Integration of geometric rough paths

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Rough Path Theory

Differential equations of the form:

$$dy_t = f(y_t)\, dx_t,\ y_0 = \xi,$$

have been widely used in the modelling of controlled systems.

The theory of rough path develops a mathematical tool to model the evolution of controlled systems, which is applicable but not restricted to Brownian motion and semi-martingales.

Young Integration

(Classical Integral) When x is a continuous path and y is a continuous path of finite length, $\int xdy$ is well-defined.

(Young Integral) When x is α-Hölder and y is β-Hölder for $\alpha + \beta > 1$, $\int xdy$ is well-defined as a Riemann-Stieltjes integral.

$$x_s (y_t - y_s) = x_s (y_u - y_s) + x_u (y_t - y_u) + O\left(|t - s|^{\alpha+\beta} \right).$$

Young's condition is sharp. When x is α-Hölder for $\alpha \leq 2^{-1}$, the integral

$$\int_{r=0}^{1} f\left(x_r\right) dx_r \approx f\left(x_0\right)\left(x_1 - x_0\right) + f'\left(x_0\right) \int_{r=0}^{1} \left(x_r - x_0\right) dx_r$$

may not be meaningfully defined.

(Lyons, Rough Path Theory) When x is α-Hölder for $\alpha \leq 2^{-1}$, one can lift x to a group valued path X (a rough path), and define the integration of X.

Regularity Structures

(Gubinelli, Controlled Rough Path) For a rough path X, ρ is called a path controlled by X if

$$\rho_t - \rho_s \approx L_s X_{s,t} \text{ for } |t - s| \ll 1.$$

(Hairer, Regularity Structure) The theory of regularity structures gives a meaning to a class of classically ill-posed stochastic partial differential equations (e.g. KPZ equation).

Slowly-varying exact one-forms

For each t, let $df_t = x_t$, then

$$\int_{r=0}^{1} df_t dy_r = x_t \left(y_1 - y_0 \right) .$$

For $s < u < t$,

$$\int_{r=s}^{t} df_s dy_r \approx \int_{r=s}^{u} df_s dy_r + \int_{r=u}^{t} df_u dy_r .$$

Exact one-forms in group setting

Suppose G_1 and G_2 are two topological groups. Let $f : G_1 \to G_2$ be a differentiable function and $X : [0,1] \to G_1$ be a differentiable path. The integral of the exact one-form df along X is given by

$$\int_{r=0}^{1} df dX_r = \int_{X_0}^{X_1} df = f\left(X_0\right)^{-1} f\left(X_1\right).$$

Slowly-varying exact one-forms in group setting

The integral is well defined when a family of exact one-forms $(df_t)_t$ vary slowly along a path X:

$$\int_{r=s}^{t} df_s dX_r \approx \int_{r=s}^{u} df_s dX_r \int_{r=u}^{t} df_u dX_r$$

for $s < u < t$.

(Signature) Let $x : [0,1] \to V$ be a continuous path of finite length. The signature of x is given by

$$\begin{aligned} S(x) &:= \left(1, X^1, \ldots, X^n, \ldots\right) \\ \text{with } X^n &:= \int_{0<u_1<\cdots<u_n<1} dx_{u_1} \cdots dx_{u_n}. \end{aligned}$$

(Chen) The signature takes values in a group, denoted as G:

$$S(x)\, S(y) = S(x * y) \text{ and } S(x)^{-1} = S(\overleftarrow{x}).$$

Lifting of Integrals

Classical integral of polynomial one-form

Let p be a polynomial one-form, and let x be a continuous path of finite length. Then

$$\begin{aligned}\int_{r=0}^{1} p\left(x_r\right) dx_r &= \sum_{k=0}^{n} \left(D^k p\right)\left(x_0\right) \int_{r=0}^{1} \frac{\left(x_r - x_0\right)^k}{k!} dx_r \\ &= \sum_{k=0}^{n} \left(D^k p\right)\left(x_0\right) X^{k+1}.\end{aligned}$$

Lifting of Integrals

The **classical integral** $\int p(x)\, dx$ is **lifted to a function on** G:

$$f : a \mapsto \sum_{k=0}^{n} \left(D^k p\right)(x_0)\, a^{k+1},$$

and

$$\int_{r=0}^{1} p(x_r)\, dx_r = f(X_1) - f(X_0) = \int_{r=0}^{1} df dX_r,$$

where X_t denotes the signature of $x|_{[0,t]}$.

Integration of rough paths

The integration of rough paths can be viewed as the integration of slowly varying exact one-forms $(dF_t)_t$ with $F_t : G_1 \to G_2$, along a continuous path X in G_1, and the integral $\int_{r=0}^{\cdot} dF_r dX_r$ obtained is a continuous path in G_2.

Thank You!

References

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