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On approximating the Gramian

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The controllability Gramian of a stable LTI system is defined as $\mathcal{P} = \int_0^\infty e^{At}BB^\mathsf{T}e^{A^\mathsf{T}t}\,\mathrm{d}t$. It can be characterized as the solution of the Lyapunov equation $A\mathcal{P} + \mathcal{P}A^\mathsf{T} + BB^\mathsf{T} = 0$. We will show that solving the Lyapunov equation with the **ADI method is equivalent to a particular integral approximation.**

Approximating the Gramian via quadrature

The Gramian $\mathcal{P}$ is approximated via the midpoint rectangle method. With $h(t) := e^{At}B$ this results in

$$\mathcal{P} \approx \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} h(t)h(t)^\mathsf{T}\,\mathrm{d}t \approx \sum_{i=1}^{N}\omega_i h\left(\frac{t_{i-1}+t_i}{2}\right) h\left(\frac{t_{i-1}+t_i}{2}\right)^\mathsf{T}$$
$$\approx [h_1,\dots,h_N]\,\mathrm{diag}(\omega_1,\dots,\omega_N)\,[h_1,\dots,h_N]^\mathsf{T},$$

with $t_0 = 0$ and quadrature weights $\omega_i = t_i - t_{i-1}$. To obtain the approximation $h_j \approx h(\frac{t_{j-1}+t_j}{2})$, $j = 1,\dots,N$, the ODE $\dot{x} = Ax$, $x(0) = B$ is solved with $j-1$ steps of the trapezoidal rule with step sizes $\omega_1,\dots,\omega_{j-1}$, followed by one backward Euler step with step size $\frac{1}{2}\omega_j$:

$$\begin{aligned}
x_0 &= B\\
x_i &= x_{i-1} + \frac{\omega_i}{2}(Ax_{i-1} + Ax_i)\\
&= \left(I - \frac{\omega_i}{2}A\right)^{-1}\left(I + \frac{\omega_i}{2}A\right)x_{i-1}\\
&= -\underbrace{\left(A + \frac{2}{\omega_i}I\right)}_{=:T_i}\underbrace{\left(A - \frac{2}{\omega_i}I\right)^{-1}}_{=:S_i}x_{i-1}.
\end{aligned}$$

With one final backward Euler step this leads to

$$h_j = \underbrace{\left(I - \frac{\omega_j}{2}\right)^{-1}}_{=:R_j} x_{j-1} = (-1)^{j-1}R_jT_{j-1}S_{j-1}\cdots T_1S_1B.$$

Approximating the Gramian via CF-ADI

In the Cholesky factor ADI method [1] the solution of the Lyapunov equation is approximated by the low-rank factorization

$$\mathcal{P} \approx [z_1, z_2, \dots, z_N][z_1, z_2, \dots, z_N]^\mathsf{T}.$$

With the so called ADI parameters $p_1,\dots,p_N \in \mathbb{C}^-$ the CF-ADI iteration is defined as follows:

$$\begin{aligned}
z_1 &= \sqrt{-2p_1}(A + p_1I)^{-1}B\\
z_i &= \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i-1}}}\underbrace{(A + p_iI)^{-1}}_{=:\widetilde{S}_i}\underbrace{(A - p_{i-1}I)}_{=:\widetilde{T}_i}z_{i-1}.
\end{aligned}$$

After j steps the iteration leads to

$$\begin{aligned}
z_j &= \sqrt{-2p_j}(A + p_jI)^{-1}\widetilde{T}_{j-1}\widetilde{S}_{j-1}\cdots\widetilde{T}_1\widetilde{S}_1B\\
&= \underbrace{\sqrt{\frac{2}{-p_j}}\left(I - \frac{1}{-p_j}A\right)^{-1}}_{=:\widetilde{R}_j}\widetilde{T}_{j-1}\widetilde{S}_{j-1}\cdots\widetilde{T}_1\widetilde{S}_1B.
\end{aligned}$$

It's equivalent

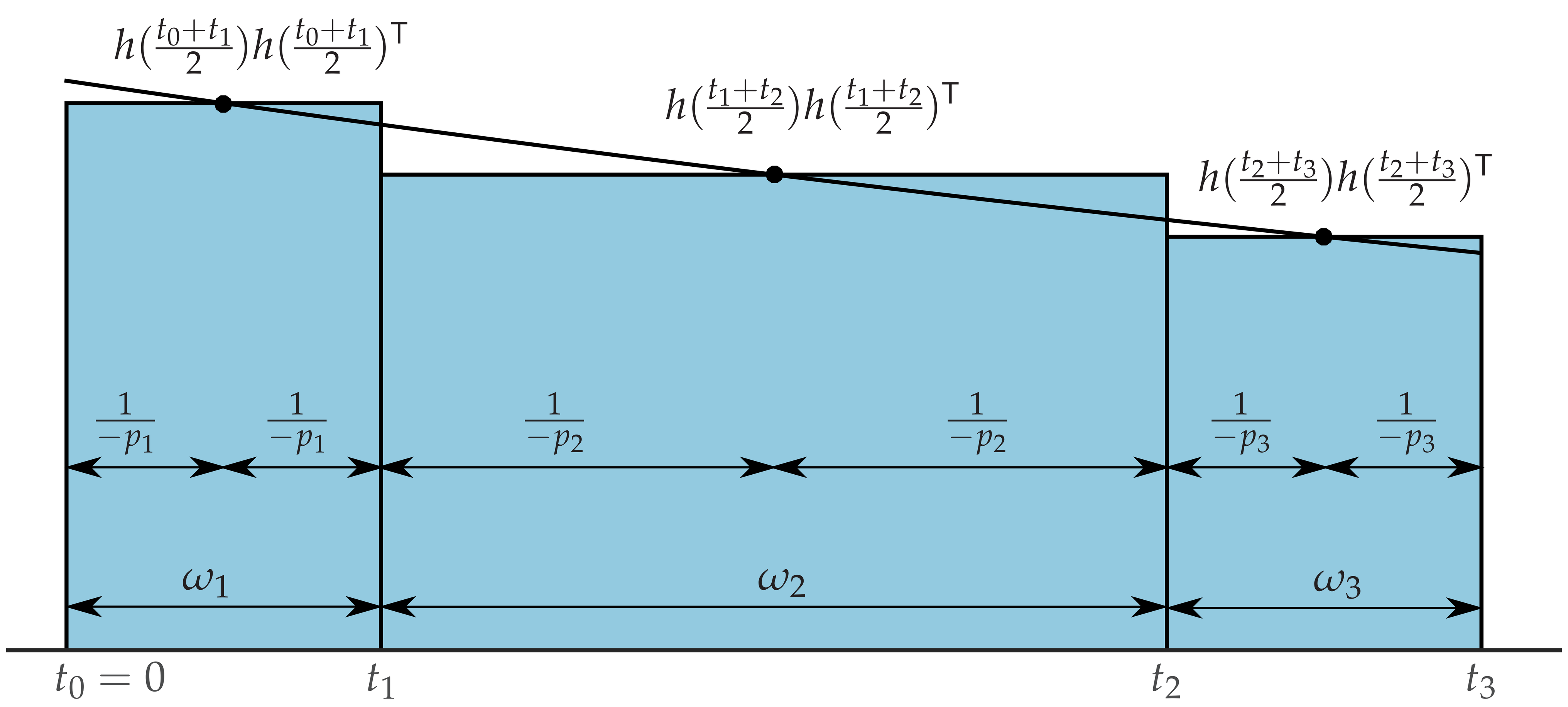
With $p_i \in \mathbb{R}^-$ and the choice

$$\omega_i = \frac{2}{-p_i}$$

the iteration matrices $T_i = \widetilde{T}_i$, $S_i = \widetilde{S}_i$ and $R_i = \widetilde{R}_i$ coincide. Thus we have

$$\sqrt{\omega_j}h_j = (-1)^{j-1}z_j$$

and the approximation to the Gramian $\mathcal{P}$ is the same for both methods.



References

[1] J.-R. Li and J. White, *Low Rank Solution of Lyapunov Equations*, SIAM Journal on Matrix Analysis and Applications Vol. 24 No. 1, pp. 260–280 (2002)
[2] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, sec. 12.4.5, SIAM (2005)