Data-driven model reduction in the Loewner framework

Thanos Antoulas

Rice University, MPI Magdeburg & Baylor College of Medicine

email: aca@rice.edu
http://www.ece.rice.edu/antoulas.aspx

London Mathematical Society – EPSRC Durham Symposium
Model Order Reduction, 7-17 August 2017

Given:

row array $(\mu_j, \mathbf{v}_j),\ j = 1, \cdots, q,$

column array $(\lambda_i, \mathbf{w}_i),\ i = 1, \cdots, k,$

the associated **Loewner matrix** is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

If $\mathbf{w}_i = \mathbf{g}(\lambda_i),\ \mathbf{v}_j = \mathbf{g}(\mu_j),$ are **samples** of $\mathbf{g}$:

Main property. Let $\mathbb{L}$ be as above.
Then $k, q \geq \deg \mathbf{g} \ \Rightarrow\ \operatorname{rank} \mathbb{L} = \deg \mathbf{g}.$

Karel Löwner (1893 - 1968)

Ch. Loewner

- Born in Bohemia
- Studied in Prague under Georg Pick
- Emigrated to the US in 1939
- Seminal paper:

Über monotone Matrixfunctionen,
Math. Zeitschrift (1934).

Given:

row array $(\mu_j, \mathbf{v}_j),\ j = 1, \cdots, q,$

column array $(\lambda_i, \mathbf{w}_i),\ i = 1, \cdots, k,$

the associated **Loewner matrix** is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

If $\mathbf{w}_i = \mathbf{g}(\lambda_i),\ \mathbf{v}_j = \mathbf{g}(\mu_j),$ are **samples** of $\mathbf{g}$:

Main property. Let $\mathbb{L}$ be as above.
Then $k, q \geq \deg \mathbf{g} \ \Rightarrow\ \operatorname{rank} \mathbb{L} = \deg \mathbf{g}.$

Karel Löwner (1893 - 1968)

Ch. Loewner

- Born in Bohemia
- Studied in Prague under Georg Pick
- Emigrated to the US in 1939
- Seminal paper:

Über monotone Matrixfunctionen,
Math. Zeitschrift (1934).

Given:

row array $(\mu_j, \mathbf{v}_j),\ j = 1, \cdots, q,$

column array $(\lambda_i, \mathbf{w}_i),\ i = 1, \cdots, k,$

the associated **Loewner matrix** is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

If $\mathbf{w}_i = \mathbf{g}(\lambda_i),\ \mathbf{v}_j = \mathbf{g}(\mu_j),$ are **samples** of $\mathbf{g}$:

Main property. Let $\mathbb{L}$ be as above.

Then $k, q \geq \deg \mathbf{g} \Rightarrow \operatorname{rank} \mathbb{L} = \deg \mathbf{g}$.

Karel Löwner (1893 - 1968)


Ch. Loewner

- Born in Bohemia
- Studied in Prague under Georg Pick
- Emigrated to the US in 1939
- Seminal paper:

Über monotone Matrixfunctionen, Math. Zeitschrift (1934).

Given:

row array $(\mu_j, \mathbf{v}_j),\ j = 1, \cdots, q,$

column array $(\lambda_i, \mathbf{w}_i),\ i = 1, \cdots, k,$

the associated **Loewner matrix** is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

If $\mathbf{w}_i = \mathbf{g}(\lambda_i),\ \mathbf{v}_j = \mathbf{g}(\mu_j),$ are **samples** of $\mathbf{g}$:

Main property. Let $\mathbb{L}$ be as above.

Then $k, q \geq \deg \mathbf{g} \ \Rightarrow\ \operatorname{rank} \mathbb{L} = \deg \mathbf{g}$.

Karel Löwner (1893 - 1968)

Ch. Loewner

- Born in Bohemia
- Studied in Prague under Georg Pick
- Emigrated to the US in 1939
- Seminal paper:

Über monotone Matrixfunctionen, Math. Zeitschrift (1934).

Given:

row array $(\mu_j, \mathbf{v}_j),\ j = 1, \cdots, q,$

column array $(\lambda_i, \mathbf{w}_i),\ i = 1, \cdots, k,$

the associated **Loewner matrix** is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

If $\mathbf{w}_i = \mathbf{g}(\lambda_i),\ \mathbf{v}_j = \mathbf{g}(\mu_j),$ are **samples** of $\mathbf{g}$:

Main property. Let $\mathbb{L}$ be as above.
Then $k, q \geq \deg \mathbf{g} \ \Rightarrow \ \mathrm{rank}\, \mathbb{L} = \deg \mathbf{g}$.

Karel Löwner (1893 - 1968)

Ch. Loewner

- Born in Bohemia
- Studied in Prague under Georg Pick
- Emigrated to the US in 1939
- Seminal paper:

Über monotone Matrixfunctionen, Math. Zeitschrift (1934).

- Lagrange basis for space of polynomials of degree at most n.

Given $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n+1$: $\lambda_i \neq \lambda_j$, $i \neq j$, define

$$\mathbf{q}_i(s) := \Pi_{i' \neq i}\,(s - \lambda_{i'}),\ \ i = 1, \cdots, n+1.$$

- For given constants α_i, $\mathbf{w}_i$, $i = 1, \cdots, n+1$, consider

$$\sum_{i=1}^{n+1} \alpha_i \, \frac{\mathbf{g} - \mathbf{w}_i}{s - \lambda_i} = 0, \ \ \alpha_i \neq 0.$$

- Solving for $\mathbf{g}$ we obtain

$$\boxed{\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\alpha_i \mathbf{w}_i}{s - \lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s - \lambda_i}}} \qquad \Rightarrow \qquad \mathbf{g}(\lambda_i) = \mathbf{w}_i.$$

This is the **barycentric Lagrange interpolation** formula.

Reference: J.-P. Berrut and L. N. Trefethen, Barycentric Lagrange interpolation, SIAM Review, vol. 46, pp. 501-517, 2004.

Rational interpolation and the Loewner matrix

• Lagrange basis for space of polynomials of degree at most n.

Given $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n+1$: $\lambda_i \neq \lambda_j$, $i \neq j$, define

$$\mathbf{q}_i(s) := \Pi_{i' \neq i} (s - \lambda_{i'}), \; i = 1, \cdots, n+1.$$

• For given constants α_i, $\mathbf{w}_i$, $i = 1, \cdots, n+1$, consider

$$\sum_{i=1}^{n+1} \alpha_i \frac{\mathbf{g} - \mathbf{w}_i}{s - \lambda_i} = 0, \;\; \alpha_i \neq 0.$$

• Solving for $\mathbf{g}$ we obtain

$$\boxed{\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\alpha_i \mathbf{w}_i}{s-\lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s-\lambda_i}}} \quad \Rightarrow \quad \mathbf{g}(\lambda_i) = \mathbf{w}_i.$$

This is the **barycentric Lagrange interpolation** formula.

Reference: J.-P. Berrut and L. N. Trefethen, Barycentric Lagrange interpolation, SIAM Review, vol. 46, pp. 501-517, 2004.

- Lagrange basis for space of polynomials of degree at most n.

Given $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n+1$: $\lambda_i \neq \lambda_j$, $i \neq j$, define

$$\mathbf{q}_i(s) := \Pi_{i' \neq i}(s - \lambda_{i'}), \; i = 1, \cdots, n+1.$$

- For given constants α_i, $\mathbf{w}_i$, $i = 1, \cdots, n+1$, consider

$$\sum_{i=1}^{n+1} \alpha_i \frac{\mathbf{g} - \mathbf{w}_i}{s - \lambda_i} = 0, \;\; \alpha_i \neq 0.$$

- Solving for $\mathbf{g}$ we obtain

$$\boxed{\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\alpha_i \mathbf{w}_i}{s-\lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s-\lambda_i}}} \quad \Rightarrow \quad \mathbf{g}(\lambda_i) = \mathbf{w}_i.$$

This is the **barycentric Lagrange interpolation** formula.

Reference: J.-P. Berrut and L. N. Trefethen, Barycentric Lagrange interpolation, SIAM Review, vol. 46, pp. 501-517, 2004.

Rational interpolation and the Loewner matrix

- Lagrange basis for space of polynomials of degree at most n.

Given $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n+1$: $\lambda_i \neq \lambda_j$, $i \neq j$, define

$$\mathbf{q}_i(s) := \Pi_{i' \neq i} (s - \lambda_{i'}),\ i = 1, \cdots, n+1.$$

- For given constants α_i, $\mathbf{w}_i$, $i = 1, \cdots, n+1$, consider

$$\sum_{i=1}^{n+1} \alpha_i \frac{\mathbf{g} - \mathbf{w}_i}{s - \lambda_i} = 0,\ \ \alpha_i \neq 0.$$

- Solving for $\mathbf{g}$ we obtain

$$\boxed{\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\alpha_i \mathbf{w}_i}{s - \lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s - \lambda_i}}} \qquad \Rightarrow \qquad \mathbf{g}(\lambda_i) = \mathbf{w}_i.$$

This is the **barycentric Lagrange interpolation** formula.

Reference: J.-P. Berrut and L. N. Trefethen, Barycentric Lagrange interpolation, SIAM Review, vol. 46, pp. 501-517, 2004.

- Lagrange basis for space of polynomials of degree at most n.

Given $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n+1$: $\lambda_i \neq \lambda_j$, $i \neq j$, define

$$\mathbf{q}_i(s) := \Pi_{i' \neq i} (s - \lambda_{i'}), \; i = 1, \cdots, n+1.$$

- For given constants α_i, $\mathbf{w}_i$, $i = 1, \cdots, n+1$, consider

$$\sum_{i=1}^{n+1} \alpha_i \frac{\mathbf{g} - \mathbf{w}_i}{s - \lambda_i} = 0, \;\; \alpha_i \neq 0.$$

- Solving for $\mathbf{g}$ we obtain

$$\boxed{\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\alpha_i \mathbf{w}_i}{s - \lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s - \lambda_i}}} \qquad \Rightarrow \qquad \mathbf{g}(\lambda_i) = \mathbf{w}_i.$$

This is the **barycentric Lagrange interpolation** formula.

Reference: J.-P. Berrut and L. N. Trefethen, Barycentric Lagrange interpolation, SIAM Review, vol. 46, pp. 501-517, 2004.

The free parameters α_i, can be determined so that additional constraints are satisfied:

$$\mathbf{g}(\mu_j) = \mathbf{v}_j,\ j = 1, \cdots, r,$$

for given $(\mu_j, \mathbf{v}_j)$, $\mu_i \neq \mu_j$. For this to hold $\mathbb{L}\mathbf{c} = 0$, where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_{n+1}}{\mu_1 - \lambda_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_r - \mathbf{w}_1}{\mu_r - \lambda_1} & \cdots & \frac{\mathbf{v}_r - \mathbf{w}_{n+1}}{\mu_r - \lambda_{n+1}} \end{bmatrix} \in \mathbb{C}^{r \times (n+1)}, \quad \mathbf{c} = \begin{bmatrix} \alpha_1, \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \in \mathbb{C}^{n+1}.$$

Reference on function approximation and ChebFun in this framework:
Y. Nakatsukasa, O. Sete, and L.N. Trefethen, The AAA algorithm for rational interpolation.

The free parameters α_i, can be determined so that additional constraints are satisfied:

$$\mathbf{g}(\mu_j) = \mathbf{v}_j,\ j = 1, \cdots, r,$$

for given $(\mu_j, \mathbf{v}_j)$, $\mu_i \neq \mu_j$. For this to hold $\mathbb{L}\mathbf{c} = 0$, where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_{n+1}}{\mu_1 - \lambda_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_r - \mathbf{w}_1}{\mu_r - \lambda_1} & \cdots & \frac{\mathbf{v}_r - \mathbf{w}_{n+1}}{\mu_r - \lambda_{n+1}} \end{bmatrix} \in \mathbb{C}^{r \times (n+1)}, \quad \mathbf{c} = \begin{bmatrix} \alpha_1, \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \in \mathbb{C}^{n+1}.$$

Reference on function approximation and ChebFun in this framework:

Y. Nakatsukasa, O. Sete, and L.N. Trefethen, The AAA algorithm for rational interpolation.

The free parameters α_i, can be determined so that additional constraints are satisfied:

$$\mathbf{g}(\mu_j) = \mathbf{v}_j,\ j = 1, \cdots, r,$$

for given $(\mu_j, \mathbf{v}_j)$, $\mu_i \neq \mu_j$. For this to hold $\mathbb{L}\mathbf{c} = 0$, where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_{n+1}}{\mu_1 - \lambda_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_r - \mathbf{w}_1}{\mu_r - \lambda_1} & \cdots & \frac{\mathbf{v}_r - \mathbf{w}_{n+1}}{\mu_r - \lambda_{n+1}} \end{bmatrix} \in \mathbb{C}^{r\times(n+1)}, \quad \mathbf{c} = \begin{bmatrix} \alpha_1, \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \in \mathbb{C}^{n+1}.$$

Reference on function approximation and ChebFun in this framework:

Y. Nakatsukasa, O. Sete, and L.N. Trefethen, The AAA algorithm for rational interpolation.

Descriptor representation of interpolants and rational approximants

Given: right data: $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i)$, $i = 1, \cdots, k$, and left data: $(\mu_j; \boldsymbol{\ell}_j^*, \mathbf{v}_j^*)$, $j = 1, \cdots, q$.

Problem: Find rational $p \times m$ matrices $\mathbf{H}(s)$, such that:

$$\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \quad \boldsymbol{\ell}_j^* \mathbf{H}(\mu_j) = \mathbf{v}_j^*,$$

where $\mathbf{H}(\lambda_i)$, $\mathbf{H}(\mu_j) \in \mathbb{C}^{p \times m}$, are for instance, S-parameters.

Right data:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad \mathbf{R} = [\mathbf{r}_1 \;\; \mathbf{r}_2, \;\; \cdots \;\; \mathbf{r}_k] \in \mathbb{C}^{m \times k}, \quad \mathbf{W} = [\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \cdots \;\; \mathbf{w}_k] \in \mathbb{C}^{p \times k},$$

Left data:

$$M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix} \in \mathbb{C}^{q \times q}, \quad \mathbf{L} = \begin{bmatrix} \boldsymbol{\ell}_1^* \\ \vdots \\ \boldsymbol{\ell}_q^* \end{bmatrix} \in \mathbb{C}^{q \times p}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_q^* \end{bmatrix} \in \mathbb{C}^{q \times m}$$

▲ A.J. Mayo and A.C. Antoulas, *A framework for the solution of the generalized realization problem*, Linear Algebra and Its Applications, vol. 425, pages 634-662 (2007).

Descriptor representation of interpolants and rational approximants

Given: right data: $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i)$, $i = 1, \cdots, k$, and left data: $(\mu_j; \boldsymbol{\ell}_j^*, \mathbf{v}_j^*)$, $j = 1, \cdots, q$.

Problem: Find rational $p \times m$ matrices $\mathbf{H}(s)$, such that:

$$\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \quad \boldsymbol{\ell}_j^* \mathbf{H}(\mu_j) = \mathbf{v}_j^*,$$

where $\mathbf{H}(\lambda_i)$, $\mathbf{H}(\mu_j) \in \mathbb{C}^{p \times m}$, are for instance, S-parameters.

Right data:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad \begin{array}{l} \mathbf{R} = [\mathbf{r}_1 \;\; \mathbf{r}_2, \;\; \cdots \;\; \mathbf{r}_k] \in \mathbb{C}^{m \times k}, \\ \mathbf{W} = [\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \cdots \;\; \mathbf{w}_k] \in \mathbb{C}^{p \times k}, \end{array}$$

Left data:

$$M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix} \in \mathbb{C}^{q \times q}, \quad \mathbf{L} = \begin{bmatrix} \boldsymbol{\ell}_1^* \\ \vdots \\ \boldsymbol{\ell}_q^* \end{bmatrix} \in \mathbb{C}^{q \times p}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_q^* \end{bmatrix} \in \mathbb{C}^{q \times m}$$

▲ A.J. Mayo and A.C. Antoulas, *A framework for the solution of the generalized realization problem*, Linear Algebra and Its Applications, vol. 425, pages 634-662 (2007).

Descriptor representation of interpolants and rational approximants

Given: right data: $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i)$, $i = 1, \cdots, k$, and left data: $(\mu_j; \boldsymbol{\ell}_j^*, \mathbf{v}_j^*)$, $j = 1, \cdots, q$.

Problem: Find rational $p \times m$ matrices $\mathbf{H}(s)$, such that:

$$\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \quad \boldsymbol{\ell}_j^* \mathbf{H}(\mu_j) = \mathbf{v}_j^*,$$

where $\mathbf{H}(\lambda_i)$, $\mathbf{H}(\mu_j) \in \mathbb{C}^{p \times m}$, are for instance, S-parameters.

Right data:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad \mathbf{R} = [\mathbf{r}_1 \;\; \mathbf{r}_2, \;\; \cdots \;\; \mathbf{r}_k] \in \mathbb{C}^{m \times k}, \quad \mathbf{W} = [\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \cdots \;\; \mathbf{w}_k] \in \mathbb{C}^{p \times k},$$

Left data:

$$M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix} \in \mathbb{C}^{q \times q}, \quad \mathbf{L} = \begin{bmatrix} \boldsymbol{\ell}_1^* \\ \vdots \\ \boldsymbol{\ell}_q^* \end{bmatrix} \in \mathbb{C}^{q \times p}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_q^* \end{bmatrix} \in \mathbb{C}^{q \times m}$$

▲ A.J. Mayo and A.C. Antoulas, *A framework for the solution of the generalized realization problem*, Linear Algebra and Its Applications, vol. 425, pages 634-662 (2007).

Descriptor representation of interpolants and rational approximants

Given: right data: $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i)$, $i = 1, \cdots, k$, and left data: $(\mu_j; \boldsymbol{\ell}_j^*, \mathbf{v}_j^*)$, $j = 1, \cdots, q$.

Problem: Find rational $p \times m$ matrices $\mathbf{H}(s)$, such that:

$$\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \quad \boldsymbol{\ell}_j^* \mathbf{H}(\mu_j) = \mathbf{v}_j^*,$$

where $\mathbf{H}(\lambda_i)$, $\mathbf{H}(\mu_j) \in \mathbb{C}^{p \times m}$, are for instance, S-parameters.

Right data:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad \begin{array}{l} \mathbf{R} = [\mathbf{r}_1 \;\; \mathbf{r}_2, \;\; \cdots \;\; \mathbf{r}_k] \in \mathbb{C}^{m \times k}, \\ \mathbf{W} = [\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \cdots \;\; \mathbf{w}_k] \in \mathbb{C}^{p \times k}, \end{array}$$

Left data:

$$M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix} \in \mathbb{C}^{q \times q}, \;\; \mathbf{L} = \begin{bmatrix} \boldsymbol{\ell}_1^* \\ \vdots \\ \boldsymbol{\ell}_q^* \end{bmatrix} \in \mathbb{C}^{q \times p}, \;\; \mathbf{V} = \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_q^* \end{bmatrix} \in \mathbb{C}^{q \times m}$$

▲ A.J. Mayo and A.C. Antoulas, *A framework for the solution of the generalized realization problem*, Linear Algebra and Its Applications, vol. 425, pages 634-662 (2007).

Descriptor representation: the Loewner pencil

Data: $\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i$, $\ell_j\mathbf{H}(\mu_j) = \mathbf{v}_j$.

The **Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1}{\mu_q-\lambda_1} & \cdots & \frac{\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k}{\mu_q-\lambda_k} \end{bmatrix}$$

$\mathbb{L}$ satisfies the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = \mathbf{VR} - \mathbf{LW}$$

The **shifted Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L}_\sigma = \begin{bmatrix} \frac{\mu_1\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1\lambda_1}{\mu_1-\lambda_1} & \cdots & \frac{\mu_1\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k\lambda_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1\lambda_1}{\mu_q-\lambda_1} & \cdots & \frac{\mu_q\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k\lambda_k}{\mu_q-\lambda_k} \end{bmatrix}$$

$\mathbb{L}_\sigma$ satisfies the Sylvester equation

$$\mathbb{L}_\sigma\Lambda - M\mathbb{L}_\sigma = M\mathbf{VR} - \mathbf{LW}\Lambda$$

▲ A.C. Antoulas, S. Lefteriu, and A.C Ionita, A tutorial introduction to the Loewner framework for model reduction, in *Model Reduction and Approximation for Complex Systems*, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia (2017).

Descriptor representation: the Loewner pencil

Data: $\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i$, $\ell_j\mathbf{H}(\mu_j) = \mathbf{v}_j$.

The **Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}$ satisfies the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = \mathbf{VR} - \mathbf{LW}$$

The **shifted Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L}_\sigma = \begin{bmatrix} \frac{\mu_1\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1\lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k\lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1\lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k\lambda_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}_\sigma$ satisfies the Sylvester equation

$$\mathbb{L}_\sigma\Lambda - M\mathbb{L}_\sigma = M\mathbf{VR} - \mathbf{LW}\Lambda$$

▲ A.C. Antoulas, S. Lefteriu, and A.C Ionita, A tutorial introduction to the Loewner framework for model reduction, in *Model Reduction and Approximation for Complex Systems*, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia (2017).

Data: $\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i$, $\ell_j\mathbf{H}(\mu_j) = \mathbf{v}_j$.

The **Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}$ satisfies the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$$

The **shifted Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L}_\sigma = \begin{bmatrix} \frac{\mu_1\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1\lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k\lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1\lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k\lambda_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}_\sigma$ satisfies the Sylvester equation

$$\mathbb{L}_\sigma\Lambda - M\mathbb{L}_\sigma = M\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\Lambda$$

▲ A.C. Antoulas, S. Lefteriu, and A.C Ionita, A tutorial introduction to the Loewner framework for model reduction, in *Model Reduction and Approximation for Complex Systems*, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia (2017).

Descriptor representation: the Loewner pencil

Data: $\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i$, $\ell_j\mathbf{H}(\mu_j) = \mathbf{v}_j$.

The **Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1\mathbf{r}_1-\ell_1\mathbf{w}_1}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{v}_1\mathbf{r}_k-\ell_1\mathbf{w}_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q\mathbf{r}_1-\ell_q\mathbf{w}_1}{\mu_q-\lambda_1} & \cdots & \frac{\mathbf{v}_q\mathbf{r}_k-\ell_q\mathbf{w}_k}{\mu_q-\lambda_k} \end{bmatrix}$$

$\mathbb{L}$ satisfies the Sylvester equation

$$\boxed{\mathbb{L}\Lambda - M\mathbb{L} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}}$$

The **shifted Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L}_\sigma = \begin{bmatrix} \frac{\mu_1\mathbf{v}_1\mathbf{r}_1-\ell_1\mathbf{w}_1\lambda_1}{\mu_1-\lambda_1} & \cdots & \frac{\mu_1\mathbf{v}_1\mathbf{r}_k-\ell_1\mathbf{w}_k\lambda_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{v}_q\mathbf{r}_1-\ell_q\mathbf{w}_1\lambda_1}{\mu_q-\lambda_1} & \cdots & \frac{\mu_q\mathbf{v}_q\mathbf{r}_k-\ell_q\mathbf{w}_k\lambda_k}{\mu_q-\lambda_k} \end{bmatrix}$$

$\mathbb{L}_\sigma$ satisfies the Sylvester equation

$$\boxed{\mathbb{L}_\sigma\Lambda - M\mathbb{L}_\sigma = M\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\Lambda}$$

▲ A.C. Antoulas, S. Lefteriu, and A.C Ionita, A tutorial introduction to the Loewner framework for model reduction, in *Model Reduction and Approximation for Complex Systems*, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia (2017).

- If the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ is regular, then

$$\mathbf{E} = -\mathbb{L}, \;\; \mathbf{A} = -\mathbb{L}_\sigma, \;\; \mathbf{B} = \mathbf{V}, \;\; \mathbf{C} = \mathbf{W}$$

is a minimal interpolant of the data, i.e., $\mathbf{H}(s)$ interpolates the data:

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V}$$

- Otherwise, if the **numerical** $\operatorname{rank} \mathbb{L} = k$, compute the rank revealing *SVD*:

$$\mathbb{L} = \mathbf{Y}\Sigma\mathbf{X}^* \approx \mathbf{Y}_k\Sigma_k\mathbf{X}_k^*$$

Theorem. A realization $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}]$, of an approximate interpolant is given as follows:

$$\mathbf{E} = -\mathbf{Y}_k^*\mathbb{L}\mathbf{X}_k, \quad \mathbf{A} = -\mathbf{Y}_k^*\mathbb{L}_\sigma\mathbf{X}_k, \quad \mathbf{B} = \mathbf{Y}_k^*\mathbf{V}, \quad \mathbf{C} = \mathbf{W}\mathbf{X}_k.$$

Remark. ▲ If we have more data than necessary, we can consider $(\mathbb{L}_\sigma, \;\mathbb{L}, \;\mathbf{V}, \;\mathbf{W})$, as a singular model of the data. **Consequence**: The original pencil $(\mathbb{L}_\sigma, \mathbb{L})$ and the projected pencil $(\mathbf{A}, \mathbf{E})$, have the same non-trivial eigenvalues.

▲ A.C. Antoulas, *The Loewner framework and transfer functions of singular/rectangular systems*, Applied Mathematics Letters, vol 54, pages 36-47, 2016.

Construction of Interpolants (Models)

- If the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ is regular, then

$$\mathbf{E} = -\mathbb{L}, \quad \mathbf{A} = -\mathbb{L}_\sigma, \quad \mathbf{B} = \mathbf{V}, \quad \mathbf{C} = \mathbf{W}$$

is a minimal interpolant of the data, i.e., $\mathbf{H}(s)$ interpolates the data:

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V}$$

- Otherwise, if the **numerical** $\mathrm{rank}\,\mathbb{L} = k$, compute the rank revealing *SVD*:

$$\mathbb{L} = \mathbf{Y}\Sigma\mathbf{X}^* \approx \mathbf{Y}_k\Sigma_k\mathbf{X}_k^*$$

Theorem. A realization [**E**, **A**, **B**, **C**], of an approximate interpolant is given as follows:

$$\mathbf{E} = -\mathbf{Y}_k^*\mathbb{L}\mathbf{X}_k, \quad \mathbf{A} = -\mathbf{Y}_k^*\mathbb{L}_\sigma\mathbf{X}_k, \quad \mathbf{B} = \mathbf{Y}_k^*\mathbf{V}, \quad \mathbf{C} = \mathbf{W}\mathbf{X}_k.$$

Remark. ▲ If we have more data than necessary, we can consider $(\mathbb{L}_\sigma,\ \mathbb{L},\ \mathbf{V},\ \mathbf{W})$, as a singular model of the data. **Consequence**: The original pencil $(\mathbb{L}_\sigma, \mathbb{L})$ and the projected pencil $(\mathbf{A}, \mathbf{E})$, have the same non-trivial eigenvalues.

▲ A.C. Antoulas, *The Loewner framework and transfer functions of singular/rectangular systems*, Applied Mathematics Letters, vol 54, pages 36-47, 2016.

- If the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ is regular, then

$$\mathbf{E} = -\mathbb{L}, \quad \mathbf{A} = -\mathbb{L}_\sigma, \quad \mathbf{B} = \mathbf{V}, \quad \mathbf{C} = \mathbf{W}$$

is a minimal interpolant of the data, i.e., $\mathbf{H}(s)$ interpolates the data:

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V}$$

- Otherwise, if the **numerical** $\operatorname{rank} \mathbb{L} = k$, compute the rank revealing *SVD*:

$$\mathbb{L} = \mathbf{Y}\Sigma\mathbf{X}^* \approx \mathbf{Y}_k\Sigma_k\mathbf{X}_k^*$$

Theorem. A realization [**E**, **A**, **B**, **C**], of an approximate interpolant is given as follows:

$$\mathbf{E} = -\mathbf{Y}_k^*\mathbb{L}\mathbf{X}_k, \quad \mathbf{A} = -\mathbf{Y}_k^*\mathbb{L}_\sigma\mathbf{X}_k, \quad \mathbf{B} = \mathbf{Y}_k^*\mathbf{V}, \quad \mathbf{C} = \mathbf{W}\mathbf{X}_k.$$

Remark. ▲ If we have more data than necessary, we can consider $(\mathbb{L}_\sigma, \mathbb{L}, \mathbf{V}, \mathbf{W})$, as a singular model of the data. **Consequence**: The original pencil $(\mathbb{L}_\sigma, \mathbb{L})$ and the projected pencil $(\mathbf{A}, \mathbf{E})$, have the same non-trivial eigenvalues.

▲ A.C. Antoulas, *The Loewner framework and transfer functions of singular/rectangular systems*, Applied Mathematics Letters, vol 54, pages 36-47, 2016.

Proof: the factorization of $\mathbb{L}$, $\mathbb{L}_\sigma$, $\mathbf{V}$, $\mathbf{W}$

▷ Recall the Loewner pencil:

$$(\mathbb{L})_{i,j} = \left[\frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}\right], \quad (\mathbb{L}_\sigma)_{i,j} = \left[\frac{\mu_i \mathbf{v}_i - \lambda_j \mathbf{w}_j}{\mu_i - \lambda_j}\right] \in \mathbb{C}^{q \times k}.$$

▷ Define: $\mathcal{R}$: generalized controllability matrix, $\mathcal{O}$: generalized observability matrix

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{E} \underbrace{\begin{bmatrix} (\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}} = -\mathbb{L} \qquad \text{and}$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{A} \underbrace{\begin{bmatrix} (\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}} = -\mathbb{L}_\sigma.$$

Also $\mathbf{V} = \mathbf{C}\mathcal{R}$, $\mathbf{W} = \mathcal{O}\mathbf{B}$.

Proof: the factorization of $\mathbb{L}$, $\mathbb{L}_\sigma$, $\mathbf{V}$, $\mathbf{W}$

▷ Recall the Loewner pencil:

$$(\mathbb{L})_{i,j} = \left[\frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}\right], \quad (\mathbb{L}_\sigma)_{i,j} = \left[\frac{\mu_i \mathbf{v}_i - \lambda_j \mathbf{w}_j}{\mu_i - \lambda_j}\right] \in \mathbb{C}^{q \times k}.$$

▷ Define: $\mathcal{R}$: generalized controllability matrix, $\mathcal{O}$: generalized observability matrix

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{E} \underbrace{\left[\, (\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \quad \cdots \quad (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \,\right]}_{\mathcal{R}} = -\mathbb{L} \qquad \text{and}$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{A} \underbrace{\left[\, (\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \quad \cdots \quad (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \,\right]}_{\mathcal{R}} = -\mathbb{L}_\sigma.$$

Also $\mathbf{V} = \mathbf{C}\mathcal{R}$, $\mathbf{W} = \mathcal{O}\mathbf{B}$.

Proof: the factorization of $\mathbb{L}$, $\mathbb{L}_\sigma$, **V**, **W**

▷ Recall the Loewner pencil:

$$(\mathbb{L})_{i,j} = \left[\frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j} \right], \quad (\mathbb{L}_\sigma)_{i,j} = \left[\frac{\mu_i \mathbf{v}_i - \lambda_j \mathbf{w}_j}{\mu_i - \lambda_j} \right] \in \mathbb{C}^{q \times k}.$$

▷ Define: $\mathcal{R}$: generalized controllability matrix, $\mathcal{O}$: generalized observability matrix

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{E} \underbrace{\begin{bmatrix} (\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}} = -\mathbb{L} \qquad \text{and}$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{A} \underbrace{\begin{bmatrix} (\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}} = -\mathbb{L}_\sigma .$$

Also $\mathbf{V} = \mathbf{C}\mathcal{R}$, $\mathbf{W} = \mathcal{O}\mathbf{B}$.

Proof: the factorization of $\mathbb{L}$, $\mathbb{L}_\sigma$, $\mathbf{V}$, $\mathbf{W}$

▷ Recall the Loewner pencil:

$$(\mathbb{L})_{i,j} = \left[\frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}\right], \quad (\mathbb{L}_\sigma)_{i,j} = \left[\frac{\mu_i \mathbf{v}_i - \lambda_j \mathbf{w}_j}{\mu_i - \lambda_j}\right] \in \mathbb{C}^{q \times k}.$$

▷ Define: $\mathcal{R}$: generalized controllability matrix, $\mathcal{O}$: generalized observability matrix

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1\mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q\mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{E} \underbrace{\begin{bmatrix} (\lambda_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}} = -\mathbb{L} \qquad \text{and}$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1\mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q\mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{A} \underbrace{\begin{bmatrix} (\lambda_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}} = -\mathbb{L}_\sigma.$$

Also $\mathbf{V} = \mathbf{C}\mathcal{R}$, $\mathbf{W} = \mathcal{O}\mathbf{B}$.

A simple example

Consider the system described by the transfer function

$$\mathbf{H}(s) = \frac{s}{s^2 + s + 1}.$$

The data are obtained by evaluating **H** at the *right* frequencies $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 1$, and at the *left* frequencies $\mu_1 - \frac{1}{2}$, $\mu_2 = -1$. The corresponding values are

$$\mathbf{W} = \begin{pmatrix} \frac{2}{7} & \frac{1}{3} \end{pmatrix}, \ \mathbf{V} = \begin{pmatrix} -\frac{2}{3} & -1 \end{pmatrix}^T.$$

With $\mathbf{R} = [1 \ \ 1] = \mathbf{L}^T$, we construct the Loewner pencil $(\mathbb{L}_\sigma, \mathbb{L})$, where:

$$\mathbb{L} = \begin{bmatrix} \frac{20}{21} & \frac{2}{3} \\ \frac{6}{7} & \frac{2}{3} \end{bmatrix}, \ \ \mathbb{L}_\sigma = \begin{bmatrix} -\frac{4}{21} & 0 \\ -\frac{4}{7} & -\frac{1}{3} \end{bmatrix}.$$

It follows that since the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ is regular, there holds

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V} = \frac{s}{s^2 + s + 1}.$$

Hence, the measurements above yield a minimal descriptor realization of the system with transfer function $\mathbf{H}(s)$.

A simple example

Consider the system described by the transfer function

$$\mathbf{H}(s) = \frac{s}{s^2 + s + 1}.$$

The data are obtained by evaluating **H** at the *right* frequencies $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 1$, and at the *left* frequencies $\mu_1 - \frac{1}{2}$, $\mu_2 = -1$. The corresponding values are

$$\mathbf{W} = \begin{pmatrix} \frac{2}{7} & \frac{1}{3} \end{pmatrix}, \ \mathbf{V} = \begin{pmatrix} -\frac{2}{3} & -1 \end{pmatrix}^T.$$

With $\mathbf{R} = [1 \ \ 1] = \mathbf{L}^T$, we construct the Loewner pencil $(\mathbb{L}_\sigma, \mathbb{L})$, where:

$$\mathbb{L} = \begin{bmatrix} \frac{20}{21} & \frac{2}{3} \\ \frac{6}{7} & \frac{2}{3} \end{bmatrix}, \ \ \mathbb{L}_\sigma = \begin{bmatrix} -\frac{4}{21} & 0 \\ -\frac{4}{7} & -\frac{1}{3} \end{bmatrix}.$$

It follows that since the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ is regular, there holds

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V} = \frac{s}{s^2 + s + 1}.$$

Hence, the measurements above yield a minimal descriptor realization of the system with transfer function $\mathbf{H}(s)$.

A simple example

Consider the system described by the transfer function

$$\mathbf{H}(s) = \frac{s}{s^2 + s + 1}.$$

The data are obtained by evaluating **H** at the *right* frequencies $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 1$, and at the *left* frequencies $\mu_1 - \frac{1}{2}$, $\mu_2 = -1$. The corresponding values are

$$\mathbf{W} = \begin{pmatrix} \frac{2}{7} & \frac{1}{3} \end{pmatrix}, \ \mathbf{V} = \begin{pmatrix} -\frac{2}{3} & -1 \end{pmatrix}^T.$$

With $\mathbf{R} = [1 \ \ 1] = \mathbf{L}^T$, we construct the Loewner pencil $(\mathbb{L}_\sigma, \mathbb{L})$, where:

$$\mathbb{L} = \begin{bmatrix} \frac{20}{21} & \frac{2}{3} \\ \frac{6}{7} & \frac{2}{3} \end{bmatrix}, \ \mathbb{L}_\sigma = \begin{bmatrix} -\frac{4}{21} & 0 \\ -\frac{4}{7} & -\frac{1}{3} \end{bmatrix}.$$

It follows that since the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ is regular, there holds

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V} = \frac{s}{s^2 + s + 1}.$$

Hence, the measurements above yield a minimal descriptor realization of the system with transfer function $\mathbf{H}(s)$.

The question now is, what happens if we collect **more** data that necessary:

$$\mathbf{\Lambda} = \mathrm{diag}\left(\begin{array}{cccc} \frac{1}{2} & 1 & \frac{3}{2} & 2 \end{array}\right), \quad \mathbf{M} = \mathrm{diag}\left(\begin{array}{cccc} -\frac{1}{2} & -1 & -\frac{3}{2} & -2 \end{array}\right).$$

In this case, the measurements are

$$\mathbf{W} = \left(\begin{array}{cccc} \frac{2}{7} & \frac{1}{3} & \frac{6}{19} & \frac{2}{7} \end{array}\right), \mathbf{V} = \left(\begin{array}{cccc} -\frac{2}{3} & -1 & -\frac{6}{7} & -\frac{2}{3} \end{array}\right)^T,$$

and with $\mathbf{R} = [1\ \ 1\ \ 1\ \ 1] = \mathbf{L}^T$, we obtain the Loewner pencil

$$\mathbb{L} = \begin{bmatrix} \frac{20}{21} & \frac{2}{3} & \frac{28}{57} & \frac{8}{21} \\ \frac{6}{7} & \frac{2}{3} & \frac{10}{19} & \frac{3}{7} \\ \frac{4}{7} & \frac{10}{21} & \frac{52}{133} & \frac{16}{49} \\ \frac{8}{21} & \frac{1}{3} & \frac{16}{57} & \frac{5}{21} \end{bmatrix}, \quad \mathbb{L}_\sigma = \begin{bmatrix} -\frac{4}{21} & 0 & \frac{4}{57} & \frac{2}{21} \\ -\frac{4}{7} & -\frac{1}{3} & -\frac{4}{19} & -\frac{1}{7} \\ -\frac{4}{7} & -\frac{8}{21} & -\frac{36}{133} & -\frac{10}{49} \\ -\frac{10}{21} & -\frac{1}{3} & -\frac{14}{57} & -\frac{4}{21} \end{bmatrix}.$$

Here the rank of $\mathbb{L}$ is 2, and we can choose **arbitrary** $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{4\times 2}$, such that $\mathbf{Y}^T\mathbf{X}$ is nonsingular:

$$\mathbf{X} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{Y}^T = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The question now is, what happens if we collect **more** data that necessary:

$$\mathbf{\Lambda} = \text{diag}\begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} & 2 \end{pmatrix}, \quad \mathbf{M} = \text{diag}\begin{pmatrix} -\frac{1}{2} & -1 & -\frac{3}{2} & -2 \end{pmatrix}.$$

In this case, the measurements are

$$\mathbf{W} = \begin{pmatrix} \frac{2}{7} & \frac{1}{3} & \frac{6}{19} & \frac{2}{7} \end{pmatrix}, \mathbf{V} = \begin{pmatrix} -\frac{2}{3} & -1 & -\frac{6}{7} & -\frac{2}{3} \end{pmatrix}^T,$$

and with $\mathbf{R} = [1 \ \ 1 \ \ 1 \ \ 1] = \mathbf{L}^T$, we obtain the Loewner pencil

$$\mathbb{L} = \begin{bmatrix} \frac{20}{21} & \frac{2}{3} & \frac{28}{57} & \frac{8}{21} \\ \frac{6}{7} & \frac{2}{3} & \frac{10}{19} & \frac{3}{7} \\ \frac{4}{7} & \frac{10}{21} & \frac{52}{133} & \frac{16}{49} \\ \frac{8}{21} & \frac{1}{3} & \frac{16}{57} & \frac{5}{21} \end{bmatrix}, \quad \mathbb{L}_\sigma = \begin{bmatrix} -\frac{4}{21} & 0 & \frac{4}{57} & \frac{2}{21} \\ -\frac{4}{7} & -\frac{1}{3} & -\frac{4}{19} & -\frac{1}{7} \\ -\frac{4}{7} & -\frac{8}{21} & -\frac{36}{133} & -\frac{10}{49} \\ -\frac{10}{21} & -\frac{1}{3} & -\frac{14}{57} & -\frac{4}{21} \end{bmatrix}.$$

Here the rank of $\mathbb{L}$ is 2, and we can choose **arbitrary** $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{4\times 2}$, such that $\mathbf{Y}^T\mathbf{X}$ is nonsingular:

$$\mathbf{X} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{Y}^T = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The question now is, what happens if we collect **more** data that necessary:

$$\boldsymbol{\Lambda} = \text{diag}\left(\begin{array}{cccc} \frac{1}{2} & 1 & \frac{3}{2} & 2 \end{array}\right), \quad \mathbf{M} = \text{diag}\left(\begin{array}{cccc} -\frac{1}{2} & -1 & -\frac{3}{2} & -2 \end{array}\right).$$

In this case, the measurements are

$$\mathbf{W} = \left(\begin{array}{cccc} \frac{2}{7} & \frac{1}{3} & \frac{6}{19} & \frac{2}{7} \end{array}\right), \ \mathbf{V} = \left(\begin{array}{cccc} -\frac{2}{3} & -1 & -\frac{6}{7} & -\frac{2}{3} \end{array}\right)^T,$$

and with $\mathbf{R} = [1 \;\; 1 \;\; 1 \;\; 1] = \mathbf{L}^T$, we obtain the Loewner pencil

$$\mathbb{L} = \begin{bmatrix} \frac{20}{21} & \frac{2}{3} & \frac{28}{57} & \frac{8}{21} \\ \frac{6}{7} & \frac{2}{3} & \frac{10}{19} & \frac{3}{7} \\ \frac{4}{7} & \frac{10}{21} & \frac{52}{133} & \frac{16}{49} \\ \frac{8}{21} & \frac{1}{3} & \frac{16}{57} & \frac{5}{21} \end{bmatrix}, \quad \mathbb{L}_\sigma = \begin{bmatrix} -\frac{4}{21} & 0 & \frac{4}{57} & \frac{2}{21} \\ -\frac{4}{7} & -\frac{1}{3} & -\frac{4}{19} & -\frac{1}{7} \\ -\frac{4}{7} & -\frac{8}{21} & -\frac{36}{133} & -\frac{10}{49} \\ -\frac{10}{21} & -\frac{1}{3} & -\frac{14}{57} & -\frac{4}{21} \end{bmatrix}.$$

Here the rank of $\mathbb{L}$ is 2, and we can choose **arbitrary** $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{4\times 2}$, such that $\mathbf{Y}^T\mathbf{X}$ is nonsingular:

$$\mathbf{X} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{Y}^T = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The question now is, what happens if we collect **more** data that necessary:

$$\boldsymbol{\Lambda} = \text{diag}\left(\begin{array}{cccc} \frac{1}{2} & 1 & \frac{3}{2} & 2 \end{array}\right), \quad \mathbf{M} = \text{diag}\left(\begin{array}{cccc} -\frac{1}{2} & -1 & -\frac{3}{2} & -2 \end{array}\right).$$

In this case, the measurements are

$$\mathbf{W} = \left(\begin{array}{cccc} \frac{2}{7} & \frac{1}{3} & \frac{6}{19} & \frac{2}{7} \end{array}\right), \; \mathbf{V} = \left(\begin{array}{cccc} -\frac{2}{3} & -1 & -\frac{6}{7} & -\frac{2}{3} \end{array}\right)^T,$$

and with $\mathbf{R} = [1 \;\; 1 \;\; 1 \;\; 1] = \mathbf{L}^T$, we obtain the Loewner pencil

$$\mathbb{L} = \begin{bmatrix} \frac{20}{21} & \frac{2}{3} & \frac{28}{57} & \frac{8}{21} \\ \frac{6}{7} & \frac{2}{3} & \frac{10}{19} & \frac{3}{7} \\ \frac{4}{7} & \frac{10}{21} & \frac{52}{133} & \frac{16}{49} \\ \frac{8}{21} & \frac{1}{3} & \frac{16}{57} & \frac{5}{21} \end{bmatrix}, \quad \mathbb{L}_\sigma = \begin{bmatrix} -\frac{4}{21} & 0 & \frac{4}{57} & \frac{2}{21} \\ -\frac{4}{7} & -\frac{1}{3} & -\frac{4}{19} & -\frac{1}{7} \\ -\frac{4}{7} & -\frac{8}{21} & -\frac{36}{133} & -\frac{10}{49} \\ -\frac{10}{21} & -\frac{1}{3} & -\frac{14}{57} & -\frac{4}{21} \end{bmatrix}.$$

Here the rank of $\mathbb{L}$ is 2, and we can choose **arbitrary** $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{4\times 2}$, such that $\mathbf{Y}^T\mathbf{X}$ is nonsingular:

$$\mathbf{X} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{Y}^T = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The projected quantities are

$$\widehat{\mathbf{W}} = \mathbf{W}\mathbf{X} = \left[\ -\frac{6}{7} \quad -\frac{1}{21}\ \right], \ \ \widehat{\mathbf{V}} = \mathbf{Y}^T\mathbf{V} = \begin{bmatrix} -\frac{1}{3} \\ \frac{11}{21} \end{bmatrix},$$

$$\widehat{\mathbb{L}} = \mathbf{Y}^T\mathbb{L}\mathbf{X} = \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} \\ \frac{18}{49} & \frac{1}{147} \end{bmatrix}, \ \ \widehat{\mathbb{L}}_\sigma = \mathbf{Y}^T\mathbb{L}_\sigma\mathbf{X} = \begin{bmatrix} 0 & \frac{1}{21} \\ -\frac{48}{49} & -\frac{19}{147} \end{bmatrix},$$

constitute a minimal realization of $\mathbf{H}(s)$, in other words,

$$\mathbf{H}(s) = \widehat{\mathbf{W}}\left(\widehat{\mathbb{L}}_\sigma - s\,\widehat{\mathbb{L}}\right)^{-1}\widehat{\mathbf{V}} = \frac{s}{s^2+s+1},$$

A.C. Antoulas, *The Loewner framework and transfer functions of singular/rectangular systems*, Applied Mathematics Letters, 54: 36-47.

The projected quantities are

$$\widehat{\mathbf{W}} = \mathbf{W}\mathbf{X} = \left[\ -\tfrac{6}{7} \quad -\tfrac{1}{21}\ \right], \quad \widehat{\mathbf{V}} = \mathbf{Y}^T\mathbf{V} = \begin{bmatrix} -\frac{1}{3} \\ \frac{11}{21} \end{bmatrix},$$

$$\widehat{\mathbb{L}} = \mathbf{Y}^T\mathbb{L}\mathbf{X} = \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} \\ \frac{18}{49} & \frac{1}{147} \end{bmatrix}, \quad \widehat{\mathbb{L}}_\sigma = \mathbf{Y}^T\mathbb{L}_\sigma\mathbf{X} = \begin{bmatrix} 0 & \frac{1}{21} \\ -\frac{48}{49} & -\frac{19}{147} \end{bmatrix},$$

constitute a minimal realization of $\mathbf{H}(s)$, in other words,

$$\mathbf{H}(s) = \widehat{\mathbf{W}}\left(\widehat{\mathbb{L}}_\sigma - s\widehat{\mathbb{L}}\right)^{-1}\widehat{\mathbf{V}} = \frac{s}{s^2+s+1},$$

A.C. Antoulas, *The Loewner framework and transfer functions of singular/rectangular systems*, Applied Mathematics Letters, 54: 36-47.

There is a *more natural way* to achieve the same result involving **no** projections. Instead it involves the **Morre-Penrose or the Drazin inverse** of

$$\boldsymbol{\Phi}(s) = \mathbb{L}_\sigma - s\,\mathbb{L}$$

The *Moore-Penrose inverse* of the (rectangular) matrix $\mathbf{M} \in \mathbb{R}^{q \times k}$, is denoted by $\mathbf{M}^{MP} \in \mathbb{R}^{k \times q}$, and satisfies:

$$\left.\begin{array}{ll} (\mathbf{a})\ \ \mathbf{M}\mathbf{M}^{MP}\mathbf{M} = \mathbf{M}, & (\mathbf{b})\ \ \mathbf{M}^{MP}\mathbf{M}\mathbf{M}^{MP} = \mathbf{M}^{MP}, \\ (\mathbf{c})\ \ \left[\mathbf{M}\mathbf{M}^{MP}\right]^T = \mathbf{M}\mathbf{M}^{MP}, & (\mathbf{d})\ \ \left[\mathbf{M}^{MP}\mathbf{M}\right]^T = \mathbf{M}^{MP}\mathbf{M}. \end{array}\right\}$$

This generalized inverse always exists and is unique.

Given a square matrix $\mathbf{M} \in \mathbb{R}^{q \times q}$, its *index* is the least nonnegative integer κ such that rank $\mathbf{M}^{\kappa+1}$ = rank $\mathbf{M}^{\kappa}$. The *Drazin inverse* of $\mathbf{M}$ is the unique matrix $\mathbf{M}^D$ which satisfies:

$$(\mathbf{a})\ \ \mathbf{M}^{\kappa+1}\mathbf{M}^D = \mathbf{M}^{\kappa}, \quad (\mathbf{b})\ \ \mathbf{M}^D\mathbf{M}\mathbf{M}^D = \mathbf{M}^D, \quad (\mathbf{c})\ \ \mathbf{M}\mathbf{M}^D = \mathbf{M}^D\mathbf{M}.$$

In the sequel we will be concerned with rectangular $n \times m$ *polynomial matrices* which have an explicit (rank revealing) factorization as follows:

$$\mathbf{M} = \mathbf{X}\boldsymbol{\Delta}\mathbf{Y}^T,$$

where $\mathbf{X}$, $\mathbf{\Delta}$, $\mathbf{Y}$ have dimension $q \times n$, $n \times n$, $n \times k$, respectively, $n \leq q, k$, and in addition they all have full rank k for almost all $s \in \mathbb{C}$. The *Moore-Penrose* generalized inverse is:

$$\mathbf{M}^{MP} = \mathbf{Y}(\mathbf{Y}^T\mathbf{Y})^{-1}\mathbf{\Delta}^{-1}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T.$$

If in addition $q = k$ and $\mathbf{Y}^T\mathbf{X}$ is *invertible*, the *Drazin generalized inverse* is:

$$\mathbf{M}^{D} = \mathbf{X}(\mathbf{Y}^T\mathbf{X})^{-1}\mathbf{\Delta}^{-1}(\mathbf{Y}^T\mathbf{X})^{-1}\mathbf{Y}^T.$$

A simple example – continuation. The quantity needed is the polynomial matrix

$$\mathbf{\Phi}(s) = \mathbb{L}_\sigma - s\,\mathbb{L} = \left[\begin{array}{cc|cc} -\frac{20s}{21} - \frac{4}{21} & -\frac{2s}{3} & \frac{4}{57} - \frac{28s}{57} & \frac{2}{21} - \frac{8s}{21} \\ -\frac{6s}{7} - \frac{4}{7} & -\frac{2s}{3} - \frac{1}{3} & -\frac{10s}{19} - \frac{4}{19} & -\frac{3s}{7} - \frac{1}{7} \\ \hline -\frac{4s}{7} - \frac{4}{7} & -\frac{10s}{21} - \frac{8}{21} & -\frac{52s}{133} - \frac{36}{133} & -\frac{16s}{49} - \frac{10}{49} \\ -\frac{8s}{21} - \frac{10}{21} & -\frac{s}{3} - \frac{1}{3} & -\frac{16s}{57} - \frac{14}{57} & -\frac{5s}{21} - \frac{4}{21} \end{array}\right] = \mathbf{X}\mathbf{\Delta}(s)\mathbf{Y}^T.$$

Let the common range of the columns of $\mathbb{L}$, $\mathbb{L}_\sigma$ be spanned by the columns of $\mathbf{X}$ and the common range of the rows of the same matrices by the rows of $\mathbf{Y}$:

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{3}{7} & \frac{8}{7} \\ -\frac{1}{2} & 1 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 1 & 0 & -\frac{7}{19} & -\frac{1}{2} \\ 0 & 1 & \frac{24}{19} & \frac{9}{7} \end{bmatrix}.$$

Then with $\boldsymbol{\Delta}(s) = \boldsymbol{\Phi}(1:2, 1:2)(s)$, there holds $\boldsymbol{\Phi}(s)^{MP} = \frac{1}{80989667}\frac{1}{s^2+s+1}\cdot$

$$\cdot\begin{bmatrix} -28\,(11610185s+7274073) & 14\,(3558666s-5604037) & 6076\,(32301s-391) & 14\,(15168851s+1670036) \\ 294\,(225182s+281171) & (-147)\,(192415s-19668) & -2058\,(29494s+15609) & -147\,(417597s+261503) \\ 3724\,(54617s+48189) & (-1862)\,(29046s-17485) & -26068\,(5715s+1523) & -1862\,(83663s+30704) \\ 98\,(2527157s+2123670) & -49\,(1250553s-876439) & -98\,(1797669s+409322) & -49\,(3777710s+1247231]) \end{bmatrix}$$

and $\boldsymbol{\Phi}(s)^{D} = \frac{1}{4897369}\frac{1}{s^2+s+1}\cdot$

$$\cdot\begin{bmatrix} -84\,(234677s+152881) & 294\,(10652s-13755) & 588\,(19079s-641) & 42\,(330545s+29086) \\ 126\,(31956s+42829) & -147\,(11885s+4) & -882\,(4184s+2255) & -63\,(67611s+42841) \\ 684\,(19079s+17063) & -798\,(4184s-2171) & -4788\,(1885s+441) & -342\,(31631s+10550) \\ 42\,(330545s+281368) & -147\,(22537s-13751) & -294\,(31631s+6124) & -21\,(533378s+157609) \end{bmatrix}.$$

It follows that

$$\boxed{\mathbf{W}\,\boldsymbol{\Phi}(s)^{MP}\,\mathbf{V} = \mathbf{W}\,\boldsymbol{\Phi}(s)^{D}\,\mathbf{V} = \mathbf{H}(s)}$$

Conclusions thus far

- Given data $(\ell_i, \mathbf{v}_i)$, $(\lambda_j, \mathbf{w}_j)$, construct the Loewner pencil $(\mathbb{L}_\sigma, \mathbb{L})$.

The quadruple $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$, where the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ may be singular, is a **natural model** of the data. The construction involves **no** computation.

- $(\mathbb{L}_\sigma, \mathbb{L})$ and the underlying $(\mathbf{A}, \mathbf{E})$ have the **same** non-trivial eigenvalues.

- The projection to a minimal realization can be chosen **arbitrarily**.

- We can define transfer functions for systems of the form $\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ and $\mathbf{E}$ are rectangular!

▲ A.C. Antoulas, S. Lefteriu, and A.C Ionita, A tutorial introduction to the Loewner framework for model reduction, in *Model Reduction and Approximation for Complex Systems*, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia (2017).

Conclusions thus far

- Given data $(\ell_i, \mathbf{v}_i)$, $(\lambda_j, \mathbf{w}_j)$, construct the Loewner pencil $(\mathbb{L}_\sigma, \mathbb{L})$.

The quadruple $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$, where the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ may be singular, is a **natural model** of the data. The construction involves **no** computation.

- $(\mathbb{L}_\sigma, \mathbb{L})$ and the underlying $(\mathbf{A}, \mathbf{E})$ have the **same** non-trivial eigenvalues.

- The projection to a minimal realization can be chosen **arbitrarily**.

- We can define transfer functions for systems of the form $\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where **A** and **E** are rectangular!

▲ A.C. Antoulas, S. Lefteriu, and A.C Ionita, A tutorial introduction to the Loewner framework for model reduction, in *Model Reduction and Approximation for Complex Systems*, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia (2017).

Conclusions thus far

- Given data $(\ell_i, \mathbf{v}_i)$, $(\lambda_j, \mathbf{w}_j)$, construct the Loewner pencil $(\mathbb{L}_\sigma, \mathbb{L})$.

The quadruple $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$, where the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ may be singular, is a **natural model** of the data. The construction involves **no** computation.

- $(\mathbb{L}_\sigma, \mathbb{L})$ and the underlying $(\mathbf{A}, \mathbf{E})$ have the **same** non-trivial eigenvalues.
- The projection to a minimal realization can be chosen **arbitrarily**.
- We can define transfer functions for systems of the form $\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where **A** and **E** are rectangular!

▲ A.C. Antoulas, S. Lefteriu, and A.C Ionita, A tutorial introduction to the Loewner framework for model reduction, in *Model Reduction and Approximation for Complex Systems*, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia (2017).

Conclusions thus far

- Given data $(\ell_i, \mathbf{v}_i)$, $(\lambda_j, \mathbf{w}_j)$, construct the Loewner pencil $(\mathbb{L}_\sigma, \mathbb{L})$.

The quadruple $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$, where the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ may be singular, is a **natural model** of the data. The construction involves **no** computation.

- $(\mathbb{L}_\sigma, \mathbb{L})$ and the underlying $(\mathbf{A}, \mathbf{E})$ have the **same** non-trivial eigenvalues.
- The projection to a minimal realization can be chosen **arbitrarily**.
- We can define transfer functions for systems of the form $\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where **A** and **E** are rectangular!

▲ A.C. Antoulas, S. Lefteriu, and A.C Ionita, A tutorial introduction to the Loewner framework for model reduction, in *Model Reduction and Approximation for Complex Systems*, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia (2017).

Conclusions thus far

- Given data $(\boldsymbol{\ell}_i, \mathbf{v}_i)$, $(\lambda_j, \mathbf{w}_j)$, construct the Loewner pencil $(\mathbb{L}_\sigma, \mathbb{L})$.

The quadruple $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$, where the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ may be singular, is a **natural model** of the data. The construction involves **no** computation.

- $(\mathbb{L}_\sigma, \mathbb{L})$ and the underlying $(\mathbf{A}, \mathbf{E})$ have the **same** non-trivial eigenvalues.
- The projection to a minimal realization can be chosen **arbitrarily**.
- We can define transfer functions for systems of the form $\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where **A** and **E** are rectangular!

▲ A.C. Antoulas, S. Lefteriu, and A.C Ionita, A tutorial introduction to the Loewner framework for model reduction, in *Model Reduction and Approximation for Complex Systems*, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia (2017).

The Loewner Algorithm

1. Consider given (frequency domain) measurements $(s_i, \phi_i), \; i = 1, \ldots, N$.

2. Partition the measurements into 2 disjoint sets

$$\text{frequencies}: \quad [s_1, \cdots, s_N] = [\lambda_1, \cdots, \lambda_k], \; [\mu_1, \cdots, \mu_q], \quad k + q = N,$$
$$\text{values}: \quad [\phi_1, \cdots, \phi_N] = [w_1, \cdots, w_k], \; [v_1, \cdots v_q] = \mathbf{W}, \; \mathbf{V}^T.$$

3. Construct the **Loewner pencil**:

$$\mathbb{L} = \left(\frac{v_i - w_j}{\mu_i - \lambda_j} \right)_{i=1,\cdots,q}^{j=1,\cdots,k}, \qquad \mathbb{L}_\sigma = \left(\frac{\mu_i v_i - \lambda_j w_j}{\mu_i - \lambda_j} \right)_{i=1,\cdots,q}^{j=1,\cdots,k}.$$

4. It follows that the **raw model** is: $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$.

5. Compute the rank revealing SVD: $\mathbb{L} \approx \mathbf{Y}\Sigma\mathbf{X}^*$ $(\Sigma \in \mathbb{R}^{r \times r})$.

6. The reduced model $(\widehat{\mathbf{C}}, \widehat{\mathbf{E}}, \widehat{\mathbf{A}}, \widehat{\mathbf{B}})$ is obtained by **projecting** the raw model $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$:

$$\widehat{\mathbf{C}} = \mathbf{W}\mathbf{X}, \quad \widehat{\mathbf{E}} = -\mathbf{Y}^*\mathbb{L}\mathbf{X}, \quad \widehat{\mathbf{A}} = -\mathbf{Y}^*\mathbb{L}_\sigma\mathbf{X}, \quad \widehat{\mathbf{B}} = \mathbf{Y}^*\mathbf{V}.$$

7. Reference: S. Lefteriu and A.C. Antoulas: A New Approach to Modeling Multiport Systems from Frequency-Domain Data, IEEE Trans. CAD, 29: 14-27 (2010).

1. Consider given (frequency domain) measurements $(s_i,\ \phi_i),\ \ i = 1, \ldots, N$.

2. Partition the measurements into 2 disjoint sets

$$\text{frequencies}: \quad [s_1, \cdots, s_N] \ = \ [\lambda_1, \cdots, \lambda_k],\ [\mu_1, \cdots, \mu_q],\ \ k + q = N,$$

$$\text{values}: \quad [\phi_1, \cdots, \phi_N] \ = \ [w_1, \cdots, w_k],\ [v_1, \cdots v_q] \ = \ \mathbf{W},\ \mathbf{V}^T.$$

3. Construct the **Loewner pencil**:

$$\mathbb{L} = \left(\frac{v_i - w_j}{\mu_i - \lambda_j}\right)_{i=1,\cdots,q}^{j=1,\cdots,k}, \qquad \mathbb{L}_\sigma = \left(\frac{\mu_i v_i - \lambda_j w_j}{\mu_i - \lambda_j}\right)_{i=1,\cdots,q}^{j=1,\cdots,k}.$$

4. It follows that the **raw model** is: $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$.

5. Compute the rank revealing SVD: $\mathbb{L} \approx \mathbf{Y}\Sigma\mathbf{X}^*$ $(\Sigma \in \mathbb{R}^{r\times r})$.

6. The reduced model $(\widehat{\mathbf{C}}, \widehat{\mathbf{E}}, \widehat{\mathbf{A}}, \widehat{\mathbf{B}})$ is obtained by **projecting** the raw model $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$:

$$\widehat{\mathbf{C}} = \mathbf{W}\mathbf{X},\ \ \widehat{\mathbf{E}} = -\mathbf{Y}^*\mathbb{L}\mathbf{X},\ \ \widehat{\mathbf{A}} = -\mathbf{Y}^*\mathbb{L}_\sigma\mathbf{X},\ \ \widehat{\mathbf{B}} = \mathbf{Y}^*\mathbf{V}.$$

7. Reference: S. Lefteriu and A.C. Antoulas: A New Approach to Modeling Multiport Systems from Frequency-Domain Data, IEEE Trans. CAD, 29: 14-27 (2010).

1. Consider given (frequency domain) measurements $(s_i,\, \phi_i),\;\; i = 1, \ldots, N$.

2. Partition the measurements into 2 disjoint sets

$$\text{frequencies}: \quad [s_1, \cdots, s_N] \;=\; [\lambda_1, \cdots, \lambda_k],\; [\mu_1, \cdots, \mu_q],\;\; k + q = N,$$
$$\text{values}: \quad [\phi_1, \cdots, \phi_N] \;=\; [w_1, \cdots, w_k],\; [v_1, \cdots v_q] \;=\; \mathbf{W},\; \mathbf{V}^T.$$

3. Construct the **Loewner pencil**:

$$\mathbb{L} = \left(\frac{v_i - w_j}{\mu_i - \lambda_j}\right)_{i=1,\cdots,q}^{j=1,\cdots,k}, \qquad \mathbb{L}_\sigma = \left(\frac{\mu_i v_i - \lambda_j w_j}{\mu_i - \lambda_j}\right)_{i=1,\cdots,q}^{j=1,\cdots,k}.$$

4. It follows that the **raw model** is: $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$.

5. Compute the rank revealing SVD: $\mathbb{L} \approx \mathbf{Y}\boldsymbol{\Sigma}\mathbf{X}^*$ $(\boldsymbol{\Sigma} \in \mathbb{R}^{r \times r})$.

6. The reduced model $(\widehat{\mathbf{C}}, \widehat{\mathbf{E}}, \widehat{\mathbf{A}}, \widehat{\mathbf{B}})$ is obtained by **projecting** the raw model $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$:

$$\widehat{\mathbf{C}} = \mathbf{W}\mathbf{X},\;\; \widehat{\mathbf{E}} = -\mathbf{Y}^*\mathbb{L}\mathbf{X},\;\; \widehat{\mathbf{A}} = -\mathbf{Y}^*\mathbb{L}_\sigma\mathbf{X},\;\; \widehat{\mathbf{B}} = \mathbf{Y}^*\mathbf{V}.$$

7. Reference: S. Lefteriu and A.C. Antoulas: A New Approach to Modeling Multiport Systems from Frequency-Domain Data, IEEE Trans. CAD, 29: 14-27 (2010).

The Loewner Algorithm

1. Consider given (frequency domain) measurements $(s_i,\ \phi_i),\ \ i = 1, \ldots, N$.

2. Partition the measurements into 2 disjoint sets

$$\text{frequencies}: \quad [s_1, \cdots, s_N] \quad = \quad [\lambda_1, \cdots, \lambda_k],\ [\mu_1, \cdots, \mu_q],\ \ k + q = N,$$
$$\text{values}: \quad [\phi_1, \cdots, \phi_N] \quad = \quad [w_1, \cdots, w_k],\ [v_1, \cdots v_q] \ = \ \mathbf{W},\ \mathbf{V}^T.$$

3. Construct the **Loewner pencil**:

$$\mathbb{L} = \left(\frac{v_i - w_j}{\mu_i - \lambda_j}\right)_{i=1,\cdots,q}^{j=1,\cdots,k}, \qquad \mathbb{L}_\sigma = \left(\frac{\mu_i v_i - \lambda_j w_j}{\mu_i - \lambda_j}\right)_{i=1,\cdots,q}^{j=1,\cdots,k}.$$

4. It follows that the **raw model** is: $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$.

5. Compute the rank revealing SVD: $\mathbb{L} \approx \mathbf{Y\Sigma X}^*$ $(\mathbf{\Sigma} \in \mathbb{R}^{r\times r})$.

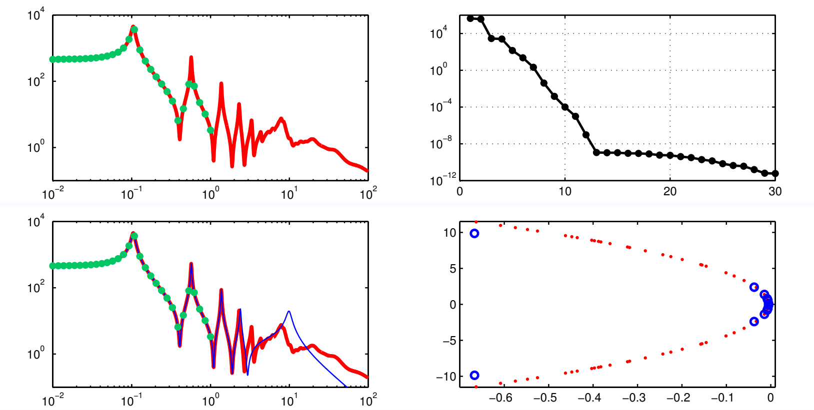
6. The reduced model $(\widehat{\mathbf{C}}, \widehat{\mathbf{E}}, \widehat{\mathbf{A}}, \widehat{\mathbf{B}})$ is obtained by **projecting** the raw model $(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$:

$$\widehat{\mathbf{C}} = \mathbf{WX},\ \ \widehat{\mathbf{E}} = -\mathbf{Y}^*\mathbb{L}\mathbf{X},\ \ \widehat{\mathbf{A}} = -\mathbf{Y}^*\mathbb{L}_\sigma\mathbf{X},\ \ \widehat{\mathbf{B}} = \mathbf{Y}^*\mathbf{V}.$$

7. Reference: S. Lefteriu and A.C. Antoulas: A New Approach to Modeling Multiport Systems from Frequency-Domain Data, IEEE Trans. CAD, 29: 14-27 (2010).

Example: (discretized) Euler-Bernoulli beam

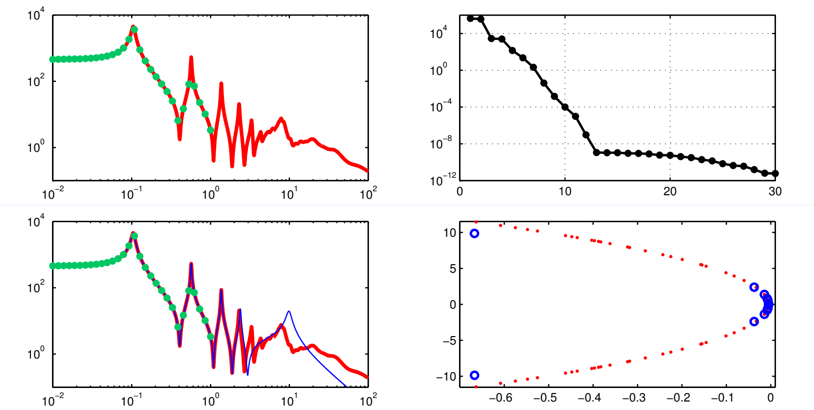
- System of order $n = 348$ (obtained after discretization) representing a clamped beam.
- $N = 60$ frequency response measurements, $s_k = j\omega_k$, with $\omega_k \in [-1, -0.01] \cup [0.01, 1]$.
- Construct 30×30 Loewner pencil and $\mathbf{Y}, \mathbf{X} \in \mathbb{R}^{30\times 12}$ from the SVD.
- Project to get reduced model of order $r = 12$.



(1,1) Original and data	(1,2) Singular values of $\mathbb{L}$.
(2,1) Original & reduced FR	(2,2) Poles original & reduced

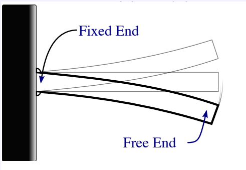
Example: (discretized) Euler-Bernoulli beam

- System of order $n = 348$ (obtained after discretization) representing a clamped beam.
- $N = 60$ frequency response measurements, $s_k = j\omega_k$, with $\omega_k \in [-1, -0.01] \cup [0.01, 1]$.
- Construct 30×30 Loewner pencil and $\mathbf{Y}, \mathbf{X} \in \mathbb{R}^{30 \times 12}$ from the SVD.
- Project to get reduced model of order $r = 12$.



(1,1) Original and data	(1,2) Singular values of $\mathbb{L}$.
(2,1) Original & reduced FR	(2,2) Poles original & reduced

An Euler-Bernoulli beam: bypassing PDE discretization



$$\text{BC}\begin{cases} w(0,t)=0,\ \frac{\partial w}{\partial x}(0,t)=0,\ E\,I\,\frac{\partial^2 w(L,t)}{\partial x^2}+c_d I\,\frac{\partial^3 w(L,t)}{\partial x^2\partial t}=0,\\ -E\,I\frac{\partial^3 w(L,t)}{\partial x^3}-c_d I\,\frac{\partial^4 w(L,t)}{\partial x^3\partial t}=u(t),\ y(t)=\frac{\partial w(L,t)}{\partial t}, \end{cases}$$

$$\frac{\partial^2 w(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[E\,I\,\frac{\partial^2 w(x,t)}{\partial x^2}+c_d I\,\frac{\partial^3 w(x,t)}{\partial x^2\partial t}\right]=0,$$

where E, I, c_d are constants. The transfer function is:

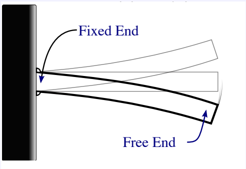
$$\boxed{\mathbf{H}(s)=\frac{s\,\mathbf{N}(s)}{(E\,I+s\,c_d I)\,m^3(s)\,\mathbf{D}(s)}} \qquad \text{with} \quad \mathbf{m}(s)=\left[\frac{-s^2}{E\,I+c_d\,I\,s}\right]^{\frac{1}{4}},$$

$$\mathbf{N}(s)=\cosh(L\,\mathbf{m}(s))\sin(L\,\mathbf{m}(s))-\sinh(L\,\mathbf{m}(s))\cos(L\,\mathbf{m}(s)) \ \text{ and}$$

$$\mathbf{D}(s)=1+\cosh(L\,\mathbf{m}(s))\cos(L\,\mathbf{m}(s))\,.$$

Parameter values: $E=69$, $GPa=6.9\cdot10^{10}N/m^2$ - Young's modulus elasticity constant, $I=(1/12)\cdot7\cdot8.5^3\cdot10^{-11}m^4$ - moment of inertia, $c_d=5\cdot10^{-4}$ - damping constant, $L=0.7m$, $b=7cm$, $h=8.5mm$ - length, base, height of the rectangular cross section.

An Euler-Bernoulli beam: bypassing PDE discretization



$$\text{BC}\begin{cases} w(0,t)=0,\ \frac{\partial w}{\partial x}(0,t)=0,\ E\,I\frac{\partial^2 w(L,t)}{\partial x^2}+c_d I\,\frac{\partial^3 w(L,t)}{\partial x^2\partial t}=0, \\ -E\,I\frac{\partial^3 w(L,t)}{\partial x^3}-c_d I\,\frac{\partial^4 w(L,t)}{\partial x^3\partial t}=u(t),\ y(t)=\frac{\partial w(L,t)}{\partial t}, \end{cases}$$

$$\frac{\partial^2 w(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[E\,I\,\frac{\partial^2 w(x,t)}{\partial x^2}+c_d I\,\frac{\partial^3 w(x,t)}{\partial x^2\partial t}\right]=0,$$

where E, I, c_d are constants. The transfer function is:

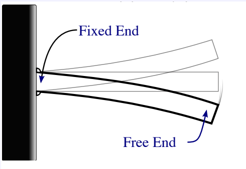
$$\mathbf{H}(s)=\frac{s\,\mathbf{N}(s)}{(E\,I+s\,c_d I)\,m^3(s)\,\mathbf{D}(s)} \qquad \text{with} \quad \mathbf{m}(s)=\left[\frac{-s^2}{E\,I+c_d\,I\,s}\right]^{\frac{1}{4}},$$

$$\mathbf{N}(s)=\cosh(L\,\mathbf{m}(s))\sin(L\,\mathbf{m}(s))-\sinh(L\,\mathbf{m}(s))\cos(L\,\mathbf{m}(s)) \ \text{ and}$$

$$\mathbf{D}(s)=1+\cosh(L\,\mathbf{m}(s))\cos(L\,\mathbf{m}(s))\,.$$

Parameter values: $E=69$, $GPa=6.9\cdot 10^{10}N/m^2$ - Young's modulus elasticity constant, $I=(1/12)\cdot 7\cdot 8.5^3\cdot 10^{-11}m^4$ - moment of inertia, $c_d=5\cdot 10^{-4}$ - damping constant, $L=0.7m$, $b=7cm$, $h=8.5mm$ - length, base, height of the rectangular cross section.

An Euler-Bernoulli beam: bypassing PDE discretization



$$\text{BC} \begin{cases} w(0,t)=0,\ \frac{\partial w}{\partial x}(0,t)=0,\ E\,I\frac{\partial^2 w(L,t)}{\partial x^2}+c_d I\,\frac{\partial^3 w(L,t)}{\partial x^2\partial t}=0, \\ -E\,I\frac{\partial^3 w(L,t)}{\partial x^3}-c_d I\,\frac{\partial^4 w(L,t)}{\partial x^3\partial t}=u(t),\ y(t)=\frac{\partial w(L,t)}{\partial t}, \end{cases}$$

$$\frac{\partial^2 w(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[E\,I\,\frac{\partial^2 w(x,t)}{\partial x^2}+c_d I\,\frac{\partial^3 w(x,t)}{\partial x^2\partial t}\right]=0,$$

where E, I, c_d are constants. The transfer function is:

$$\mathbf{H}(s)=\frac{s\,\mathbf{N}(s)}{(E\,I+s\,c_d I)\,m^3(s)\,\mathbf{D}(s)} \quad \text{with} \quad \mathbf{m}(s)=\left[\frac{-s^2}{E\,I+c_d\,I\,s}\right]^{\frac{1}{4}},$$

$$\mathbf{N}(s)=\cosh(L\,\mathbf{m}(s))\sin(L\,\mathbf{m}(s))-\sinh(L\,\mathbf{m}(s))\cos(L\,\mathbf{m}(s)) \ \text{ and}$$

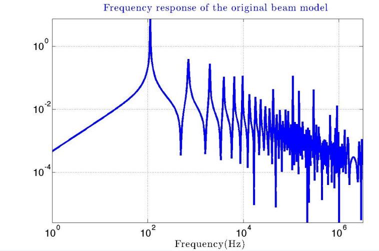
$$\mathbf{D}(s)=1+\cosh(L\,\mathbf{m}(s))\cos(L\,\mathbf{m}(s)).$$

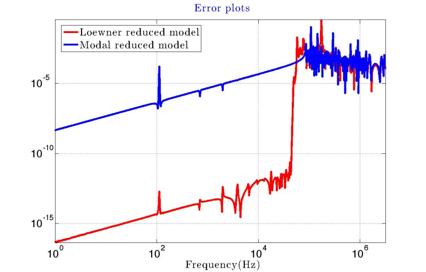
Parameter values: $E=69$, $GPa=6.9\cdot 10^{10}N/m^2$ - Young's modulus elasticity constant, $I=(1/12)\cdot 7\cdot 8.5^3\cdot 10^{-11}m^4$ - moment of inertia, $c_d=5\cdot 10^{-4}$ - damping constant, $L=0.7m$, $b=7cm$, $h=8.5mm$ - length, base, height of the rectangular cross section.

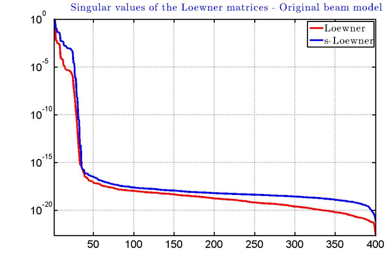
Plots

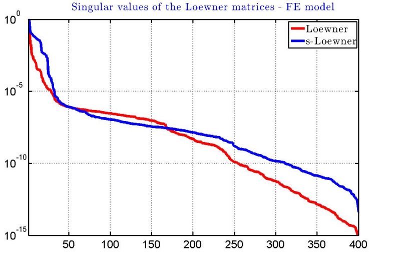
Reduction methods:

1. **Modal truncation.**
2. **FEM followed by Loewner.**
3. **Loewner based on the transfer function.**



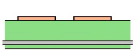


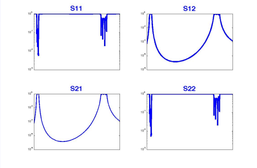
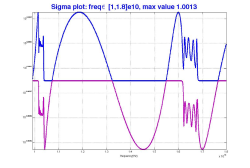
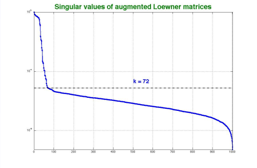
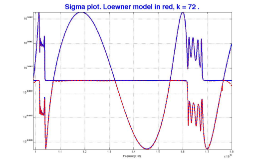
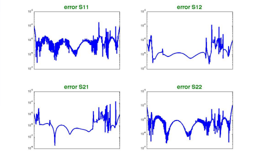
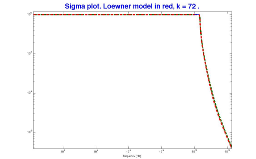




Reduced model from frequency response measurements (microstrip device)

1001 S-parameter measurements between 10-18 GHz (CST).



Data frequency response $\|\mathbf{S}_{i,j}\|$, $i, j = 1, 2$.	Data two singular values.
	
Singular values of 1001 $\times$ 1001 Loewner matrix	Singular-value fit of model $k = 72$
	
S-parameter-error: $\in [10^{-6}, 10^{-4}]$	Two singular values of model: $\omega \in [0, 10\text{THz}]$
	

Summary: MOR in the Loewner framework

- Given is: measured or simulated data (DNS)

Examples		Measurements
0.	various examples	n = small
1.	clamped beam	$n = 346$
2.	semi-conductor device	no model
3.	Euler-Bernoulli beam	$n = \infty$

- Key tool: **Loewner pencil** (followed by a **projection**).

Types of systems
Linear (SISO and MIMO)
Linear parametrized
Linear switched systems
Bilinear
General quadratic bilinear

- Given data, we construct with **no computation**, a singular, high-order model in descriptor form ⇒ natural way to construct reduced models.

⇒ **SVD of the Loewner pencil provides trade-off between accuracy and complexity; resulting model complexity n = (numerical) rank of Loewner matrix.**

⇒ can deal with **nonlinear systems**.

- Philosophy: **Collect data and extract desired information**

Summary: MOR in the Loewner framework

▲ Given is: measured or simulated data (DNS)

Examples		Measurements
0.	various examples	$n =$ small
1.	clamped beam	$n = 346$
2.	semi-conductor device	no model
3.	Euler-Bernoulli beam	$n = \infty$

▲ Key tool: **Loewner pencil** (followed by a **projection**).

Types of systems
Linear (SISO and MIMO)
Linear parametrized
Linear switched systems
Bilinear
General quadratic bilinear

▲ Given data, we construct with **no computation**, a singular, high-order model in descriptor form ⇒ natural way to construct reduced models.

⇒ **SVD of the Loewner pencil provides trade-off between accuracy and complexity; resulting model complexity $n =$ (numerical) rank of Loewner matrix.**

⇒ can deal with **nonlinear systems**.

▲ Philosophy: **Collect data and extract desired information**

Summary: MOR in the Loewner framework

- Given is: measured or simulated data (DNS)

Examples		Measurements
0.	various examples	$n =$ small
1.	clamped beam	$n = 346$
2.	semi-conductor device	no model
3.	Euler-Bernoulli beam	$n = \infty$

- Key tool: **Loewner pencil** (followed by a **projection**).

Types of systems
Linear (SISO and MIMO)
Linear parametrized
Linear switched systems
Bilinear
General quadratic bilinear

- Given data, we construct with **no computation**, a singular, high-order model in descriptor form $\Rightarrow$ natural way to construct reduced models.

$\Rightarrow$ **SVD of the Loewner pencil provides trade-off between accuracy and complexity; resulting model complexity $n =$ (numerical) rank of Loewner matrix.**

$\Rightarrow$ can deal with **nonlinear systems**.

- Philosophy: **Collect data and extract desired information**

Summary: MOR in the Loewner framework

- Given is: measured or simulated data (DNS)

Examples		Measurements
0.	various examples	$n =$ small
1.	clamped beam	$n = 346$
2.	semi-conductor device	no model
3.	Euler-Bernoulli beam	$n = \infty$

- Key tool: **Loewner pencil** (followed by a **projection**).

Types of systems
Linear (SISO and MIMO)
Linear parametrized
Linear switched systems
Bilinear
General quadratic bilinear

- Given data, we construct with **no computation**, a singular, high-order model in descriptor form $\Rightarrow$ natural way to construct reduced models.

$\Rightarrow$ **SVD of the Loewner pencil provides trade-off between accuracy and complexity; resulting model complexity $n =$ (numerical) rank of Loewner matrix.**

$\Rightarrow$ can deal with **nonlinear systems**.

- Philosophy: **Collect data and extract desired information**

Summary: MOR in the Loewner framework

- Given is: measured or simulated data (DNS)

	Examples	Measurements
0.	various examples	$n =$ small
1.	clamped beam	$n = 346$
2.	semi-conductor device	no model
3.	Euler-Bernoulli beam	$n = \infty$

- Key tool: **Loewner pencil** (followed by a **projection**).

Types of systems
Linear (SISO and MIMO)
Linear parametrized
Linear switched systems
Bilinear
General quadratic bilinear

- Given data, we construct with **no computation**, a singular, high-order model in descriptor form ⇒ natural way to construct reduced models.

⇒ **SVD of the Loewner pencil provides trade-off between accuracy and complexity; resulting model complexity $n =$ (numerical) rank of Loewner matrix.**

⇒ can deal with **nonlinear systems**.

- Philosophy: **Collect data and extract desired information**

Summary: MOR in the Loewner framework

- Given is: measured or simulated data (DNS)

	Examples	Measurements
0.	various examples	$n =$ small
1.	clamped beam	$n = 346$
2.	semi-conductor device	no model
3.	Euler-Bernoulli beam	$n = \infty$

Types of systems
Linear (SISO and MIMO)
Linear parametrized
Linear switched systems
Bilinear
General quadratic bilinear

- Key tool: **Loewner pencil** (followed by a **projection**).

- Given data, we construct with **no computation**, a singular, high-order model in descriptor form $\Rightarrow$ natural way to construct reduced models.

$\Rightarrow$ **SVD of the Loewner pencil provides trade-off between accuracy and complexity; resulting model complexity $n =$ (numerical) rank of Loewner matrix.**

$\Rightarrow$ can deal with **nonlinear systems**.

- Philosophy: **Collect data and extract desired information**

Summary: MOR in the Loewner framework

- Given is: measured or simulated data (DNS)

Examples		Measurements
0.	various examples	$n =$ small
1.	clamped beam	$n = 346$
2.	semi-conductor device	no model
3.	Euler-Bernoulli beam	$n = \infty$

- Key tool: **Loewner pencil** (followed by a **projection**).

Types of systems
Linear (SISO and MIMO)
Linear parametrized
Linear switched systems
Bilinear
General quadratic bilinear

- Given data, we construct with **no computation**, a singular, high-order model in descriptor form $\Rightarrow$ natural way to construct reduced models.

$\Rightarrow$ **SVD of the Loewner pencil provides trade-off between accuracy and complexity; resulting model complexity $n =$ (numerical) rank of Loewner matrix**.

$\Rightarrow$ can deal with **nonlinear systems**.

- Philosophy: **Collect data and extract desired information**

Some additional references

1. A.C. Antoulas, Approximation of large-scale dynamical systems, Series in Design and Control, **DC-6**, SIAM Philadelphia 2005 (reprinted 2008).
2. A.C. Antoulas, C.A. Beattie and S. Gugercin, Data-driven model reduction methods and applications, Series in Computational Science and Engineering, SIAM, Philadelphia (in preparation) 2017.
3. U. Baur, P. Benner, L. Feng, *Model order reduction for linear and nonlinear systems: a system-theoretic perspective*, Arch. Computat. Methods Eng., 21: 331-358 (2014).
4. G. Flagg and S. Gugercin, *Multipoint Volterra Series Interpolation and H2 Optimal Model Reduction of Bilinear Systems*, SIAM Journal on Matrix Analysis and Applications, Vol. 36, Issue: 2, 549-579 (2015).
5. B. Kramer and S. Gugercin. " The Eigensystem Realization Algorithm from Tangentially Interpolated Data". Mathematical and Computer Modelling of Dynamical Systems, Vol. 22, pp. 282-306, 2016.
6. B. Peherstorfer, S. Gugercin, and K. Willcox. "Data-driven Reduced Model Construction with Time-domain Loewner Models."Accepted to appear in SIAM Journal on Scientific Computing, 2017.

See also Sara Gründel's talk tomorrow at 17h.

1. A.C. Antoulas, Approximation of large-scale dynamical systems, Series in Design and Control, **DC-6**, SIAM Philadelphia 2005 (reprinted 2008).

2. A.C. Antoulas, C.A. Beattie and S. Gugercin, Data-driven model reduction methods and applications, Series in Computational Science and Engineering, SIAM, Philadelphia (in preparation) 2017.

3. U. Baur, P. Benner, L. Feng, *Model order reduction for linear and nonlinear systems: a system-theoretic perspective*, Arch. Computat. Methods Eng., 21: 331-358 (2014).

4. G. Flagg and S. Gugercin, *Multipoint Volterra Series Interpolation and H2 Optimal Model Reduction of Bilinear Systems*, SIAM Journal on Matrix Analysis and Applications, Vol. 36, Issue: 2, 549-579 (2015).

5. B. Kramer and S. Gugercin. " The Eigensystem Realization Algorithm from Tangentially Interpolated Data". Mathematical and Computer Modelling of Dynamical Systems, Vol. 22, pp. 282-306, 2016.

6. B. Peherstorfer, S. Gugercin, and K. Willcox. "Data-driven Reduced Model Construction with Time-domain Loewner Models."Accepted to appear in SIAM Journal on Scientific Computing, 2017.

See also Sara Gründel's talk tomorrow at 17h.