Some recent developments on ROMs in computational fluid dynamics

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Introduction and outline of the talk

- reduced basis model order reduction, with a focus on applications in **computational fluid dynamics**;
- tackle **convection dominated** problems;

- **part 1:** reduced order stabilization techniques for incompressible CFD:
  - **S. Ali**, F. Ballarin, and G. Rozza. Stabilized reduced basis methods for parametrized Stokes and Navier-Stokes equations. In preparation, 2017.

- **part 2:** certification of reduced basis for a Smagorinsky turbulence model:
  - T. Chacón Rebollo, **E. Delgado Ávila**, M. Gómez Mármol, F. Ballarin, G. Rozza. On a certified Smagorinsky reduced basis turbulence model. Submitted, 2017.

- **part 3:** weighted reduced basis methods for uncertainty quantification:
  - **D. Torlo**, F. Ballarin, and G. Rozza. Stabilized weighted reduced basis methods for parametrized advection dominated problems with random inputs. In preparation, 2017.

Warm up: stabilization for parametrized advection dominated elliptic problems

$$L(\boldsymbol{\mu})y(\boldsymbol{\mu}) = -\varepsilon(\boldsymbol{\mu})\Delta y(\boldsymbol{\mu}) + \boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla y(\boldsymbol{\mu}) = f(\boldsymbol{\mu}) \qquad \text{in } \Omega,$$

s.t. suitable boundary conditions on $\partial\Omega$.

- (local) **Péclet number** $\text{Pe}_K := \frac{\|\boldsymbol{\beta}(\boldsymbol{\mu})\| h_K}{2\varepsilon(\boldsymbol{\mu})} \gg 1$ for advection dominated problems, being K a cell of the triangulation of Ω and h_K its diameter.
- define bilinear and linear forms associated to the problem

$$a(y, v; \boldsymbol{\mu}) = \int_\Omega \varepsilon(\boldsymbol{\mu})\nabla y(\boldsymbol{\mu}) \cdot \nabla v + \boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla y(\boldsymbol{\mu}) v$$

$$F(v; \boldsymbol{\mu}) = \int_\Omega f(\boldsymbol{\mu}) v$$

- standard FE discretization may produce unphysical solutions $\rightarrow$ strongly consistent **stabilizations**, e.g. SUPG

$$a_{\text{stab}}(y, v; \boldsymbol{\mu}) = a(y, v; \boldsymbol{\mu}) + \sum_K \delta_K \int_K L(\boldsymbol{\mu})y \; \frac{h_K}{\|\boldsymbol{\beta}(\boldsymbol{\mu})\|} L_{SS}(\boldsymbol{\mu})v,$$

for $L_{SS}(\boldsymbol{\mu}) = \boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla v$.

Warm up: "stabilized" reduced basis greedy algorithm

```
Sample Ξ_train ⊂ D
Pick arbitrary μ^1 ∈ Ξ_train
Define S_0 = ∅, V_0 = ∅
for N = 1, ..., N_max
    Perform a PDE solve to compute y(μ^N)
    S_N = S_{N-1} ∪ {μ^N}
    V_N = V_{N-1} ⊕ {y(μ^N)}
    μ^{N+1} = arg max_{μ∈Ξ_train} Δ_N(μ)
    if Δ_N(μ^{N+1}) ≤ tol
        break
    end
end
```

where $\Delta_N(\boldsymbol{\mu})$ is a sharp, *inexpensive a posteriori error bound* for $\|y(\boldsymbol{\mu}) - y_N(\boldsymbol{\mu})\|_{\mathbb{V}}$, being $y_N(\boldsymbol{\mu})$ **the RB solution of dimension N.**

J. S. Hesthaven, G. Rozza, B. Stamm. *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*. SpringerBriefs in Mathematics. Springer International Publishing, 2015

RB online system: to stabilize or not to stabilize?

- **offline** stabilization by SUPG:

$$\text{find } y(\boldsymbol{\mu}) \in V \text{ s.t.} \quad a_{\text{stab}}(y(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = F_{\text{stab}}(v; \boldsymbol{\mu}), \quad \forall v \in V,$$

  being V a FE space.
- **online**:
  - do stabilize also *online*, to guarantee consistency $\rightarrow$ ***Offline-Online* stabilized RB method**

$$\text{find } y_N(\boldsymbol{\mu}) \in V_{N,\text{stab}} \text{ s.t.} \quad a_{\text{stab}}(y_N(\boldsymbol{\mu}), v_N; \boldsymbol{\mu}) = F_{\text{stab}}(v_N; \boldsymbol{\mu}), \quad \forall v_N \in V_{N,\text{stab}}$$

  - do not stabilize *online*, to avoid assembly of all stabilization terms and (possibly) gain in performance $\rightarrow$ ***Offline-only* stabilized RB method**

$$\text{find } y_N(\boldsymbol{\mu}) \in V_N \text{ s.t.} \quad a(y_N(\boldsymbol{\mu}), v_N; \boldsymbol{\mu}) = F(v_N; \boldsymbol{\mu}), \quad \forall v_N \in V_N$$

  - note that $V_{N,\text{stab}}$ and V_N may be different, because the greedy procedure may pick different snapshots with vs without online stabilization.

P. Pacciarini and G. Rozza, Stabilized reduced basis method for parametrized advection–diffusion PDEs, Comput. Methods Appl. Mech. Engrg., 274:1-18, 2014.

Incompressible CFD

$$\begin{cases} -\nu \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = 0 & \text{in } \Omega \end{cases}$$

- **Reynolds number** $\text{Re} := \frac{\overline{u}L}{\nu} \gg 1$ for advection dominated problems, being $\overline{u}$ the magnitude of a characteristic velocity, L a characteristic length of Ω, ν viscosity of fluid.
- we can adapt the previous **greedy algorithm** to this case. For the sake of simplicity in Part 1 we will use an indicator based on the residual rather than a proper error estimator $\Delta_N(\boldsymbol{\mu})$;
- requires careful treatment of the **incompressibility constraint**, as explained in the next slide.

The classical reduced order inf-sup stabilization

- inf-sup condition is **not** necessarily preserved by Galerkin projection in the online phase.
- reduced velocity space **enrichment** by supremizer solutions,

$$V_N = GS(\{\boldsymbol{u}(\boldsymbol{\mu}^i)\}_{i=1}^N) \quad \oplus \quad GS(\{S^{\boldsymbol{\mu}^i} p(\boldsymbol{\mu}^i)\}_{i=1}^N),$$
$$Q_N = GS(\{p(\boldsymbol{\mu}^i)\}_{i=1}^N),$$

where $S^{\boldsymbol{\mu}} : Q \to V$ is the **supremizer operator** given by

$$(S^{\boldsymbol{\mu}} q, \boldsymbol{w})_V = b(q, \boldsymbol{w}; \boldsymbol{\mu}), \quad \forall \boldsymbol{w} \in V.$$

in order to fullfil an ***inf-sup condition at the reduced-order level*** too:

$$\beta_N(\boldsymbol{\mu}) = \inf_{\underline{\mathbf{q}}_N \neq \underline{\mathbf{0}}} \sup_{\underline{\mathbf{v}}_N \neq \underline{\mathbf{0}}} \frac{\underline{\mathbf{q}}_N^T B_N(\boldsymbol{\mu}) \underline{\mathbf{v}}_N}{\|\underline{\mathbf{v}}_N\|_{V_N} \|\underline{\mathbf{q}}_N\|_{Q_N}} \geq \tilde{\beta}_N > 0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

where $B_N(\boldsymbol{\mu})$ is the reduced-order matrix associated to the divergence term. *(Rozza, Veroy. CMAME (2007), Rozza et al, Numerische Mathematik (2013). Ballarin et al. IJNME (2015))*, residual-based stabilization procedures *(Caiazzo, Iliescu et al. JCP (2014), **Ali** et al (2017))*.

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Part 1: stabilization for incompressible CFD

$$\begin{cases} -\nu\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = 0 & \text{in } \Omega \end{cases}$$

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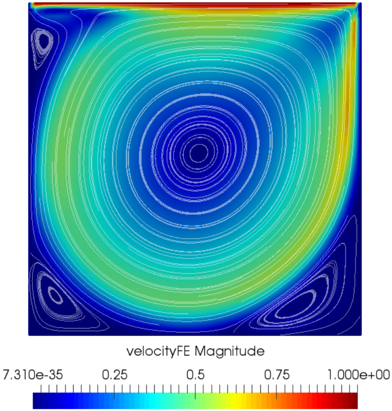
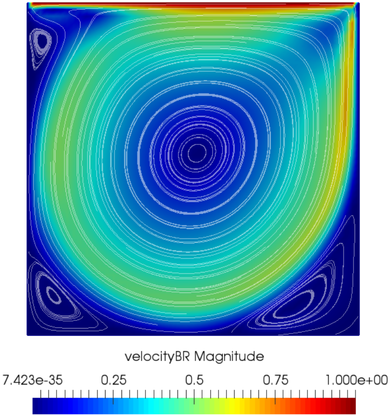
- **stabilization** for:
  - **advection terms** (Brezzi-Pitkaranta, Franca-Hughes, SUPG, Galerkin Least Squares, Douglas Wang);
  - velocity-pressure FE space pairs which are **not inf-sup stable** $\rightarrow$ same motivation for which supremizers are added to the reduced velocity space;
- discuss **Offline-Online** stabilization vs **Offline-only** stabilization;
- discuss the interplay between **inf-sup stabilization techniques** and supremizer enrichment of the reduced velocity space.

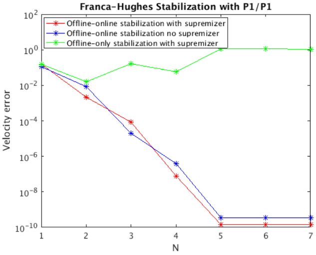
S. Ali, F. Ballarin, and G. Rozza. Stabilized reduced basis methods for parametrized Stokes and Navier-Stokes equations. In preparation, 2017

Lid-driven cavity flow test case

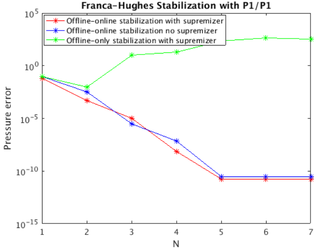




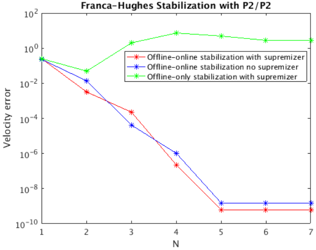
Lid-driven cavity flow – Stokes



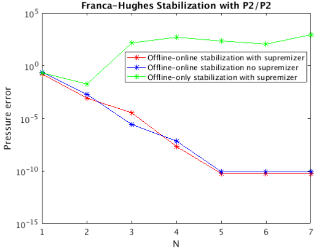
a: Velocity for $\mathbb{P}_1/\mathbb{P}_1$



b: Pressure for $\mathbb{P}_1/\mathbb{P}_1$



c: Velocity for $\mathbb{P}_2/\mathbb{P}_2$

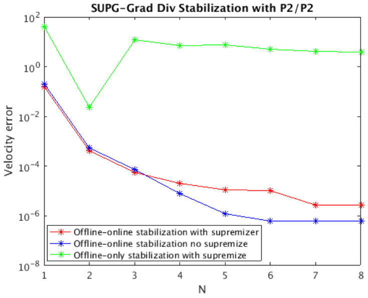


d: Pressure for $\mathbb{P}_2/\mathbb{P}_2$

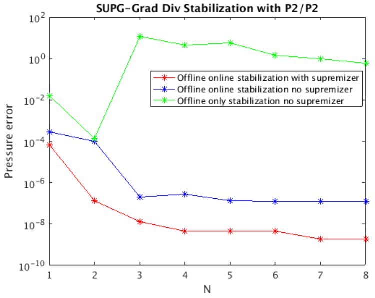
Lid-driven cavity flow – Navier Stokes, low Reynolds

Cavity flow for $\mathrm{Re} \in [10, 500]$.

Error analysis for $\mathrm{Re} = 250$:



a: SUPG and grad-div: velocity error

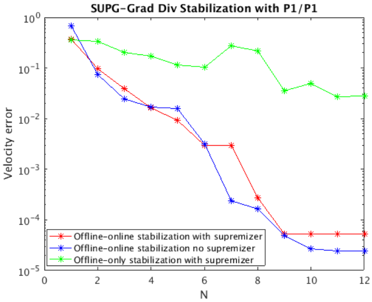


b: SUPG and grad-div: pressure error

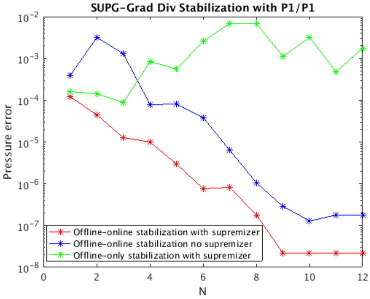
Lid-driven cavity flow – Navier Stokes, moderate Reynolds

Cavity flow for $\mathrm{Re} \in [2500, 3500]$.

Error analysis for $\mathrm{Re} = 3000$:



a: SUPG and grad-div: velocity error



b: SUPG and grad-div: pressure error

Part 1, conclusion:

stabilized RB methods can handle simple test problems with high(er) Reynolds numbers

Part 2, what's next:

step up our game, and provide certified a posteriori error bounds

Part 2: certified reduced basis for a Smagorinsky turbulence model

$$\begin{cases} -\text{div}\left[(\nu + \nu_T(\boldsymbol{u}))\,\nabla \boldsymbol{u}\right] + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \text{div } \boldsymbol{u} = 0 & \text{in } \Omega \end{cases}$$

where

$$\nu_T(\boldsymbol{u}) = C_S^2 \sum_K h_K^2 \; \left\| \nabla \boldsymbol{u}_{|_K} \right\| \; \mathbb{1}_K$$

being K a cell of the triangulation of Ω, h_K its diameter, C_S the Smagorinsky constant

- Smagorinsky turbulence model → "physically based" **stabilization** technique instead of general purpose SUPG in the previous Part; it's the simplest **Large Eddy Simulation** turbulence model;
- interested in *Offline-Online* stabilization → reduce the Smagorinsky stabilization term, and provide **error bounds** for the full Smagorinsky model;
- Smagorinsky turbulence model provides stabilization for the convective term but not for the inf-sup condition → will always use **supremizers**.

T. Chacón Rebollo, E. Delgado Ávila, M. Gómez Mármol, F. Ballarin, G. Rozza. On a certified Smagorinsky reduced basis turbulence model. Submitted, 2017

Parametrization and notation

- we consider one parameter μ s.t. $\nu = \dfrac{1}{\mu} \to$ Reynolds number
- for the following analysis, we reformulate the problem as

$$\text{find } U(\mu) = (\boldsymbol{u}(\mu), p(\mu)) \in H^1_0(\Omega) \times L^2_0(\Omega) \text{ s.t.}$$
$$A(U(\mu), V; \mu) = F(V) \quad \forall V \in H^1_0(\Omega) \times L^2_0(\Omega)$$

- in particular

$$A(U, V; \mu) = \frac{1}{\mu} A_0(U, V) + A_1(U, V) + A_2(U, U, V) + A_3(U, U, V)$$

where

$$A_0(U, V) = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, d\Omega$$
$$A_1(U, V) = -\int_\Omega \left[\text{div } \boldsymbol{v} \; p - \text{div } \boldsymbol{u} \; q\right] d\Omega$$
$$A_2(Z, U, V) = \int_\Omega (\boldsymbol{z} \cdot \nabla \boldsymbol{u}) \boldsymbol{v} \, d\Omega$$
$$A_3(Z, U, V) = \int_\Omega \nu_T(\boldsymbol{z}) \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, d\Omega$$

for $\quad U = (\boldsymbol{u}, p), \quad V = (\boldsymbol{v}, q), \quad Z = (\boldsymbol{z}, \cdot)$

Well-posedness analysis of the Smagorinsky turbulence model

- based on **Brezzi-Rappaz-Raviart theory**;
- denote by $\partial_1 A(U, V; \mu)(Z)$ the directional derivative of $A(U, V; \mu)$, at U in direction Z;
- requires $\partial_1 A(\cdot, V; \mu)(Z)$ to be inf-sup stable and bounded;

Proposition [Chacón, Delgado, Gómez, B., Rozza (2017)]

$\partial_1 A(U, \cdot; \mu)(\cdot)$ is inf-sup stable and bounded under a small condition assumption on boundary data and forcing terms.

Technical ingredients:

- a (sort of) energy norm needs to be introduced for the velocity, induced by the inner product

$$(\boldsymbol{w}, \boldsymbol{v})_T = \int_\Omega \left[\frac{1}{\overline{\mu}} + \overline{\nu}_T\right] \nabla \boldsymbol{w} : \nabla \boldsymbol{v} \, d\Omega$$

  being

$$\overline{\nu}_T = \nu_T(\boldsymbol{u}(\overline{\mu}))$$

  and

$$\overline{\mu} = \arg\min_{\mu} \left\{ C_S^2 \sum_K h_K^2 \min_{\boldsymbol{x} \in K} \left[\left\| \nabla \boldsymbol{u}_{|_K} \right\| (\boldsymbol{x}) \, \mathbb{1}_K(\boldsymbol{x}) \right] \right\}$$

- data need to be small with respect to the T-norm.

A posteriori error estimation for RB Smagorinsky turbulence model

- RB residual: $\mathcal{R}(V;\mu) = F(V;\mu) - A(U_N(\mu), V;\mu)$, $\forall V \in X = H_0^1(\Omega) \times L^2(\Omega)$;
- dual norm of $\mathcal{R}$:

$$\varepsilon_N(\mu) = \|\mathcal{R}(\cdot;\mu)\|_{X'};$$

- inf-sup constant $\beta_N(\mu)$ and continuity constant $\gamma_N(\mu)$ for $\partial_1 A(U_N(\mu), \cdot;\mu)(\cdot)$;
- a posteriori error bound

$$\Delta_N(\mu) = \frac{\beta_N(\mu)}{2\rho_T}\left[1 - \sqrt{1 - \tau_N(\mu)}\right]$$

where

$$\tau_N(\mu) = \frac{4\varepsilon_N(\mu)\rho_T}{\beta_N(\mu)^2}$$

and ρ_T is an upper bound of the Lipschitz constant for $\partial_1 A(\cdot, V;\mu)(Z)$

Theorem [Chacón, Delgado, Gómez, B., Rozza (2017)]

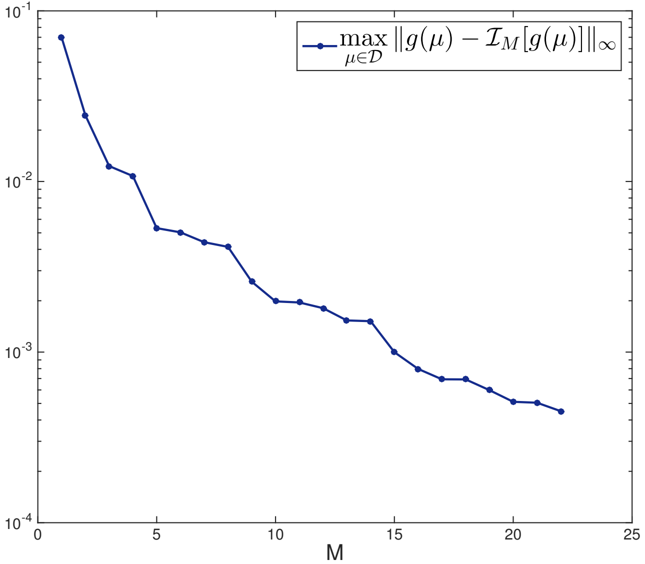
If $\beta_N(\mu) > 0$ and $\tau_N(\mu) \leq 1$, then the following a posteriori error bound holds:

$$\|U(\mu) - U_N(\mu)\|_X \leq \Delta_N(\mu),$$

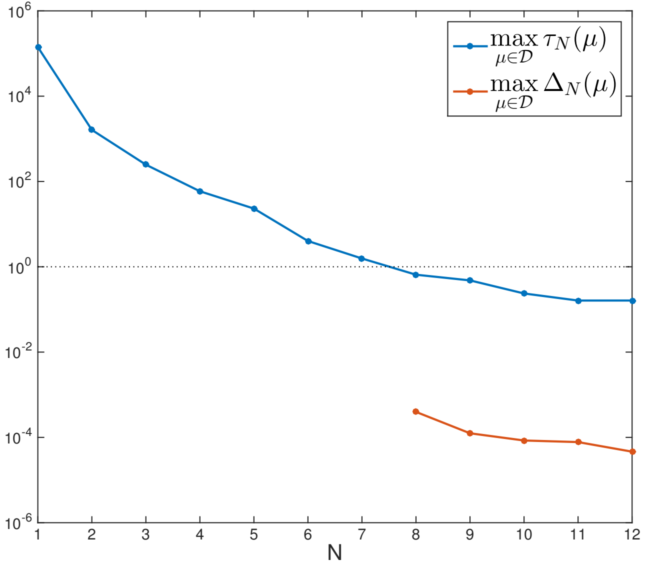
with effectivity

$$\frac{\Delta_N(\mu)}{\|U(\mu) - U_N(\mu)\|_X} \leq \frac{2\gamma_N(\mu)}{\beta_N(\mu)} + \tau_N(\mu)$$

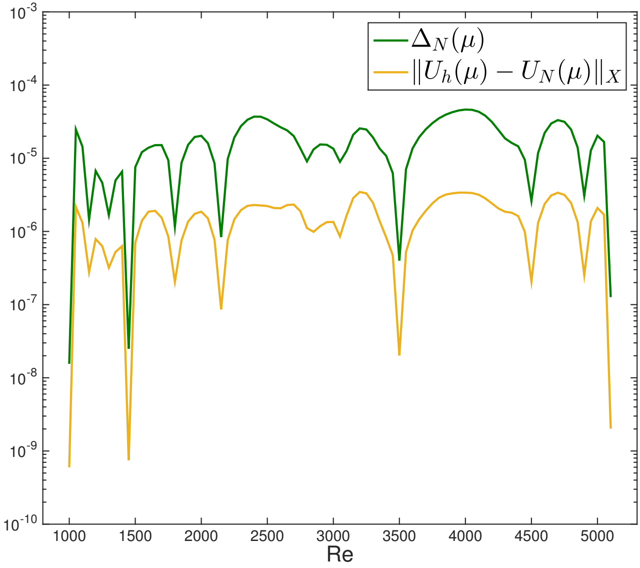
Lid-driven cavity flow – EIM approximation of the Smagorinsky term



Lid-driven cavity flow – Greedy convergence, Re $\in [1000, 5100]$



Lid-driven cavity flow – Error analysis, Re $\in$ [1000, 5100]



Lid-driven cavity flow – Speedup analysis

FE dof: 23003
EIM dof: 22, RB dof: 36

	$\mu = 1610$	$\mu = 2751$	$\mu = 3886$	$\mu = 4521$
T_{FE}	638.02s	1027.62s	1369.49s	1583.08s
T_{online}	0.47s	0.47s	0.47s	0.49s
speedup	1349	2182	2899	3243
$\|\boldsymbol{u} - \boldsymbol{u}_N\|_T$	$1.91 \cdot 10^{-6}$	$1.87 \cdot 10^{-6}$	$3.28 \cdot 10^{-6}$	$6.26 \cdot 10^{-7}$
$\|p - p_N\|_{L^2}$	$1.18 \cdot 10^{-7}$	$3.65 \cdot 10^{-7}$	$3.78 \cdot 10^{-7}$	$8.34 \cdot 10^{-8}$

Part 3: parametrized stochastic partial differential equations

- $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, a **domain**;

- $(A, \mathcal{F}, P)$ a complete **probability** space;
- $\boldsymbol{\mu} : (A, \mathcal{F}) \to (\mathcal{D}, \mathcal{B})$, a **random boldsymbol**:
  - $\mathcal{D} \subset \mathbb{R}^p$, a compact set in the **parameter space**;
  - $\boldsymbol{\mu}(\omega) = (\mu_1(\omega), \ldots, \mu_p(\omega))$ independent identically distributed and absolutely continuous **random variables**;

- $H_0^1(\Omega) \subset \mathbb{V} \subset H^1(\Omega)$;
- $S(\Omega) := L^2(A) \bigotimes \mathbb{V}$;
- $u : \Omega \times A \to \mathbb{R}$, i.e. $u \in S(\Omega)$, a **random field**;
- elliptic PDE, e.g., advection–diffusion stochastic equation

$$-\varepsilon(\boldsymbol{\mu}(\omega))\Delta u(\boldsymbol{\mu}(\omega)) + \boldsymbol{\beta}(\boldsymbol{\mu}(\omega)) \cdot \nabla u(\boldsymbol{\mu}(\omega)) = f(\boldsymbol{\mu}(\omega)) \qquad \text{in } \Omega,$$

s.t. suitable boundary conditions on $\partial\Omega$.

Weighted reduced basis methods: motivation

The introduction of a **weight** in the greedy algorithm reflects our desire of minimizing the squared norm error of the reduced order approximation, i.e.

$$\mathbb{E}\left[\|u - u_N\|_{\mathbb{V}}^2\right] = \int_A \|u(\boldsymbol{\mu}(\omega)) - u_N(\boldsymbol{\mu}(\omega))\|_{\mathbb{V}}^2 \; dP(\omega) =$$

$$= \int_{\mathcal{D}} \|u(\boldsymbol{\mu}) - u_N(\boldsymbol{\mu})\|_{\mathbb{V}}^2 \rho(\boldsymbol{\mu}) \; d\boldsymbol{\mu},$$

being $\rho : A \to \mathbb{R}$ the **probability density distribution** of $\boldsymbol{\mu}$.
Thus,

$$\mathbb{E}\left[\|u - u_N\|_{\mathbb{V}}^2\right] \leq \int_{\mathcal{D}} \Delta_N(\boldsymbol{\mu})^2 \rho(\boldsymbol{\mu}) d\boldsymbol{\mu},$$

This motivates the choice of the weight

$$w(\boldsymbol{\mu}) = \sqrt{\rho(\boldsymbol{\mu})}$$

and the introduction of the error bound

$$\Delta_N^w(\boldsymbol{\mu}) := \Delta_N(\boldsymbol{\mu})\sqrt{\rho(\boldsymbol{\mu})}.$$

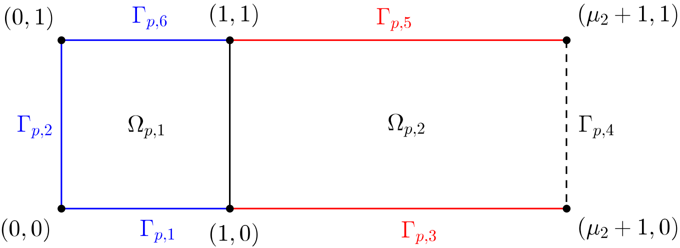
P. Chen, A. Quarteroni, and G. Rozza. A weighted reduced basis method for elliptic partial differential equations with random input data. SIAM Journal on Numerical Analysis, 51(6):3163–3185, 2013.

Weighted reduced basis methods: the greedy algorithm

```
Properly sample Ξ_train ⊂ D
Pick arbitrary μ^1 ∈ Ξ_train
Define S_0 = ∅, V_0 = ∅
for N = 1, ..., N_max
    Perform a PDE solve to compute u(μ^N)
    S_N = S_{N-1} ∪ {μ^N}
    V_N = V_{N-1} ⊕ {u(μ^N)}
    μ^{N+1} = arg max_{μ∈Ξ_train} Δ_N^w(μ)
    if Δ_N^w(μ^{N+1}) ≤ tol
        break
    end
end
```

P. Chen, A. Quarteroni, and G. Rozza. A weighted reduced basis method for elliptic partial differential equations with random input data. SIAM Journal on Numerical Analysis, 51(6):3163–3185, 2013.

Test case: Graetz problem

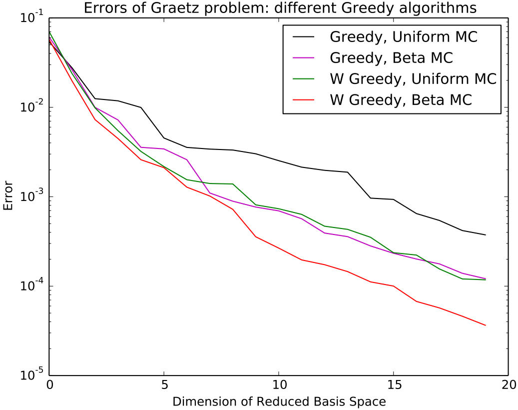


$$\begin{cases} -\frac{1}{\mu_1}\Delta u(\boldsymbol{\mu}) + 4y(1-y)\partial_x u(\boldsymbol{\mu}) = 0 & \text{in } \Omega_p(\boldsymbol{\mu}) \\ u(\boldsymbol{\mu}) = 0 & \text{on } \Gamma_{p,1}(\boldsymbol{\mu}) \cup \Gamma_{p,2}(\boldsymbol{\mu}) \cup \Gamma_{p,6}(\boldsymbol{\mu}) \\ u(\boldsymbol{\mu}) = 1 & \text{on } \Gamma_{p,3}(\boldsymbol{\mu}) \cup \Gamma_{p,5}(\boldsymbol{\mu}) \\ \frac{1}{\mu_1}\frac{\partial u}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma_{p,4}(\boldsymbol{\mu}). \end{cases}$$

$$\mu_1 \sim 10^{1+5\cdot X_1} \quad \text{where } X_1 \sim \text{Beta}(4,2), \quad \mu_1 \in [10^1, 10^6]$$
$$\mu_2 \sim 0.5 + 3.5X_2 \quad \text{where } X_2 \sim \text{Beta}(3,4), \quad \mu_2 \in [0.5, 4]$$

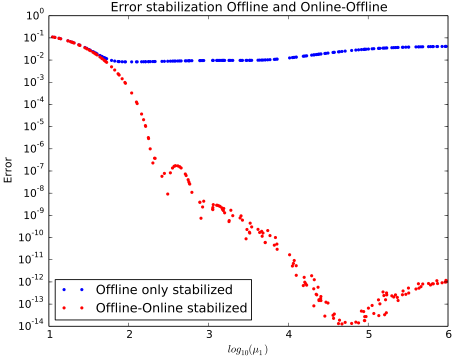
D. Torlo, F. Ballarin, and G. Rozza. Stabilized weighted reduced basis methods for parametrized advection dominated problems with random inputs. Submitted, 2017

Test case: **stabilized RB vs stabilized weighted RB**



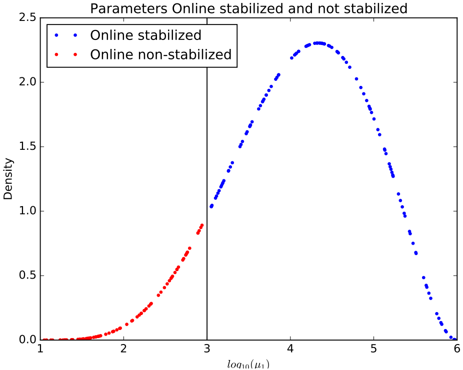
- both **weighting** and **correct sampling** are necessary to obtain good results;
- **weighted** Greedy with sampling from **distribution** guarantees best results;
- **weighted** Greedy with **uniform** sampling is comparable to standard greedy with **sampling from distribution**; both are better than Greedy with **uniform** sampling.

Test case: **selective online stabilization**



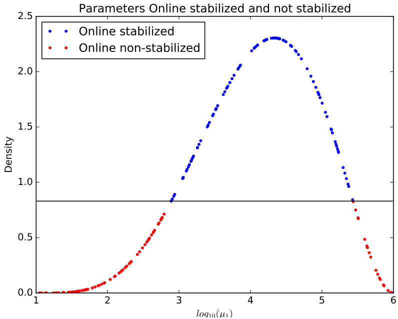
- for low Péclet number ($\mu_1 \leq 10^2$), *Offline-Online* stabilization and *Offline only* stabilization produce very similar results. Thus, we would prefer the **less expensive** *Offline only* stabilization procedure;
- in the regions where the density of $\boldsymbol{\mu}$ is very small, even a large error would be **less relevant** in terms of the probabilistic mean error;
- $\Rightarrow$ enable the more expensive online stabilization only for parameters with high **density** (which would affect more the mean error) or parameters with large **Péclet numbers** (were the more expensive assembly is fully justified by the convection dominated regime)

Test case: **selective online stabilization**



Threshold $\widetilde{\mu}_1$	Error	Percentage non-stabilized
10^1	$7.9673 \cdot 10^{-4}$	0%
$10^{1.5}$	$8.0704 \cdot 10^{-4}$	10%
10^2	$10.0060 \cdot 10^{-4}$	20%
$10^{2.5}$	$18.2806 \cdot 10^{-4}$	33%
10^3	$33.4593 \cdot 10^{-4}$	45%
10^6	0.021128	100%

Test case: selective online stabilization



Threshold $\widetilde{\nu}$	Threshold $\widetilde{\rho}$	Error	Percentage non-stabilized
0	0	$7.9673 \cdot 10^{-4}$	0%
0.001	0.02233	$9.3222 \cdot 10^{-4}$	15%
0.002	0.04423	$9.6456 \cdot 10^{-4}$	17%
0.005	0.09094	$14.7861 \cdot 10^{-4}$	21%
0.01	0.13877	$15.9482 \cdot 10^{-4}$	25%
0.02	0.21433	$25.6017 \cdot 10^{-4}$	30%
0.05	0.38244	$49.1931 \cdot 10^{-4}$	38%
0.1	0.89068	$66.7488 \cdot 10^{-4}$	45%
1	∞	0.021128	100%

Conclusion

- **reduced basis methods** for problems in computational fluid dynamics;
- increase **Reynolds** number;

- **part 1** on reduced order stabilization techniques:
  - how to add online strongly consistent **stabilization**;
  - **interplay** with supremizer enrichment of the velocity space;

- **part 2** on certification of RB for the most simple LES turbulence model:
  - rigorous **a posteriori** error bounds to be used during the greedy algorithm;
  - deal with the nonlinear term introduced by the Smagorinsky turbulence model (EIM);

- **part 3** on weighted reduced basis methods for uncertainty quantification:
  - need to **weigh** and **sample** from relevant distribution during the construction stage;
  - opportunity to **selectively enable** online stabilization based either on probability density function or on the Péclet number.

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Thanks for your attention!

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