrm{d}x) = (2\pi)^{-d/2}\mathrm{d}x$ denote the normalised Lebesgue measure on $\mathbb{R}^d$.

- The pointwise multiplier

$$Ef(x) := e(x)f(x)$$

with $e(x) := \exp(-\frac{1}{4}|x|^2)$, is unitary from $L^2(\mathbb{R}^d, \gamma)$ onto $L^2(\mathbb{R}^d, m)$.

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Consequently,

- The operator

$$U := \delta \circ E$$

is unitary from $L^2(\mathbb{R}^d, \gamma)$ onto $L^2(\mathbb{R}^d, m)$.

By [Segal 56] the operator $U$ establishes a unitary equivalence

$$\mathscr{W} = U^{-1} \circ \mathscr{F} \circ U$$

of the Fourier-Plancherel transform $\mathscr{F}$ as a unitary operator on $L^2(\mathbb{R}^d, m)$,

$$\mathscr{F} f(y) = \int_{\mathbb{R}^d} f(x) \exp(-ix \cdot y) \, \mathrm{d}m(x),$$

with the unitary operator $\mathscr{W}$ on $L^2(\mathbb{R}^d, \gamma)$, defined for polynomials $f$ by

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We have the following representation of this operator in terms of the *second quantisation* functor $\Gamma$:

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NB.: Without the dilation $\delta$, this identity would not come out right.

## Gaussian position and momentum

On $L^2(\mathbb{R}^d, \gamma)$, consider the Gaussian position and Gaussian momentum operators

$$Q = (q_1, \dots, q_d), \qquad P = (p_1, \dots, p_d),$$

where

$$q_j := U^{-1} \circ x_j \circ U, \qquad p_j := U^{-1} \circ D_j \circ U.$$

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We have

$$\frac{1}{2}(P^2 + Q^2) = L + \frac{d}{2}I,$$

with $L$ the OU operator (in the physics literature: $L$ is the 'boson number operator', $\frac{1}{2}(P^2 + Q^2)$ the 'quantum harmonic oscillator', and $\frac{d}{2}$ the 'ground state energy').

# 4. The Weyl calculus

For Schwartz functions $a : \mathbb{R}^{2d} \to \mathbb{C}$ define

$$a(X, D)f(x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widehat{a}(u, v) \exp(i(uX + vD))f(x) \, m(\mathrm{d}u) \, m(\mathrm{d}v).$$

Here

- $\widehat{a}$ is the Fourier-Plancherel transform of $a$,

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Here

- $\widehat{a}$ is the Fourier-Plancherel transform of $a$,
- the unitary operators $\exp(i(uX + vD))$ on $L^2(\mathbb{R}^d, \gamma)$ are defined through the action

$$\exp(i(uX + vD))f(x) := \exp(iux + \tfrac{1}{2}iuv)f(v + x)$$

(formally apply the Baker-Campbell-Hausdorff formula and use the commutator relations, or note that it gives a unitary representation of the Heisenberg group; see [Hall 13]).

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By explicit computation,

$$a(Q,P)f(x) = \int_{\mathbb{R}^d} K_a(x,y) f(y)\,\mathrm{d}y,$$

where

$$K_a(x,y) := \frac{1}{(2\sqrt{2}\pi)^d} \int_{\mathbb{R}^d} a\Big(\frac{x+y}{2\sqrt{2}},\xi\Big) \times \exp\Big(i\xi\Big(\frac{x-y}{\sqrt{2}}\Big)\Big) \exp\big(\tfrac{1}{4}|x|^2 - \tfrac{1}{4}|y|^2\big)\,\mathrm{d}\xi.$$

Recall the identity $\frac{1}{2}(P^2 + Q^2) = L + \frac{1}{2}I$. Thus one would expect

$$e^{-zL} = e^{-\frac{1}{2}z} \exp(-\tfrac{1}{2}z(P^2 + Q^2)).$$

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Rather, the Weyl calculus gives:

**Theorem 1.**

$$e^{-zL} = \Big(1 + \frac{1-e^{-z}}{1+e^{-z}}\Big)^d \exp\Big(-\frac{1-e^{-z}}{1+e^{-z}}(P^2 + Q^2)\Big).$$

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NB.: The RHS can be computed explicitly using the integral representation for the Weyl calculus.

*Sketch of the proof:* By elementary computation, the integral representation for $a_s(Q,P)$, with $s = \frac{1-e^{-t}}{1+e^{-t}}$, reduces to the Mehler formula for $e^{-tL}$.

Theorem 1 suggests the study of the family of operators

$$\exp(-s(P^2 + Q^2)), \quad \mathrm{Re} s > 0.$$

With

$$a_s(x) = \exp(-s(|x|^2 + |y|^2))$$

we obtain

$$\begin{aligned}\exp(-s(P^2 + Q^2))f(x) &= \int_{\mathbb{R}^d} K_{a_s}(x,y)f(y)\mathrm{d}y \\ &= \frac{1}{2^d(2\pi s)^{d/2}} \int_{\mathbb{R}^d} \exp(-\tfrac{1}{8s}(1-s)^2(|x|^2+|y|^2) + \tfrac{1}{4}(\tfrac{1}{s} - s)xy)f(y)\mathrm{d}\gamma(y).\end{aligned}$$

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$$= \frac{1}{2^d(2\pi s)^{d/2}}\int_{\mathbb{R}^d} \exp(-\tfrac{1}{8s}(1-s)^2(|x|^2+|y|^2)+\tfrac{1}{4}(\tfrac{1}{s}-s)xy)f(y)\mathrm{d}\gamma(y).$$

Note the symmetry in $x$ and $y$.

Consider the Gaussian measure in $\mathbb{R}^d$ with variance $\tau$,

$$\gamma_\tau(dx) := (2\pi\tau)^{-d/2} \exp(-|x|^2/2\tau)\, dx.$$

Define, for $\mathrm{Re}\, s > 0$,

$$r_\pm(s) := \frac{1}{2}\mathrm{Re}\,(\frac{1}{s} \pm s).$$

**Theorem 2.** *Let $p, q \in [1, \infty)$ and let $\alpha, \beta > 0$. If $1 - \frac{2}{\alpha p} + r_+(s) > 0$, $\frac{2}{\beta q} - 1 + r_+(s) > 0$, and*

$$(*) \qquad (r_-(s))^2 \le (1 - \frac{2}{\alpha p} + r_+(s))(\frac{2}{\beta q} - 1 + r_+(s)),$$

*then $\exp(-s(P^2 + Q^2))$ is bounded from $L^p(\mathbb{R}^d, \gamma_\alpha)$ to $L^q(\mathbb{R}^d, \gamma_\beta)$.*

The case '$<$' in $(*)$ follows by a simple application of Hölder's inequality!

To get the result with '$\leq$' in $(*)$, a Schur estimate is used instead:

**Lemma.** (Schur estimate) *Let $p, q, r \in [1, \infty)$ be such that $\frac{1}{r} = 1 - (\frac{1}{p} - \frac{1}{q})$. If $K \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $\phi, \psi : \mathbb{R} \to (0, \infty)$ are integrable functions such that*

$$\sup_{y \in \mathbb{R}^d} \Big( \int_{\mathbb{R}^d} |K(y,x)|^r \frac{\psi^{r/q}(y)}{\phi^{r/p}(x)} \,\mathrm{d}x \Big)^{1/r} =: C_1 < \infty,$$

*and*

$$\sup_{x \in \mathbb{R}^d} \Big( \int_{\mathbb{R}^d} |K(y,x)|^r \frac{\psi^{r/q}(y)}{\phi^{r/p}(x)} \,\mathrm{d}y \Big)^{1/r} =: C_2 < \infty$$

*then*

$$T_K f(y) := \int_{\mathbb{R}} K(y,x) f(x) \,\mathrm{d}x \quad (f \in C_c(\mathbb{R}))$$

*defines a bounded operator $T_K$ from $L^p(\mathbb{R}^d, \phi(x)\,\mathrm{d}x)$ to $L^q(\mathbb{R}^d, \psi(x)\,\mathrm{d}x)$ with norm*

$$\|T_K\|_{L^p(\mathbb{R}^d, \phi(x)\,\mathrm{d}x) \to L^q(\mathbb{R}^d, \psi(x)\,\mathrm{d}x)} \leq C_1^{1-\frac{r}{q}} C_2^{\frac{r}{q}}.$$

**Proposition.** *Theorem 2 implies Epperson's $L^p$-$L^q$ boundedness criterion.*

*Proof.* Substitute $z = x + iy$ and check that Epperson's criterion implies the positivity conditions of Theorem 2.

This involves only elementary (but quite miraculous) high-school algebra.

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This involves only elementary (but quite miraculous) high-school algebra. The crucial thing is to recognise (we used MAPLE) that

$$\underbrace{\begin{aligned}(q-1)((1-(x^2+y^2))^2+4y^2)^2+(2-p-q)(1-(x^2+y^2))^2((1+x)^2+y^2)^2\\ -(2-p-q+pq)4y^2((1+x)^2+y^2)^2+(p-1)((1+x)^2+y^2)^4\end{aligned}}_{\text{the positivity condition in Epperson's criterion}}$$

factors as

$$\underbrace{\left[4((1+x)^2+y^2)^2\right]}_{\geq 0} \times \underbrace{\left[(p-q)x(1+x^2+y^2)+(2p+2q-4)x^2-(pq-2p-2q+4)y^2\right]}_{\text{the positivity condition in Theorem 2}}. \qquad \square$$

**Corollary.** *For $p \in (1, \infty)$, the operator-valued function*

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*Proof.* For $q = p$ and $z = x + iy$, Epperson's $L^p$-$L^q$ criterion reduces to

$$p^2\left(\frac{x^2}{x^2 + y^2} - 1\right) + 4p - 4 > 0,$$

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$$p^2(\frac{x^2}{x^2 + y^2} - 1) + 4p - 4 > 0,$$

which is equivalent to saying that $s \in \Sigma_{\phi_p}$. □

NB: $s = \frac{1-e^{-z}}{1+e^{-z}}$ *maps the Epperson region $E_p$ to $\Sigma_{\phi_p}$!*

Thus we recover that $E_p$ is the $L^p$-domain of holomorphy of $e^{-zL}$.

For $p = 1$ the following is due to [Bakry, Bolley, Gentil 12] by very different methods (they get contractivity):

**Corollary.** *Let* $p \in [1, 2]$. *For all* $\mathrm{Re}\, z > 0$ *the operator* $\exp(-zL)$ *maps* $L^p(\mathbb{R}^d, \gamma_{2/p})$ *into* $L^2(\mathbb{R}^d, \gamma)$.

For $p = 1$ the following is due to [Bakry, Bolley, Gentil 12] by very different methods (they get contractivity):

**Corollary.** *Let* $p \in [1, 2]$*. For all* $\mathrm{Re}\, z > 0$ *the operator* $\exp(-zL)$ *maps* $L^p(\mathbb{R}^d, \gamma_{2/p})$ *into* $L^2(\mathbb{R}^d, \gamma)$*.*

*As a consequence, the semigroup generated by* $-L$ *extends to an analytic* $C_0$*-semigroup on* $L^p(\mathbb{R}^d, \gamma_{2/p})$ *of angle* $\frac{1}{2}\pi$*.*

(Recall that $-L$ extends to an analytic $C_0$-semigroup on $L^p(\mathbb{R}^d, \gamma)$ of non-trivial angle $\phi_p$.)

## 5. Work in progress

Much of this can be generalised to the setting of a Weyl pair $(A, B)$ of two densely defined linear operators on a Banach space $X$ such that

(a) $iA$ and $iB$ generate bounded $C_0$-groups on $X$

(b) $e^{isA}e^{itB} = e^{ist}e^{itB}e^{isA}$ for all $s, t \in \mathbb{R}$

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**Proposition.** *If $(A, B)$ be a Weyl pair,*

1. $-(A^2 + B^2) + \frac{1}{2}$ *generates an bounded analytic semigroup on* $X$
   ($\leftrightarrow$ OU operator in $d = 1$)

2. $\exp(i(uA + iB))$ *is unitary for all* $u, v \in \mathbb{R}$
   ($\leftrightarrow$ Schrödinger representation)

Thus a Weyl calculus $a \mapsto a(A, B)$ can be defined.

We recover the formula

$$e^{-tL} = (1+s)\exp(-s(A^2+B^2))$$

with $s = \frac{1-e^{-t}}{1+e^{-t}}$.

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Conjecturally (work in progress),

- $A^2 + B^2$ has a bounded $H^\infty$-calculus
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THANK YOU FOR YOUR ATTENTION!

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