Introducing *IR Tools*

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Joint work with **P. C. Hansen** and **J. Nagy**

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UNIVERSITY OF BATH

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Outline

1. Introduction
   - Discrete inverse problems
   - Putting *IR Tools* into place
2. Test problems
   - Image deblurring
   - Computed tomography
   - Inverse Interpolation
3. Iterative Solvers
   - Enhancing classical iterative methods
   - Regularization, projection, hybrid methods
4. Conclusions

Some backrgound

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Numerical solution of $Ax^* + e = b$

- discretization of Fredholm integral equation of the first kind
- $A \in \mathbb{R}^{M \times N}$, $b \in \mathbb{R}^M$
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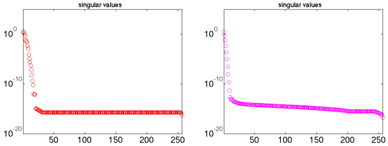

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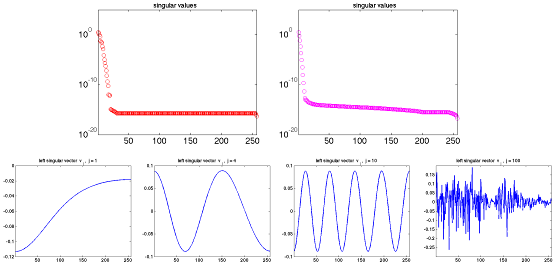

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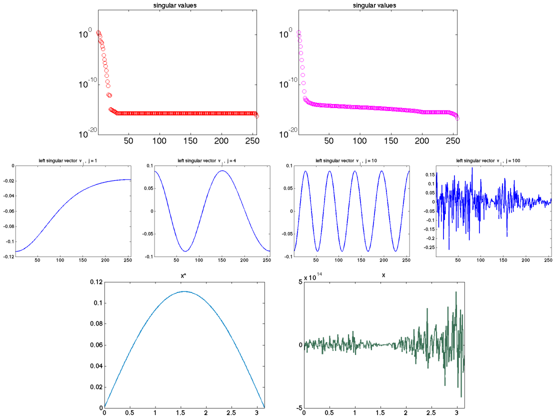

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- "small-scale" problems (direct)
  - TSVD
  - Tikhonov-regularization

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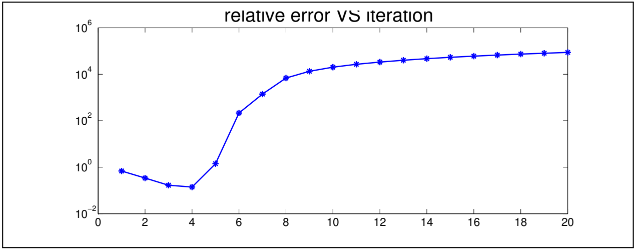

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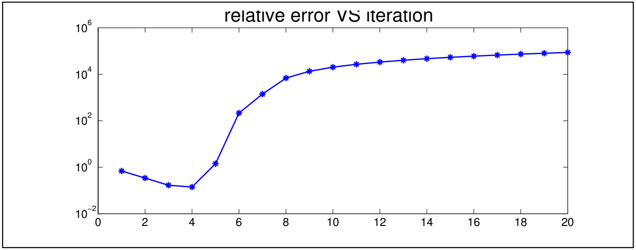

  - Tikhonov(-like) regularization, solved iteratively

$$\min_x \left\{ \|Ax - b\|_2^2 + \lambda^2 \Omega(x) \right\}, \quad \Omega(x) = \|x\|_2^2,\ \|Lx\|_2^2,\ \|x\|_1,\ \mathrm{TV}(x)$$

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- easy to use: almost identical calls to iterative solvers and test-problem generators; naming convention for all functions; default options;
- flexible (control over the parameters) and expandable.

Test problems: the PRxxx functions

Generating a test problem:

1. Define A, b^*, x^*.

2. Add noise to $b^* = Ax^*$: $b = b^* + e$.

3. Visualise the data.

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     image deblurring: spatially (in)variant blur
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     inverse interpolation
   - PRnmr
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Something more about PRblur

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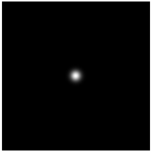
available

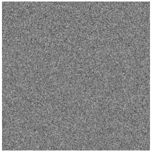
Something more about `PRblur`


PSF

$*$


exact

$+$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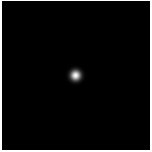
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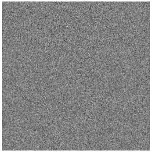
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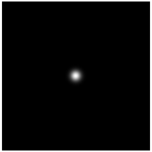
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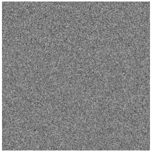
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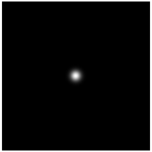

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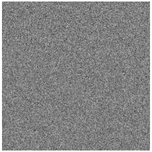

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More advanced call:

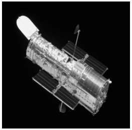
$$[\mathtt{A}, \mathtt{b}, \mathtt{x}, \mathtt{ProbInfo}] = \mathtt{PRblur}(\mathtt{n}, \mathtt{options});$$

Exploring the `PRblur` options

Exploring the PRblur options

- default options (spatially invariant, medium level, reflective b.c.)

x^*



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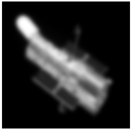

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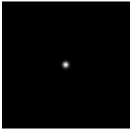

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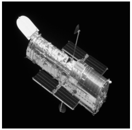

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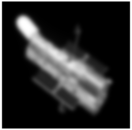

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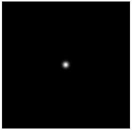

- shaking blur (spatially variant, mild level, zero b.c.)

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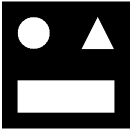

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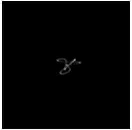

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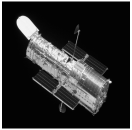

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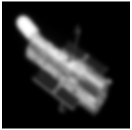

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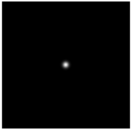

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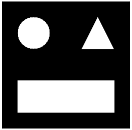

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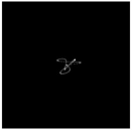

- rotation blur (spatially variant, severe level, periodic b.c.)
  [Hansen, Nagy, and Tigkos. *Rotational image deblurring with sparse matrices*, BIT, 2014]

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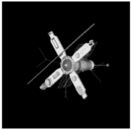

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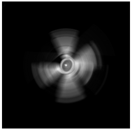

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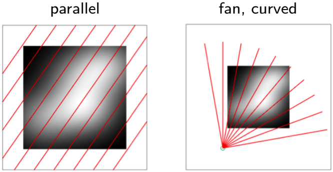

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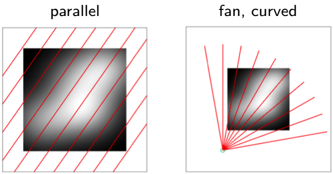

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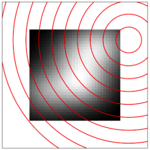

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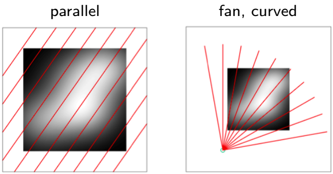

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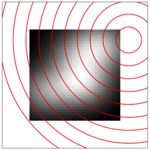
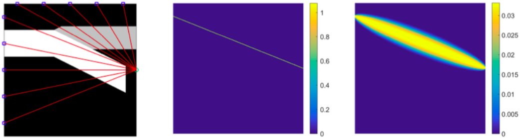


- Seismic travel-time tomography (image courtesy: Hansen, Jorgensen, *AIR Tools II*)



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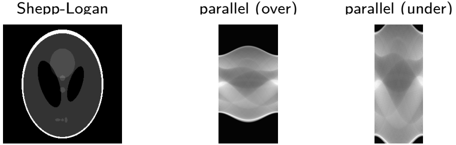

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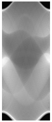

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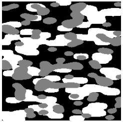
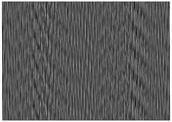
threephases





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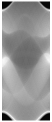

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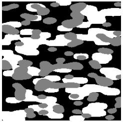
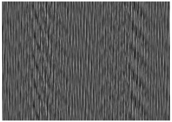
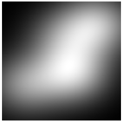
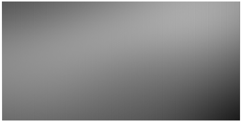




- `[A, b, x, ProbInfo] = PRseismic(n, opt);` choosing `wavemodel`, `p`...

smooth





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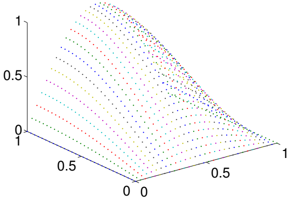

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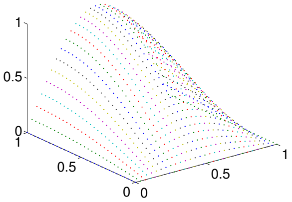

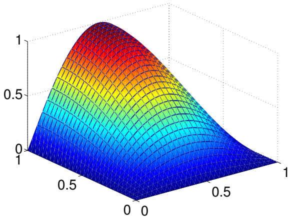

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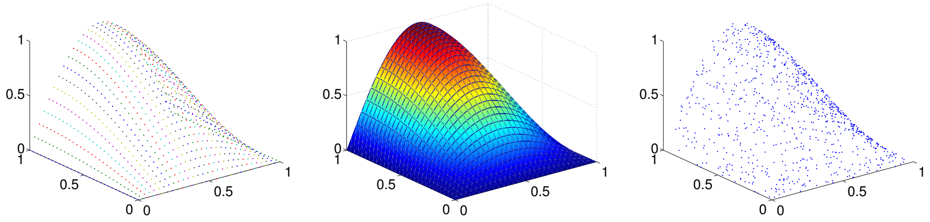

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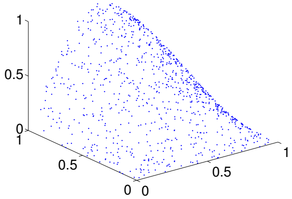

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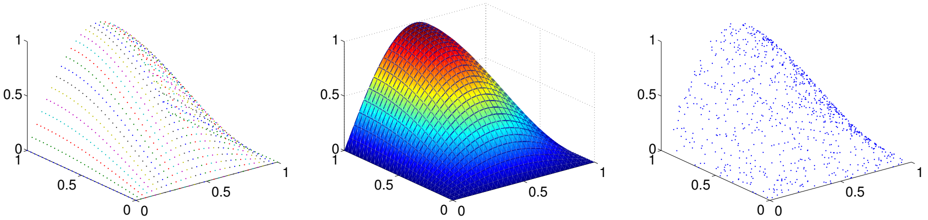

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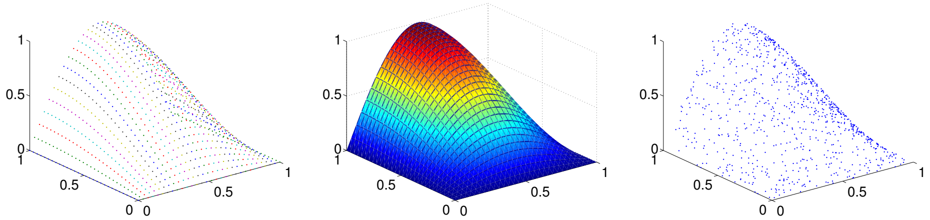

options.InterpMethod: 'linear', 'nearest', 'cubic', 'spline'.

Iterative Solvers: the IRxxx functions

$$\min_{x\in\mathbb{R}^N} \|Ax-b\|_2^2 \qquad \text{(LS)}$$
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solver	problem	notes
IRart	(LS)	
IRsirt	(LS)	
IRmrnsd	(LS)	$x \geq 0$
IRfista	(cLS)	$x \in \mathcal{C}$, $\Omega(x) = \|x\|_1$
Krylov methods		
IRcgls	(LS)	$x \in \hat{\mathcal{K}}_k$
	(cLS)	$x \in \hat{\mathcal{K}}_k$, $\Omega(x) = \|(L)x\|_2^2$
IRenrich	(LS)	$x \in \mathcal{K}_k + \mathcal{W}_p$
IRrrgmres	(LS)	$M = N$, $x \in \hat{\mathcal{K}}_k$
IRnnfcgls	(LS)	$x \geq 0$
IRhybrid_{lsqr}{gmres}	(cLS)	$x \in \hat{\mathcal{K}}_k$
IRhybrid_fgmres	(cLS)	$\Omega(x) = \|x\|_1$, $x \in \hat{\mathcal{K}}_k$
IRrestart	(cLS)	$x \in \mathcal{C} \cap \hat{\mathcal{K}}_k$

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- nonnegativty
  [G. and Wiaux. *Fast nonnegative least squares through flexible Krylov subspaces*, SISC, 2017]
  Apply flexible CGLS to: $XA^T(Ax - b)$, $x \geq 0$.

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These two approaches are equivalent!

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- Generalized cross validation (GCV)
  [Chung, Nagy and O'Leary, *A weighted-GCV method for Lanczos-hybrid regularization*, ETNA, 2008]

$$\min_{\lambda} G(\lambda)\,, \quad G(\lambda) = \frac{\|(I - AA_\lambda^\sharp)b\|_2^2}{(\mathrm{trace}(I - AA_\lambda^\sharp))^2}$$

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Consider a standard form transformation

$$\min_x \left\{ \|b - Ax\|_2^2 + \lambda \|L_m(x - x^*)\|_2^2 \right\}$$

$$\min_{\widetilde{x}} \{ \|b - \widetilde{A}_m \widetilde{x}\|_2^2 + \widetilde{\lambda} \|\widetilde{x} - \widetilde{x}^*\|_2^2 \}$$

- $\widetilde{A}_m = AL_m^{-1}$
- $\widetilde{x}^* = L_m x^*$
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With an iteratively reweighted-norm approach:

$$\|x\|_1 \approx \|\mathcal{W}x\|_2^2 = \|\mathcal{W}_m x\|_2^2, \quad \text{with} \quad \mathcal{W}_m = L_m = \text{diag}\left(\frac{1}{\sqrt{|x_{m-1}|}}\right).$$

Consider a standard form transformation

$$\min_x \left\{\|b - Ax\|_2^2 + \lambda\|L_m(x - x^*)\|_2^2\right\}$$

$$\min_{\widetilde{x}}\{\|b - \widetilde{A}_m\widetilde{x}\|_2^2 + \widetilde{\lambda}\|\widetilde{x} - \widetilde{x}^*\|_2^2\}$$

- $\widetilde{A}_m = AL_m^{-1}$
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Efficiently handled by Flexible Arnoldi, in a hybrid fashion.
[G. and Nagy, *Generalized AT method for sparse reconstruction*, SISC, 2014]

Regularization matrix choice for hybrid methods

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Other possible approaches: restarted Krylov methods.

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Some references:

P. C. Hansen.
Discrete Inverse Problems: Insight and Algorithms.
SIAM, 2010.

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THANKS FOR YOUR ATTENTION!