Multilevel discrete least squares polynomial apporixmation

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Contents

PDEs with random parameters

Consider a differential problem

$$\mathcal{L}(u;\mathbf{y}) = \mathcal{G} \qquad (*)$$

depending on a set of random parameters
$\mathbf{y} = (y_1, \ldots, y_N) \in \Gamma \subset \mathbb{R}^N$ with joint probability measure μ on Γ.

We assume that (*) has a unique solution $u(\mathbf{y})$, in some suitable function space V, and we focus on a Quantity of Interest $Q : V \to \mathbb{R}$.

Goal: approximate the whole response function

$$\mathbf{y} \mapsto f(\mathbf{y}) := Q(u(\mathbf{y})) : \Gamma \to \mathbb{R}$$

by multivariate polynomials.

Possibly derive approximated statistics as $\mathbb{E}[f]$, $Var[f]$, etc.

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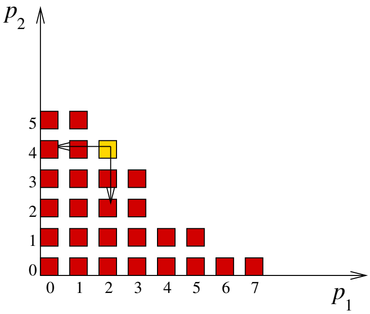
Polynomial approximation on downward closed sets

Assume $f \in L^2_\mu(\Gamma)$. We seek an approximation of f in a finite dimensional polynomial subspace

$$V_\Lambda = \mathrm{span}\left\{\prod_{n=1}^N y_n^{p_n}, \quad \text{with } \mathbf{p} = (p_1, \ldots, p_N) \in \Lambda\right\}$$

with $\Lambda \subset \mathbb{N}^N$ a downward closed index set.



Definition. An index set Λ is downward closed if

$$\mathbf{p} \in \Lambda \text{ and } \mathbf{q} \leq \mathbf{p} \quad \Longrightarrow \quad \mathbf{q} \in \Lambda.$$

Contents

Weighted discrete least squares approximation

1. Sample independently M points $\left(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(M)}\right) \in \Gamma^M$ from a distribution $\nu \ll \mu$, with density $\rho = \frac{d\nu}{d\mu}$;
2. define the weight function $w(\mathbf{y}) = \frac{1}{\rho(\mathbf{y})}$;
3. find weighted discrete least squares approximation on V_Λ

$$\hat{\Pi}_M f = \operatorname*{argmin}_{v \in V_\Lambda} \|f - v\|_M \quad \text{with} \quad \|g\|_M^2 = \frac{1}{M} \sum_{j=1}^{M} w\left(\mathbf{y}^{(j)}\right) g\left(\mathbf{y}^{(j)}\right)^2.$$

Here: $\mathbb{E}\left[\|g\|_M^2\right] = \int_\Gamma w(\mathbf{y}) g(\mathbf{y})^2 \nu(d\mathbf{y}) = \int_\Gamma g(\mathbf{y})^2 \mu(d\mathbf{y}) = \|g\|_{L^2_\mu}^2$.

Algebraic system: let $\{\phi_j\}_{j=1}^{|\Lambda|}$ be a basis of V_Λ, orthonormal w.r.t. μ, and $\hat{\Pi}_M f(\mathbf{y}) = \sum_{j=1}^{|\Lambda|} c_j \phi_j(\mathbf{y})$. Then, $\mathbf{c} = \left(c_1, \ldots, c_{|\Lambda|}\right)^T$ satisfies

$$G\mathbf{c} = \hat{\mathbf{f}}, \qquad G_{i,j} = (\phi_i, \phi_j)_M, \quad \hat{f}_i = (f, \phi_i)_M.$$

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Optimally of discrete least squares approximation

Theorem ([Cohen-Migliorati 2017][Cohen-Davenport-Leviatan 2013])

For arbitrary $r > 0$ define

$$\kappa_r := \frac{1/2(1-\log 2)}{1+r} \quad \text{and} \quad K_{\Lambda,w} := \sup_{\mathbf{y}\in\Gamma}\left(w(\mathbf{y})\sum_{j=1}^{|\Lambda|}\phi_i(\mathbf{y})^2\right).$$

If $\dfrac{M}{\log M} \geq \dfrac{K_{\Lambda,w}}{\kappa_r}$, then

- $P\left(\|G-I\| \leq \frac{1}{2}\right) > 1 - 2M^{-r}$.
- $\|f - \hat{\Pi}_M f\|_{L^2_\mu} \leq (1+\sqrt{2}) \inf_{v\in V_\Lambda} \|f-v\|_{L^\infty_{\sqrt{w}}}$ with prob. $> 1 - 2M^{-r}$.
- $\mathbb{E}\left[\|f - \hat{\Pi}^c_M f\|^2_{L^2_\mu}\right] \leq C_M \inf_{v\in V_\Lambda} \|f-v\|^2_{L^2_\mu} + 2\|f\|^2_{L^2_\mu} M^{-r}$

  where $\hat{\Pi}^c_M f = \hat{\Pi}_M f \cdot \mathbf{1}_{\{\|G-I\|\leq\frac{1}{2}\}}$ and $C_M = \left(1 + \frac{4\kappa_r}{\log M}\right) \xrightarrow{M\to\infty} 1$.

Sufficient number of points - uniform measure

- Uniform measure: $\mu = \mathcal{U}\left(\prod_{i=1}^{N} \Gamma_i\right)$

[Chkifa-Cohen-Migliorati-Nobile-Tempone 2015] When sampling from the same distribution ($\nu = \mu$ and $w = 1$), then

$$|\Lambda| \leq K_{\Lambda,1} \leq |\Lambda|^2.$$

Hence, (unweighted) discrete least square is stable and optimally convergent under the condition

$$\frac{M}{\log M} \geq \frac{|\Lambda|^2}{\kappa_r} \qquad \text{(quadratic proportionality).}$$

Sufficient number of points - optimal measure

- [Cohen-Migliorati 2017] For arbitrary μ, when sampling from the optimal measure

$$\frac{d\nu^*}{d\mu}(\mathbf{y}) = \rho^*(\mathbf{y}) = \frac{1}{|\Lambda|}\sum_{j=1}^{|\Lambda|}\phi_j(\mathbf{y})^2 \quad \Longrightarrow \quad K_{\Lambda,w^*} = |\Lambda|.$$

Hence, weighted discrete least squares stable and optimal with

$$\frac{M}{\log M} \geq \frac{|\Lambda|}{\kappa_r} \qquad \text{(linear proportionality).}$$

- Sampling algorithms from the optimal distribution are available (marginalization [Cohen-Migliorati 2017], acceptance rejection [H.-Nobile-Tempone-Wolfers, 2017])

However, the optimal distribution depends on Λ. Not good for adaptive algorithms

Sufficient number of points - Chebyshev measure

- Alternatively, for uniform measure μ (or more generally a product measure $\mu = \otimes_{j=1}^{N} \mu_j$, with μ_j doubling measure, i.e. $\mu_j(2I) = L\mu_j(I)$) one can sample from the arcsin (Chebyshev) distribution.

$$K_{\Lambda,w} \leq C^N |\Lambda|, \qquad \frac{M}{\log M} \geq \frac{C^N}{\kappa_r} |\Lambda|.$$

Still linear scaling but with a constant exponentially dependent on N.

Advantage: the sampling measure does not depend on Λ.

Good for adaptivity.

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Multilevel least squares approximation

In practice $f(\mathbf{y}) = Q(u(\mathbf{y}))$ can not be evaluated exactly as it requires the solution of a differential equation.

- We introduce a sequence of approximations f_{n_ℓ}, $n_\ell \in \mathbb{N}$ with increasing cost, s.t.

$$\lim_{\ell \to \infty} \|f - f_{n_\ell}\|_{L^2_\mu} = 0,$$

(or possibly a stronger norm)

- Similarly, we introduce a sequence of nested downward closed sets

$$\Lambda_{m_0} \subset \Lambda_{m_1} \subset \ldots \subset \Lambda_{m_k} \subset \ldots$$

such that

$$\lim_{k \to \infty} \inf_{v \in V_{\Lambda_{m_k}}} \|f - v\|_{L^2_\mu} = 0.$$

Correspondingly, for each Λ_{m_k} we introduce a weighted discrete least squares projector $\hat{\Pi}_{M_k}$ using $\frac{M_k}{\log M_k} = O(|\Lambda_{m_k}|)$ random points.

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Multilevel least squares approximation

Multilevel formula: given maximum level $L \in \mathbb{N}$

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\begin{aligned}
S_L f &= \sum_{k+\ell \le L} (\hat{\Pi}_{M_k} - \hat{\Pi}_{M_{k-1}})(f_{n_\ell} - f_{n_{\ell-1}}) \\
&= \sum_{\ell=0}^{L} \hat{\Pi}_{M_{L-\ell}}(f_{n_\ell} - f_{n_{\ell-1}}).
\end{aligned}
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- In the multilevel formula one might consider more general index sets $(k,\ell) \in \mathcal{I} \subset \mathbb{R}^2$. However, one can always recast to $k+\ell \le L$ by properly choosing $\{n_\ell\}$ and $\{m_k\}$.
- **Question:** How to properly choose $\{n_\ell\}$ and $\{m_k\}$?
- **Issue**: Since the least squares projection is random, we have to ensure that it is stable and optimally convergent on all levels. (Need union bound on failure probabilities)

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Assumptions for ML

- For the Multilevel algorithm to be effective, we have to rely on certain "mixed regularity".
- Let $(F, \|\cdot\|_F) \hookrightarrow (L^2_\mu, \|\cdot\|_{L^2_\mu})$ be a normed vector space of "smooth" functions (e.g. Hölder / Sobolev / analytic regularity).

Assumptions for ML

- **Assumption 1 (regularity):** $f, f_{n_\ell} \in F$ for all $\ell \in \mathbb{N}$
- **Assumption 2 (PDE discretization):** the sequence $\{f_{n_\ell}\}$ is s.t.

$$\|f - f_{n_\ell}\|_{L^2_\mu} \lesssim n_\ell^{-\beta_w}, \qquad \|f - f_{n_\ell}\|_F \lesssim n_\ell^{-\beta_s}$$

and, for a single $\mathbf{y} \in \Gamma$, the cost of computing $f_{n_\ell}(\mathbf{y})$ is

$$\mathrm{Work}(f_{n_\ell}) \lesssim n_\ell^{\gamma}.$$

- **Assumption 3 (polynomial approximability):** the sequence $\{\Lambda_{m_k}\}$ is s.t.

$$\dim\left(V_{\Lambda_{m_k}}\right) = |\Lambda_{m_k}| \lesssim m_k^{\sigma},$$

$$\inf_{v \in V_{\Lambda_{m_k}}} \|f - v\|_{L^\infty_{\sqrt{w}}} \lesssim m_k^{-\alpha_p} \|f\|_F, \quad \forall f \in F,$$

(Alternatively $\inf_{v \in V_{\Lambda_{m_k}}} \|f - v\|_{L^2_\mu} \lesssim m_k^{-\alpha_e} \|f\|_F, \quad \forall f \in F$).

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Tuning the ML least squares algorithm

We now choose

$$n_\ell = C \exp\left(\frac{\ell}{\gamma + \beta_s}\right), \quad \ell = 0, \dots, L \quad \text{(space discr.)}$$

$$m_k = C \exp\left(\frac{k}{\sigma + \alpha_p}\right), \quad k = 0, \dots, L \quad \text{(Polynomial approx.)}$$

$$\frac{m_k^\sigma}{\kappa_L} \leq \frac{M_k}{\log M_k} \leq \frac{2 m_k^\sigma}{\kappa_L}, \quad k = 0, \dots, L \quad \text{(sample size with } r = L\text{)}$$

By taking $r = L$ we guarantee that

$$P\left(\exists k : \|G_k - I\| > \frac{1}{2}\right) \leq \sum_{k=0}^{L} P\left(\|G_k - I\| > \frac{1}{2}\right) \lesssim L^{-L}.$$

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Complexity result

Theorem ([H.-Nobile-Tempone-Wolfers 2017])

Given $\epsilon > 0$ and $\beta_s = \beta_w$, we can choose $L \in \mathbb{N}$ such that

$$\|f - S_L f\|_{L^2_\mu} \leq \epsilon, \qquad \text{with prob.} \geq 1 - C\epsilon^{\log|\log \epsilon|},$$

$$\text{Work}(S_L f) \lesssim \epsilon^{-\lambda} |\log \epsilon|^t \log|\log \epsilon|,$$

with

$$\lambda = \max\left(\sigma/\alpha_p, \gamma/\beta_s\right),$$

$$t = \begin{cases} 2 & \text{if } \gamma/\beta_s < \sigma/\alpha_p, \\ 3 + \sigma/\alpha_p & \text{if } \gamma/\beta_s = \sigma/\alpha_p, \\ 1 & \text{if } \gamma/\beta_s > \sigma/\alpha_p. \end{cases}$$

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Analogous result holds in expectation with α_p replaced by α_e.

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Improved complexity in the case $\gamma/\beta_s > \sigma/\alpha$

In the case $\gamma/\beta_s > \sigma/\alpha$ and $\beta_w > \beta_s$ the complexity can be improved by taking

$$m_k = C \exp\left(\frac{k}{\sigma + \alpha_p} + \frac{L(\beta_w - \beta_s)}{\alpha(\gamma + \beta_s)}\right).$$

In this case the complexity result becomes

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which always improves the single level rate $\lambda_{SL} = \frac{\gamma}{\beta_w} + \frac{\sigma}{\alpha_p}$.

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Application to random elliptic PDEs

Consider

$$\begin{cases} -\operatorname{div}(a(\mathbf{y})\nabla u(\mathbf{y})) = g, & \text{in } D \subset \mathbb{R}^d \\ u(\mathbf{y}) = 0, & \text{on } \partial D \end{cases}$$

with $\mathbf{y} \in \Gamma = [-1,1]^N$ and Q linear bounded functional in $L^2(D)$ (e.g. $Q(u) = \int_D u$).

Goal: approximate $f(\mathbf{y}) = Q(u(\mathbf{y}))$.

Assumptions:

- $0 < a_{min} \leq a(\mathbf{x},\mathbf{y}) \leq a_{max}, \quad \forall (\mathbf{x},\mathbf{y}) \in D \times \Gamma.$
- g and D sufficiently smooth.

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- $0 < a_{min} \leq a(\mathbf{x},\mathbf{y}) \leq a_{max}, \quad \forall(\mathbf{x},\mathbf{y}) \in D \times \Gamma.$
- g and D sufficiently smooth.

Application to random elliptic PDEs

Consider

$$\begin{cases} -\operatorname{div}(a(\mathbf{y})\nabla u(\mathbf{y})) = g, & \text{in } D \subset \mathbb{R}^d \\ u(\mathbf{y}) = 0, & \text{on } \partial D \end{cases}$$

with $\mathbf{y} \in \Gamma = [-1,1]^N$ and Q linear bounded functional in $L^2(D)$ (e.g. $Q(u) = \int_D u$).

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Application to random elliptic PDEs

Proposition

Let u_n be a finite element approximation of order $r \geq 1$ with maximal element diameter $h = n^{-1}$ and $f_n(\mathbf{y}) = Q(u_n(\mathbf{y}))$.

- If $a \in C^r(D \times \Gamma)$, then

$$\|f - f_n\|_{L^2_\mu(\Gamma)} \lesssim h^{r+1}, \qquad \|f - f_n\|_{C^{r-1}(\Gamma)} \lesssim h^2.$$

- If $a \in C^{r,s}(D \times \Gamma) = \{v : D \times \Gamma \to \mathbb{R} : \|\partial_x^{\mathbf{r}} \partial_y^{\mathbf{s}} v\|_{C^0(D\times\Gamma)} < \infty,$ $\forall |\mathbf{r}|_1 \leq r,\ |\mathbf{s}|_1 \leq s\}$, then

$$\|f - f_n\|_{C^p(\Gamma)} \lesssim h^{r+1}, \qquad \forall p = 0, \ldots, s.$$

ML least squares complexity – mixed regularity

Consider the coefficient

$$a(\mathbf{x}, \mathbf{y}) = 1 + \|\mathbf{x}\|_2^r + \|\mathbf{y}\|_2^s \in C^{r-1,1}(D) \otimes C^{s-1,1}(\Gamma).$$

- smoother space: $F = C^{s-1,1}(\Gamma)$;
- spatial approximation: continuous finite elements of degree r,
  - error: $\|f - f_n\|_{L^2_\mu} = O\left(n^{-(r+1)}\right) = \|f - f_n\|_{C^{s-1,1}} \implies \beta_w = \beta_s = r + 1$;
  - cost: $\mathrm{Work}(f_n) = n^d$ with optimal solver $\implies \gamma = d$;
- Polynomial approximation: $V_{\Lambda_m} = \mathbb{P}_m$= polynomial space of total degree m,
  - error: $\|f - \Pi_{\mathbb{P}_m} f\|_{L^\infty} = O\left(m^{-s}\right), \implies \alpha_p = s$;
  - cost: $\dim\left(V_{\Lambda_m}\right) = \binom{m+N}{N} \lesssim m^N, \implies \sigma = N$.

ML least squares complexity – mixed regularity

- Complexity of single level method

$$\mathrm{Work}_{\mathsf{SL}} = \mathcal{O}\left(\epsilon^{-\frac{d}{r+1}-\frac{N}{s}}\log\epsilon^{-1}\right).$$

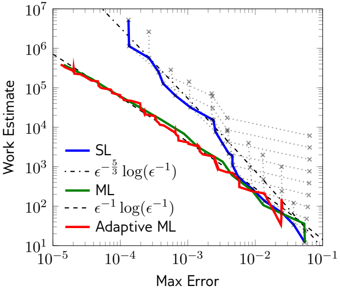
- Complexity of multilevel method

$$\mathrm{Work}_{\mathsf{ML}} = \mathcal{O}\left(\epsilon^{-\max\left\{\frac{d}{r+1},\frac{N}{s}\right\}}\left(\log\epsilon^{-1}\right)^{t}\right),$$

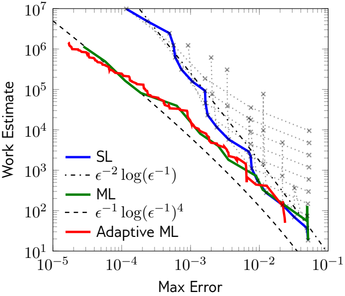
with

$$t = \begin{cases} 1, & \text{if } \frac{d}{r+1} > \frac{N}{s}, \\ 3+\frac{d}{r+1}, & \text{if } \frac{d}{r+1} = \frac{N}{s}, \\ 2, & \text{if } \frac{d}{r+1} < \frac{N}{s}. \end{cases}$$

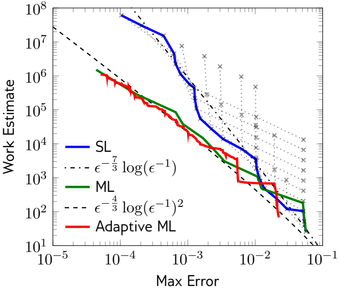
- In our experiment: $d=2, r=1, s=3$ and $N=2,3,4,6$.



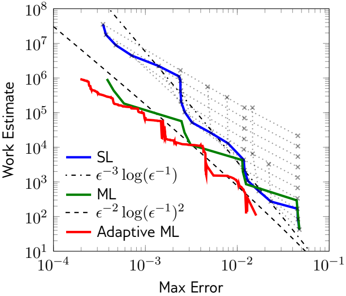
(a) $N = 2$



(b) $N = 3$



(c) $N = 4$



(d) $N = 6$

Contents

Conclusions

- We have derived a multilevel discrete least squares method for polynomial approximation of an output quantity of interest of a random PDE.
- The method uses the classical "Combination technique" and sparsifies sequences of polynomial approximations, obtained by weighted discrete least squares, and sequences of spatial discretizations of the underlying PDE.
- In particular, we have proposed a way to select the number of sample points on each level to guarantee the overall stability and accuracy of the ML formula with high probability.
- Complexity analysis carries over to infinite dimensional problems (different choice of polynomial spaces).
- We are currently working on adaptive algorithms for infinite dimensional problems.

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Thank you for your attention.

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Sketch of the proof

- Bound on M_k: use that $\sqrt{M_k} \leq \frac{M_k}{\log M_k} \leq \frac{2m_k^\sigma}{\kappa_L}$ and $\kappa_L \approx 1/(L+1)$

$$M_k \leq \frac{2}{\kappa_L} m_k^\sigma \log M_k \lesssim (L+1)e^{\frac{k\sigma}{\sigma+\alpha_p}}$$
$$\lesssim (L+1)\log(L+1)e^{\frac{k\sigma}{\sigma+\alpha_p}}(k+1)$$

- Bound on total work:

$$\mathrm{Work}(S_L f) \lesssim \sum_{\ell=0}^{L} M_{L-\ell} n_\ell^\gamma$$
$$\lesssim (L+1)\log(L+1)e^{\frac{L\sigma}{\sigma-\alpha_p}} \sum_{\ell=0}^{L} \exp\left\{-l\left(\frac{\sigma}{\sigma-\alpha_p} - \frac{\gamma}{\gamma+\beta_s}\right)\right\}(L-\ell+1)$$

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Sketch of the proof

- Bound on the error in probability:

$$\begin{aligned}\|f - S_L f\|_{L^2_\mu} &= \|f - f_L + \sum_{\ell=0}^{L}(Id - \hat{\Pi}_{M_{L-\ell}})(f_\ell - f_{\ell-1})\|_{L^2_\mu} \\ &\leq \|f - f_L\|_{L^2_\mu} + \sum_{\ell=0}^{L} \|Id - \hat{\Pi}_{M_{L-\ell}}\|_{F\to L^2_\mu} \|f_\ell - f_{\ell-1}\|_F \\ &\lesssim e^{-\frac{L\beta_w}{\gamma+\beta_s}} + e^{-\frac{L\alpha}{\sigma+\alpha}} \sum_{\ell=0}^{L} \exp\left\{\ell\left(\frac{\alpha}{\sigma+\alpha_p} - \frac{\beta_s}{\gamma+\beta_s}\right)\right\}\end{aligned}$$

Again split the three cases $\gamma/\beta_s <, =, > \sigma/\alpha_p$ and notice that the first term $e^{-\frac{L\beta_w}{\gamma+\beta_s}}$ is always negligible as $\beta_w > \beta_s$.

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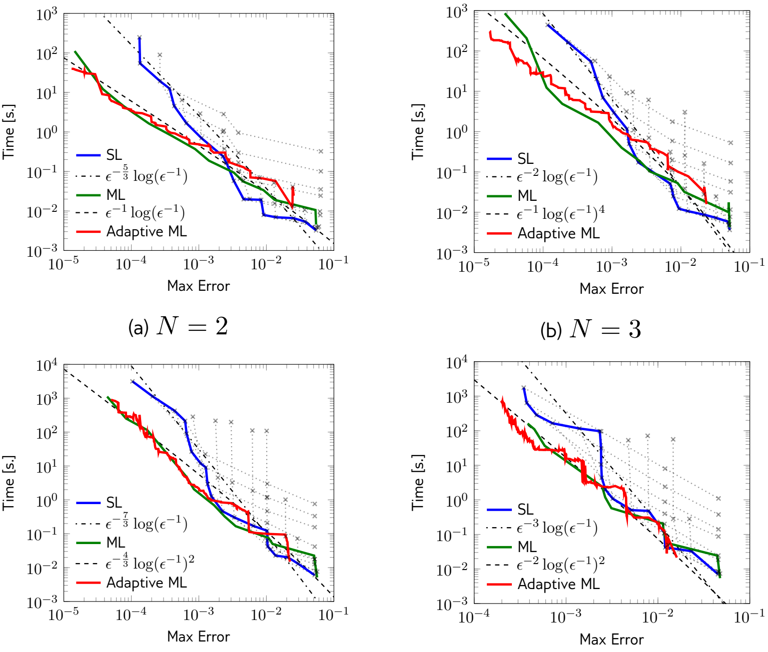

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